Plan

Random Walks on Bratteli Diagrams

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Université d'Orléans

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Introduction

My talk is based on the following article:

A. Connes and E. J. Woods, Hyperfinite von Neumann algebras and Poisson boundaries of time dependent random walks, Pacific J. Math. 137 (1989), no 2, 225-243.

It contains the statement of the two theorems which I am going to describe:

1. the description of a state on a hyperfinite von Neumann algebra (due to A. Connes);
2. the ergodic decomposition of a Markov measure via harmonic functions (a classical result in J. Neveu 64).

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Cartan subalgebras

Definition (Vershik, Feldman-Moore)

An abelian subalgebra B of a von Neumann algebra A is called a *Cartan subalgebra* if

1. B is a masa;
2. B is regular;
3. there exists a faithful normal conditional expectation $E_B : A \to B$.

Note that E_B is unique.

The basic example is $A = M_n(\mathbb{C})$ and $B = D_n(\mathbb{C})$, the subalgebra of diagonal matrices. The next theorem says that the general case looks like this basic example.

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Feldman-Moore theorem

Theorem (Feldman-Moore 77)

Let B be a Cartan subalgebra of a von Neumann algebra A on a separable Hilbert space. Then there exists a countable standard measured equivalence relation R on (X, μ), a twist $\sigma \in Z^2(R, \mathbf{T})$ and an isomorphism of A onto $W^*(R, \sigma)$ carrying B onto the diagonal subalgebra $L^\infty(X, \mu)$. The twisted relation (R, σ) is unique up to isomorphism.

In our basic example, $X = \{1, \ldots, n\}$ and $R = X \times X$ (the twist σ is trivial). The most general finite dimensional von Neumann algebra is given by an arbitrary equivalence relation R on a finite set X.

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Normal states on type I von Neumann algebras

The following result is well known:

Theorem (textbook)

Let φ be a normal state on $\mathcal{B}(H)$, where H is a Hilbert space. Then there exists a Cartan subalgebra B such that $\varphi = \mu \circ E_B$, where μ is the restriction of φ to B.

Indeed, $\varphi = \mathrm{Tr}(\Omega .)$, where Ω is a positive trace-class operator. The Cartan subalgebra B is determined by an orthonormal basis of eigenvectors of Ω. The probability measure μ is given by the eigenvalues of Ω.

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Normal states on hyperfinite von Neumann algebras 1

It may not be as well known that this result remains valid for any hyperfinite von Neumann algebra.

Theorem (Connes 75, version 1)

Let φ be a faithful normal state on a hyperfinite von Neumann algebra A. Then there exists a Cartan subalgebra B such that $\varphi = \mu \circ E_B$, where μ is the restriction of φ to B.

The proof proceeds in two steps.

1) First one shows the existence of an increasing sequence (A_n) of f.d. subalgebras with weakly dense union such that each A_n is globally invariant under the modular automorphism group σ^φ of the state φ.

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Nested Cartan subalgebras

2) Second one constructs a Cartan subalgebra B_n of A_n such that

$$\varphi_{|A_n} = \varphi_{|B_n} \circ E_{B_n}$$

and $(A_{n-1}, B_{n-1}) \subset (A_n, B_n)$ in the sense that

- $B_{n-1} \subset B_n$
- the normalizer of B_{n-1} in A_{n-1} is contained in the normalizer of B_n in A_n.

Because of the invariance under the modular group, there exists an expectation $F_n : A_n \to A_{n-1}$ such that $\varphi_{|A_n} = \varphi_{|A_{n-1}} \circ F_n$.

Inclusions and expectations are conveniently described by Bratteli diagrams.

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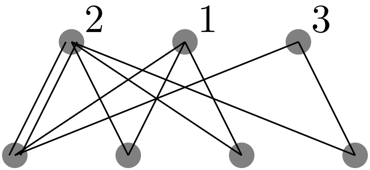
Bratteli diagram of an inclusions of f.d. C*-algebras

The following diagram represents an inclusion of (A_1, B_1) in (A_2, B_2) where $A_1 = M_2(\mathbb{C}) \oplus \mathbb{C} \oplus M_3(\mathbb{C})$,
$B_1 = D_2(\mathbb{C}) \oplus \mathbb{C} \oplus D_3(\mathbb{C})$ and
$A_2 = M_8(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_5(\mathbb{C})$,
$B_2 = D_8(\mathbb{C}) \oplus D_3(\mathbb{C}) \oplus D_3(\mathbb{C}) \oplus D_5(\mathbb{C})$.

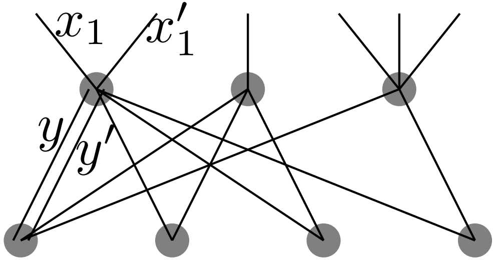


Theorem (Bratteli 72)

These diagrams classify all inclusions of f.d. C-algebras.*

path model of the inclusion

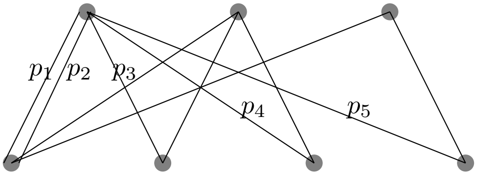
X_i is the set of paths x_i ending at level i; R_i is the set of pairs (x_i, x_i') ending at the same vertex.



$$j(f)(yx_1, y'x_1') = \begin{cases} f(x_1, x_1') & \text{if} \quad y = y' \\ 0 & \text{if} \quad y \neq y' \end{cases}$$

Expectations of f.d. C*-algebras

The following diagram represents a faithful expectation $F : A_2 \to A_1$:



where $p_i > 0$ and $p_1 + p_1 + p_3 + p_4 + p_5 = 1$.

Theorem

These diagrams classify all faithful expectations of f.d. C-algebras.*

Construction of the Cartan subalgebra

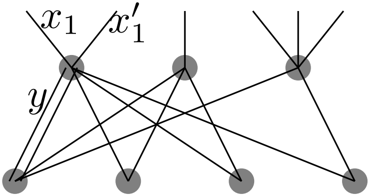
Proof. One chooses a Cartan subalgebra $B_1 \subset A_1$. The problem is to find a Cartan subalgebra $B_2 \subset A_2$ such that $(A_1, B_1) \subset (A_2, B_2)$ and making the following diagram commutative

$$\begin{array}{ccc} A_2 & \xrightarrow{E_2} & B_2 \\ F \downarrow & & \downarrow F_{|B_2} \\ A_1 & \xrightarrow{E_1} & B_1 \end{array}$$

which is always possible.

path model of the expectation

This gives the following expression of the expectation.



$$F(g)(x_1, x'_1) = \sum_y p(y)g(yx_1, yx'_1)$$

where the sum runs over all edges y emanating from the common vertex $r(x_1) = r(x'_1)$.

Normal states on hyperfinite von Neumann algebras 2

The Bratteli diagram description of the inclusions and the expectations given above leads to an equivalent but more picturesque description of a faithful normal state on a hyperfinite von Neumann algebra.

Theorem (Connes 75, version 2)

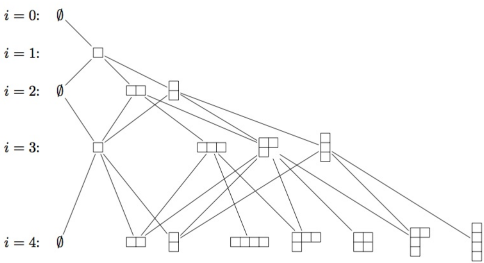
Each faithful normal state on a hyperfinite von Neumann algebra can be described as a random walk on a Bratteli diagram.

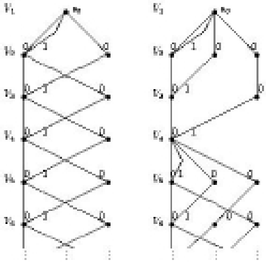
Let us explain this statement.

Bratteli diagrams

Definition

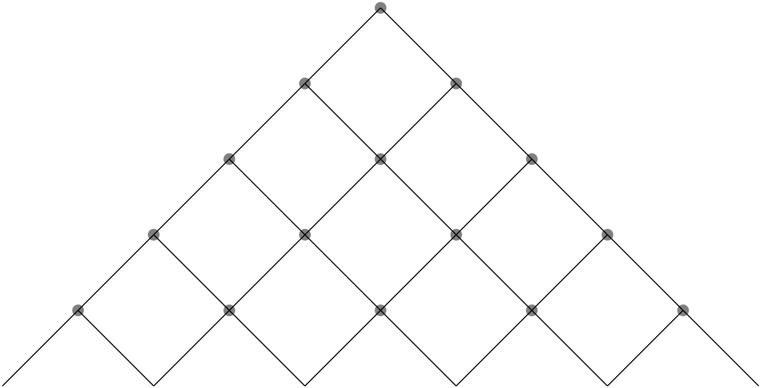
A Bratteli diagram is a directed graph $E \rightrightarrows V$ where $V = \coprod_{n=0}^{\infty} V(n)$, $E = \coprod_{n=1}^{\infty} E(n)$ and for each $n \geq 1$, $s(E(n)) = V(n-1)$ and $r(E(n)) = V(n)$, where $s(e)$ and $r(e)$ are respectively the source and the range of the edge e.





Pascal triangle

We shall use the Pascal triangle as an illustration.



Random Walk on a Bratteli diagram

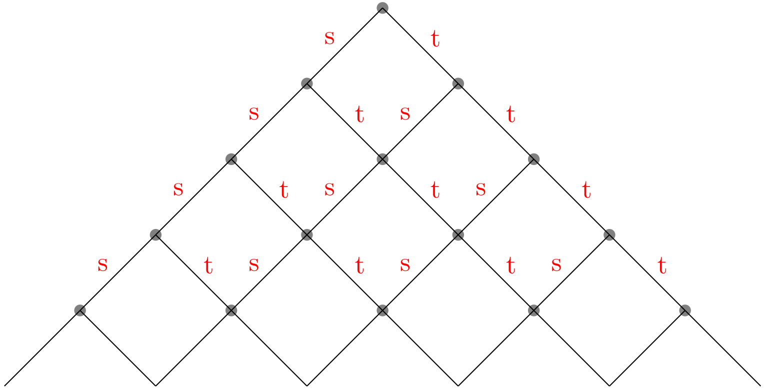
Definition

Let $E \rightrightarrows V$ be a a Bratteli diagram.

- A transition probability is a map p assigning to each vertex $v \in V$ a probability measure $p(v)$ on the set of edges $E_v = s^{-1}(v)$ emanating from v.
- An initial probability measure is a probability measure ν_0 on the set of initial vertices $V(0)$.
- A random walk is a pair (p, ν_0), where p is a transition probability and ν_0 is an initial probability measure.

Transition probability on Pascal triangle

Let $0 < t < 1$ and $s = 1 - t$:



This is the simple random walk on $\mathbb{Z}$.

The NC probability space of a random walk

Let (p, ν_0) be a random walk on a Bratteli diagram $E \rightrightarrows V$. We define:

- X, called the path space, is the space of infinite paths $x = \ldots x_2 x_1$
- R is the tail equivalence relation on X: $(x, y) \in R$ iff there is N such that $x_n = y_n$ for $n > N$.
- $\mu = \nu_0 p$ is the Markov measure on X constructed from (p, ν_0).

As we shall see, μ is quasi-invariant under R and Connes' theorem can be rephrased as

Theorem (Connes 75, version 2)

Let φ be a faithful normal state on a hyperfinite von Neumann algebra A. Then there exists a random walk (p, ν_0) on a Bratteli diagram $E \rightrightarrows V$ such that (A, φ) is isomorphic to $(W^*(R), \mu \circ E)$.

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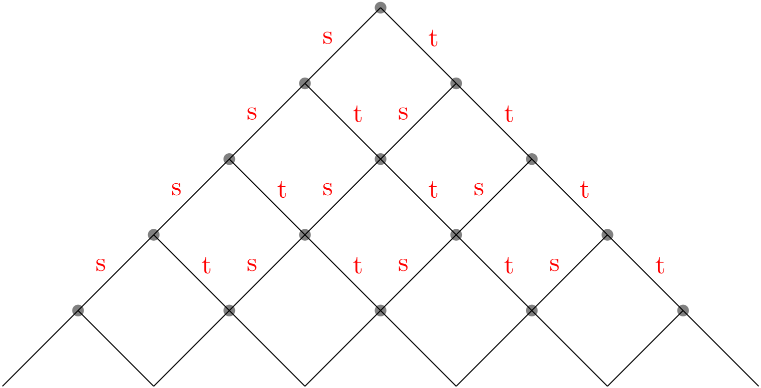
The NC space of the simple random walk.

Let $0 < t < 1$ and $s = 1 - t$:



This random walk gives the hyperfinite factor of type II_1 (for any t): the Markov measure gives a trace.

The RN derivative of a measured equivalence relation.

The invariants of the above von Neumann algebra A can be computed through the faithful normal state φ. At the level of the measured equivalence relation (X, R, μ), the most significative object is the Radon-Nikodým derivative $D_\mu = \frac{d(r^*\mu)}{d(s^*\mu)}$.

Therefore, the first task is to compute this RN derivative in terms of the random walk. We first give a definition.

Definition

Let G be a group. A map $D : R \to G$ is called a quasi-product cocycle if there exist a map $q : E \to G$, called a potential of D, such that $D(za, zb) = q(b)^{-1}q(a)$ where $q(a_n \dots a_2 a_1) = q(a_n) \dots q(a_2)q(a_1)$.

The RN derivative of a random walk.

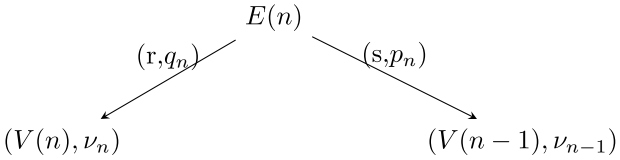
A crucial observation is that the RN derivative of a random walk does not depend on the transition probability p but only on the backward transition probability!

Theorem

The above RN derivative D_μ is the quasi-product cocycle defined by the backward transition probability $q : E \to \mathbb{R}_+^*$.

The following diagram illustrates the backward transition probability:

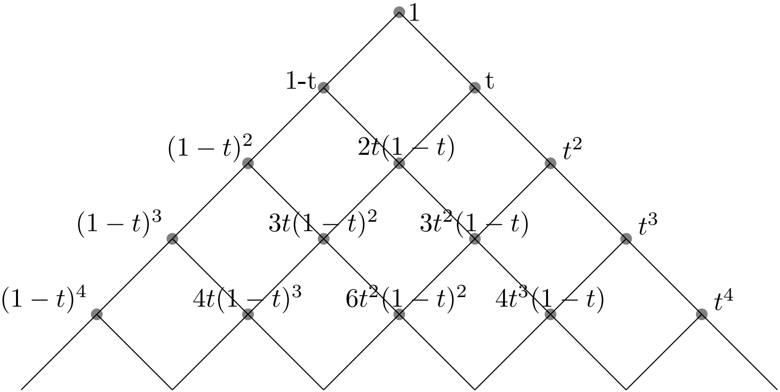
The backward transition probability.



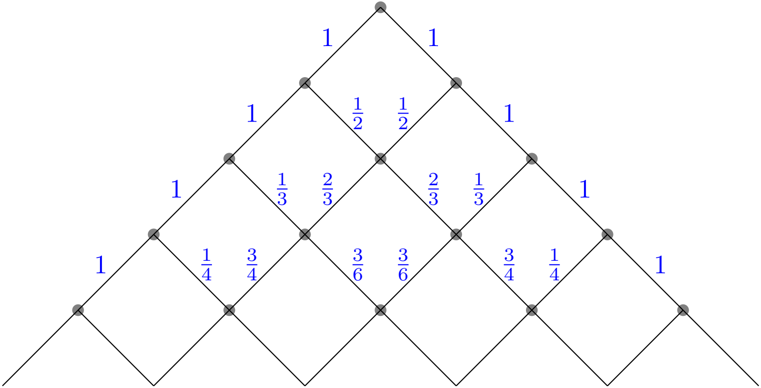
The measures ν_n on $V(n)$, constructed by induction, are the one-dimensional distributions of the random walk (recall that the initial one-dimensional distribution ν_0 is given). The backward transition probability corresponds to the disintegration along the r-fibers of the lifted measure $\nu_{n-1} \circ p_n$ on $E(n)$:

$$\nu_{n-1} \circ p_n = \nu_n \circ q_n$$

One-dimensional distributions on the Pascal triangle



Backward transition probability on the Pascal triangle



The backward transition probability does not depend on t.

The Mackey range

It is known that the flow of weights of the von Neumann algebra $W^*(R)$ is the Mackey range of the cocycle $D : R \to \mathbb{R}_+^*$. Let us recall its construction. One first defines the equivalence relation $R(D)$ on $X \times \mathbb{R}_+^*$:

$$((x,s),(y,t)) \in R(D) \Leftrightarrow (x,y) \in R \quad \text{and} \quad s = D(x,y)t$$

The Mackey range of the cocycle D is the standard quotient

$$\Omega = (X \times \mathbb{R}_+^*)//R(D)$$

(or space of ergodic components of $\mu \times \mathrm{Haar}$ with respect to $R(D)$) defined by

$$L^\infty(\Omega) = L^\infty(X \times \mathbb{R}_+^*)^{R(D)}$$

It is naturally an $\mathbb{R}_+^*$-space.

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Matrix-valued random walks

Since D is a quasi-product cocycle, the equivalence relation $R(D)$ is also given by a (Borel) Bratteli diagram. Therefore, our problem is reduced to the computation of the ergodic components of a Markov measure. Moreover, this procedure also works when $\mathbb{R}_+^*$ is replaced by an arbitrary locally compact group G and the quasi-product cocycle is defined by a labeling $\Phi : E \to G$.

Definition (Connes-Woods 1989)

A matrix-valued random walk on a group consists of:

- A Bratteli diagram (V, E);
- a group G;
- a map $\Phi : E \to G$ and
- a random walk (p, ν_0) on (V, E).

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Adams-Elliott-Giordano theorem

Theorem (Elliott-Giordano 93 and Adams-Elliott-Giordano 94)

Any amenable G-space is the Mackey range of a matrix-valued random walk.

Remarks

1. This is a sort of converse to a well known result of Zimmer: the action of an arbitrary locally compact group on a Poisson boundary is amenable.
2. Even when the G-space is reduced to a point, the theorem is not trivial.
3. Part of the joint project with T. Giordano is to extend the theorem to the case when G is a groupoid and to give applications.

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Borel Bratteli diagrams

Random walks on Bratteli diagrams are examples of time-dependent Markov chains. In order to recover the general theory of Markov chains, it suffices to introduce Borel Bratteli diagrams.

Definition

- A Borel graph is a graph $E \rightrightarrows V$ where the sets of edges E and the set of vertices V are endowed with a Borel structure and the source and range maps are Borel.
- A Borel Bratteli diagram is a Bratteli diagram which is a Borel graph.

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Transition probabilities

Markov chains are defined by a transition probability.

Definition

Let $E \rightrightarrows V$ be a Borel Bratteli diagram.

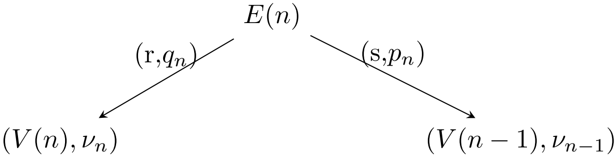
- A transition probability p assigns to each $v \in V(n-1)$ a probability measure p_v on $s^{-1}(v)$ and the map $v \mapsto p_v$ is Borel.
- A backward transition probability q assigns to each $w \in V(n)$ a probability measure q^w on $r^{-1}(w)$ and the map $w \mapsto q^w$ is Borel.

Going into the future and into the past

Starting from an initial measure ν_0 on $V(0)$, we can construct by induction the the one-dimensional distributions ν_n on $V(n)$ and the backward transition probability q by the formula

$$\nu_{n-1} \circ p_n = \nu_n \circ q_n$$

of the diagram



seen earlier.

Ergodic decomposition

Recall that the Mackey range of a matrix-valued random walk is the space of ergodic components of a Markov measure on the path space of a Bratteli diagram.

We consider now the general problem: given a random walk (p, ν_0), where p is a transition probability and ν_0 an initial measure on a Borel Bratteli diagram, identify the space of ergodic components of the associated Markov measure on the path space under the tail equivalence relation.

Definition

The tail boundary of a random walk on a Bratteli diagram is defined as the space of ergodic components of the Markov measure.

q-measures

Given a Borel Bratteli diagram (V, E), we define the space X of infinite paths. More generally, define for all n the space $X_{|n}$ of infinite paths starting at level n. We have a sequence of quotient maps

$$X \xrightarrow{\pi_1} X_{|1} \xrightarrow{\pi_2} X_{|2} \xrightarrow{\pi_3} \dots \xrightarrow{\pi_n} X_{|n} \xrightarrow{\pi_{n+1}} \dots$$

A backward transition probability q defines an inductive system of expectations

$$B(X) \xrightarrow{q_1} B(X_{|1}) \xrightarrow{q_2} \dots \xrightarrow{q_n} B(X_{|n}) \xrightarrow{q_{n+1}}$$

Definition

A q-measure is a measure on X which factors through all expectations $q_n \dots q_2 q_1$.

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A backward transition probability q defines an inductive system of expectations

$$B(X) \xrightarrow{q_1} B(X_{|1}) \xrightarrow{q_2} \dots \xrightarrow{q_n} B(X_{|n}) \xrightarrow{q_{n+1}}$$

Definition

A q-measure is a measure on X which factors through all expectations $q_n \dots q_2 q_1$.

q-measures

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Markov measures and q-measures

The key of the identification of the tail boundary is the following observation.

Theorem

For a bounded measure μ on X with one-dimensional distributions ν_n, the following conditions are equivalent:

1. *μ is a q-measure*
2. *μ is the Markov measure of a random walk (p, ν_0) where p and q are related by $\nu_{n-1} \circ p_n = \nu_n \circ q_n$.*

It is important for the sequel to note that the sequence (ν_n) is q-harmonic, i.e. $\nu_{n-1} = s_*(\nu_n \circ q_n)$ and that μ can be reconstructed from q and (ν_n) (as well as from p and ν_0).

Bounded harmonic sequences

Definition

Let (p, ν_0) be a random walk. A bounded harmonic sequence is a sequence (h_n) where $h_n \in L^\infty(V(n), \nu_n)$, $h_{n-1} = p_n(h_n \circ r)$ and $\sup \|h_n\|_\infty < \infty$. These sequences form a Banach space $H(p, \nu_0)$, which is the projective limit of

$$L^\infty(V(0), \nu_0) \xleftarrow{p_1} L^\infty(V(1), \nu_1) \xleftarrow{p_2} \dots \xleftarrow{p_n} L^\infty(V(n), \nu_n) \xleftarrow{p_{n+1}} \dots$$

where the maps are the expectations defined by the transition probability p.

Identification of the tail boundary

This gives the ergodic decomposition of a Markov measure under tail equivalence:

Theorem (Neveu 1964)

Let (p, ν_0) be a random walk. We denote by X its infinite path space and by R the tail equivalence relation and by μ the Markov measure. There is a natural isomorphism between the Banach spaces $L^\infty(X, \mu)^R$ and $H(p, \nu_0)$.

It suffices to specialize the bijection given earlier between q-measures μ' and q-harmonic sequences (ν'_n) to

$$\mu' = f\mu \quad \Leftrightarrow \quad (\nu'_n = h_n \nu_n)$$

where (ν_n) its one-dimensional distributions, $f \in L^\infty(X, \mu)^R$ and $(h_n) \in H(p, \nu_0)$.

Further Developments

Let me conclude by mentioning three developments.

1. Extension of the Connes-Woods construction to an arbitrary Markov chain.
2. Matrix-valued random walks on a groupoid and extension of the Adams-Elliott-Giordano's theorem.
3. Random walks on P-graphs, where P is an arbitrary semigroup rather than the semigroup of integers.

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The End

Thank you for your attention!