Random Walks on Bratteli Diagrams

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Yerevan, September 5, 2016

1. Hyperfinite von Neumann Algebras
2. Markov Chains
3. Further Developments
My talk is based on the following article:


It contains the statement of the two theorems which I am going to describe:

1. the description of a state on a hyperfinite von Neumann algebra (due to A. Connes);
2. the ergodic decomposition of a Markov measure via harmonic functions (a classical result in J. Neveu 64).
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Cartan subalgebras

Definition (Vershik, Feldman-Moore)

An abelian subalgebra $B$ of a von Neumann algebra $A$ is called a **Cartan subalgebra** if

1. $B$ is a masa;
2. $B$ is regular;
3. there exists a faithful normal conditional expectation $E_B : A \to B$.

Note that $E_B$ is unique.

The basic example is $A = M_n(\mathbb{C})$ and $B = D_n(\mathbb{C})$, the subalgebra of diagonal matrices. The next theorem says that the general case looks like this basic example.
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Theorem (Feldman-Moore 77)

Let $B$ be a Cartan subalgebra of a von Neumann algebra $A$ on a separable Hilbert space. Then there exists a countable standard measured equivalence relation $R$ on $(X, \mu)$, a twist $\sigma \in Z^2(R, T)$ and an isomorphism of $A$ onto $W^*(R, \sigma)$ carrying $B$ onto the diagonal subalgebra $L^\infty(X, \mu)$. The twisted relation $(R, \sigma)$ is unique up to isomorphism.

In our basic example, $X = \{1, \ldots, n\}$ and $R = X \times X$ (the twist $\sigma$ is trivial). The most general finite dimensional von Neumann algebra is given by an arbitrary equivalence relation $R$ on a finite set $X$.
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The following result is well known:

**Theorem (textbook)**

Let $\varphi$ be a normal state on $\mathcal{B}(H)$, where $H$ is a Hilbert space. Then there exists a Cartan subalgebra $B$ such that $\varphi = \mu \circ E_B$, where $\mu$ is the restriction of $\varphi$ to $B$.

Indeed, $\varphi = \text{Tr}(\Omega)$, where $\Omega$ is a positive trace-class operator. The Cartan subalgebra $B$ is determined by an orthonormal basis of eigenvectors of $\Omega$. The probability measure $\mu$ is given by the eigenvalues of $\Omega$. 
Normal states on type I von Neumann algebras

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Normal states on hyperfinite von Neumann algebras

It may not be as well known that this result remains valid for any hyperfinite von Neumann algebra.

**Theorem (Connes 75, version 1)**

Let $\varphi$ be a faithful normal state on a hyperfinite von Neumann algebra $A$. Then there exists a Cartan subalgebra $B$ such that $\varphi = \mu \circ E_B$, where $\mu$ is the restriction of $\varphi$ to $B$.

The proof proceeds in two steps.

1) First one shows the existence of an increasing sequence $(A_n)$ of f.d. subalgebras with weakly dense union such that each $A_n$ is globally invariant under the modular automorphism group $\sigma^\varphi$ of the state $\varphi$. 
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1) First one shows the existence of an increasing sequence $(A_n)$ of f.d. subalgebras with weakly dense union such that each $A_n$ is globally invariant under the modular automorphism group $\sigma^\varphi$ of the state $\varphi$. 
2) Second one constructs a Cartan subalgebra $B_n$ of $A_n$ such that

$$\varphi|_{A_n} = \varphi|_{B_n} \circ E_{B_n}$$

and $(A_{n-1}, B_{n-1}) \subset (A_n, B_n)$ in the sense that

- $B_{n-1} \subset B_n$
- the normalizer of $B_{n-1}$ in $A_{n-1}$ is contained in the normalizer of $B_n$ in $A_n$.

Because of the invariance under the modular group, there exists an expectation $F_n : A_n \to A_{n-1}$ such that $\varphi|_{A_n} = \varphi|_{A_{n-1}} \circ F_n$.

Inclusions and expectations are conveniently described by Bratteli diagrams.
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Nested Cartan subalgebras

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Inclusions and expectations are conveniently described by Bratteli diagrams.
The following diagram represents an inclusion of $(A_1, B_1)$ in $(A_2, B_2)$ where $A_1 = M_2(\mathbb{C}) \oplus \mathbb{C} \oplus M_3(\mathbb{C})$, $B_1 = D_2(\mathbb{C}) \oplus \mathbb{C} \oplus D_3(\mathbb{C})$ and $A_2 = M_8(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_5(\mathbb{C})$, $B_2 = D_8(\mathbb{C}) \oplus D_3(\mathbb{C}) \oplus D_3(\mathbb{C}) \oplus D_5(\mathbb{C})$.

**Theorem (Bratteli 72)**

*These diagrams classify all inclusions of f.d. C*-algebras.*
path model of the inclusion

$X_i$ is the set of paths $x_i$ ending at level $i$; $R_i$ is the set of pairs $(x_i, x'_i)$ ending at the same vertex.

\[
j(f)(yx_1, y'x'_1) = \begin{cases} 
  f(x_1, x'_1) & \text{if } y = y' \\
  0 & \text{if } y \neq y'
\end{cases}
\]
Expectations of f.d. $C^*$-algebras

The following diagram represents a faithful expectation $F: A_2 \rightarrow A_1$:

where $p_i > 0$ and $p_1 + p_1 + p_3 + p_4 + p_5 = 1$.

**Theorem**

*These diagrams classify all faithful expectations of f.d. $C^*$-algebras.*
Construction of the Cartan subalgebra

Proof. One chooses a Cartan subalgebra $B_1 \subset A_1$. The problem is to find a Cartan subalgebra $B_2 \subset A_2$ such that $(A_1, B_1) \subset (A_2, B_2)$ and making the following diagram commutative

\[
\begin{array}{ccc}
A_2 & \xrightarrow{E_2} & B_2 \\
\downarrow F & & \downarrow F_{|B_2} \\
A_1 & \xrightarrow{E_1} & B_1 
\end{array}
\]

which is always possible.
This gives the following expression of the expectation.

\[
F(g)(x_1, x'_1) = \sum_y p(y)g(yx_1, yx'_1)
\]

where the sum runs over all edges \( y \) emanating from the common vertex \( r(x_1) = r(x'_1) \).
The Bratteli diagram description of the inclusions and the expectations given above leads to an equivalent but more picturesque description of a faithful normal state on a hyperfinite von Neumann algebra.

**Theorem (Connes 75, version 2)**

*Each faithful normal state on a hyperfinite von Neumann algebra can be described as a random walk on a Bratteli diagram.*

Let us explain this statement.
**Definition**

A Bratteli diagram is a directed graph $E \Rightarrow V$ where $V = \bigsqcup_{n=0}^{\infty} V(n)$, $E = \bigsqcup_{n=1}^{\infty} E(n)$ and for each $n \geq 1$, $s(E(n)) = V(n - 1)$ and $r(E(n)) = V(n)$, where $s(e)$ and $r(e)$ are respectively the source and the range of the edge $e$. 

![Bratteli diagram example](image)
Pascal triangle

We shall use the Pascal triangle as an illustration.
Definition

Let $E \rightrightarrows V$ be a Bratteli diagram.

- A **transition probability** is a map $p$ assigning to each vertex $v \in V$ a probability measure $p(v)$ on the set of edges $E_v = s^{-1}(v)$ emanating from $v$.

- An **initial probability measure** is a probability measure $\nu_0$ on the set of initial vertices $V(0)$.

- A **random walk** is a pair $(p, \nu_0)$, where $p$ is a transition probability and $\nu_0$ is an initial probability measure.
Let $0 < t < 1$ and $s = 1 - t$:

This is the **simple random walk** on $\mathbb{Z}$. 
The NC probability space of a random walk

Let \((p, \nu_0)\) be a random walk on a Bratteli diagram \(E \nrightarrow V\). We define:

- \(X\), called the **path space**, is the space of infinite paths
  \[ x = \ldots x_2 x_1 \]
- \(R\) is the **tail equivalence relation** on \(X\): \((x, y) \in R\) iff there is \(N\) such that \(x_n = y_n\) for \(n > N\).
- \(\mu = \nu_0 p\) is the **Markov measure** on \(X\) constructed from \((p, \nu_0)\).

As we shall see, \(\mu\) is **quasi-invariant** under \(R\) and Connes’ theorem can be rephrased as

**Theorem (Connes 75, version 2)**

Let \(\varphi\) be a faithful normal state on a hyperfinite von Neumann algebra \(A\). Then there exists a random walk \((p, \nu_0)\) on a Bratteli diagram \(E \nrightarrow V\) such that \((A, \varphi)\) is isomorphic to \((W^*(R), \mu \circ E)\).
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The NC space of the simple random walk.

Let $0 < t < 1$ and $s = 1 - t$:

This random walk gives the hyperfinite factor of type $II_1$ (for any $t$): the Markov measure gives a trace.
The RN derivative of a measured equivalence relation.

The invariants of the above von Neumann algebra $A$ can be computed through the faithful normal state $\varphi$. At the level of the measured equivalence relation $(X, R, \mu)$, the most significative object is the Radon-Nikodým derivative $D_\mu = \frac{d(r^*\mu)}{d(s^*\mu)}$.

Therefore, the first task is to compute this RN derivative in terms of the random walk. We first give a definition.

**Definition**

Let $G$ be a group. A map $D : R \rightarrow G$ is called a quasi-product cocycle if there exist a map $q : E \rightarrow G$, called a potential of $D$, such that $D(za, zb) = q(b)^{-1}q(a)$ where $q(a_n \ldots a_2a_1) = q(a_n) \ldots q(a_2)q(a_1)$. 
A crucial observation is that the RN derivative of a random walk does not depend on the transition probability $p$ but only on the backward transition probability!

**Theorem**

The above RN derivative $D_\mu$ is the quasi-product cocycle defined by the **backward transition probability** $q : E \rightarrow \mathbb{R}^*_+$. 

The following diagram illustrates the backward transition probability:
The backward transition probability.

The measures $\nu_n$ on $V(n)$, constructed by induction, are the one-dimensional distributions of the random walk (recall that the initial one-dimensional distribution $\nu_0$ is given). The backward transition probability corresponds to the disintegration along the $r$-fibers of the lifted measure $\nu_{n-1} \circ p_n$ on $E(n)$:

$$\nu_{n-1} \circ p_n = \nu_n \circ q_n$$
One-dimensional distributions on the Pascal triangle

\[
\begin{array}{cccccc}
1-t & t & 3t(1-t)^2 & 3t^2(1-t) & t^3 & t^4 \\
(1-t)^2 & 2t(1-t) & 6t^2(1-t)^2 & 4t^3(1-t) & t^4 \\
(1-t)^3 & 3t(1-t)^2 & 4t^3(1-t) & 4t^3(1-t)^3 & 4t^4(1-t)^4 \\
(1-t)^4 & 3t^2(1-t) & 4t^3(1-t)^2 & 4t^4(1-t)^3 & 4t^5(1-t)^4 \\
\end{array}
\]
The backward transition probability does not depend on $t$. 
The Mackey range

It is known that the flow of weights of the von Neumann algebra $\mathcal{W}^*(R)$ is the **Mackey range** of the cocycle $D : R \to \mathbb{R}^*_+$. Let us recall its construction. One first defines the equivalence relation $R(D)$ on $X \times \mathbb{R}^*_+$:

$$((x, s), (y, t)) \in R(D) \iff (x, y) \in R \quad \text{and} \quad s = D(x, y)t$$

The Mackey range of the cocycle $D$ is the standard quotient

$$\Omega = (X \times \mathbb{R}^*_+)/R(D)$$

(or space of **ergodic components** of $\mu \times \text{Haar}$ with respect to $R(D)$) defined by

$$L^\infty(\Omega) = L^\infty(X \times \mathbb{R}^*_+)/R(D)$$

It is naturally an $\mathbb{R}^*_+$-space.
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It is naturally an $\mathbb{R}^*_+$-space.
Matrix-valued random walks

Since $D$ is a quasi-product cocycle, the equivalence relation $R(D)$ is also given by a (Borel) Bratteli diagram. Therefore, our problem is reduced to the computation of the ergodic components of a Markov measure. Moreover, this procedure also works when $\mathbb{R}_+^*$ is replaced by an arbitrary locally compact group $G$ and the quasi-product cocycle is defined by a labeling $\Phi : E \to G$.

Definition (Connes-Woods 1989)

A matrix-valued random walk on a group consists of:

- A Bratteli diagram $(V, E)$;
- a group $G$;
- a map $\Phi : E \to G$ and
- a random walk $(\rho, \nu_0)$ on $(V, E)$. 
Matrix-valued random walks

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Adams-Elliott-Giordano theorem

Theorem (Elliott-Giordano 93 and Adams-Elliott-Giordano 94)

Any amenable $G$-space is the Mackey range of a matrix-valued random walk.

Remarks

1. This is a sort of converse to a well known result of Zimmer: the action of an arbitrary locally compact group on a Poisson boundary is amenable.

2. Even when the $G$-space is reduced to a point, the theorem is not trivial.

3. Part of the joint project with T. Giordano is to extend the theorem to the case when $G$ is a groupoid and to give applications.
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Random walks on Bratteli diagrams are examples of time-dependent Markov chains. In order to recover the general theory of Markov chains, it suffices to introduce Borel Bratteli diagrams.

**Definition**

- A Borel graph is a graph $E \rightarrow V$ where the sets of edges $E$ and the set of vertices $V$ are endowed with a Borel structure and the source and range maps are Borel.
- A Borel Bratteli diagram is a Bratteli diagram which is a Borel graph.
Random walks on Bratteli diagrams are examples of time-dependent Markov chains. In order to recover the general theory of Markov chains, it suffices to introduce Borel Bratteli diagrams.

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- A **Borel Bratteli diagram** is a Bratteli diagram which is a Borel graph.
Markov chains are defined by a transition probability.

**Definition**

Let $E \supseteq V$ be a Borel Bratteli diagram.

- A **transition probability** $p$ assigns to each $v \in V(n-1)$ a probability measure $p_v$ on $s^{-1}(v)$ and the map $v \mapsto p_v$ is Borel.

- A **backward transition probability** $q$ assigns to each $w \in V(n)$ a probability measure $q^w$ on $r^{-1}(w)$ and the map $w \mapsto q^w$ is Borel.
Going into the future and into the past

Starting from an initial measure $\nu_0$ on $V(0)$, we can construct by induction the one-dimensional distributions $\nu_n$ on $V(n)$ and the backward transition probability $q$ by the formula

$$\nu_{n-1} \circ p_n = \nu_n \circ q_n$$

of the diagram

seen earlier.
Recall that the Mackey range of a matrix-valued random walk is the space of ergodic components of a Markov measure on the path space of a Bratteli diagram.

We consider now the general problem: given a random walk \((p, \nu_0)\), where \(p\) is a transition probability and \(\nu_0\) an initial measure on a Borel Bratteli diagram, identify the space of ergodic components of the associated Markov measure on the path space under the tail equivalence relation.

**Definition**

The **tail boundary** of a random walk on a Bratteli diagram is defined as the space of ergodic components of the Markov measure.
Given a Borel Bratteli diagram \((V, E)\), we define the space \(X\) of infinite paths. More generally, define for all \(n\) the space \(X|_n\) of infinite paths starting at level \(n\). We have a sequence of quotient maps

\[
X \xrightarrow{\pi_1} X|_1 \xrightarrow{\pi_2} X|_2 \xrightarrow{\pi_3} \ldots \xrightarrow{\pi_n} X|_n \xrightarrow{\pi_{n+1}} \ldots
\]

A backward transition probability \(q\) defines an inductive system of expectations

\[
B(X) \xrightarrow{q_1} B(X|_1) \xrightarrow{q_2} \ldots \xrightarrow{q_n} B(X|_n) \xrightarrow{q_{n+1}}
\]

**Definition**

A \textit{\(q\)-measure} is a measure on \(X\) which factors through all expectations \(q_n \ldots q_2 q_1\).
**q-measures**

Given a Borel Bratteli diagram \((V, E)\), we define the space \(X\) of infinite paths. More generally, define for all \(n\) the space \(X_{|n}\) of infinite paths starting at level \(n\). We have a sequence of quotient maps

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**Definition**

A **\(q\)-measure** is a measure on \(X\) which factors through all expectations \(q_n \ldots q_2q_1\).
The key of the identification of the tail boundary is the following observation.

**Theorem**

For a bounded measure $\mu$ on $X$ with one-dimensional distributions $\nu_n$, the following conditions are equivalent:

1. $\mu$ is a $q$-measure
2. $\mu$ is the Markov measure of a random walk $(p, \nu_0)$ where $p$ and $q$ are related by $\nu_{n-1} \circ p_n = \nu_n \circ q_n$.

It is important for the sequel to note that the sequence $(\nu_n)$ is $q$-harmonic, i.e. $\nu_{n-1} = s_*(\nu_n \circ q_n)$ and that $\mu$ can be reconstructed from $q$ and $(\nu_n)$ (as well as from $p$ and $\nu_0$).
Definition

Let \((p, \nu_0)\) be a random walk. A **bounded harmonic sequence** is a sequence \((h_n)\) where \(h_n \in L^\infty(V(n), \nu_n)\), \(h_{n-1} = p_n(h_n \circ r)\) and \(\sup \|h_n\|_\infty < \infty\). These sequences form a Banach space \(H(p, \nu_0)\), which is the projective limit of

\[
L^\infty(V(0), \nu_0) \xleftarrow{p_1} L^\infty(V(1), \nu_1) \xleftarrow{p_2} \ldots \xleftarrow{p_n} L^\infty(V(n), \nu_n) \xleftarrow{p_{n+1}} \ldots
\]

where the maps are the expectations defined by the transition probability \(p\).
Identification of the tail boundary

This gives the ergodic decomposition of a Markov measure under tail equivalence:

**Theorem (Neveu 1964)**

Let \((p, \nu_0)\) be a random walk. We denote by \(X\) its infinite path space and by \(R\) the tail equivalence relation and by \(\mu\) the Markov measure. There is a natural isomorphism between the Banach spaces \(L^\infty(X, \mu)^R\) and \(H(p, \nu_0)\).

It suffices to specialize the bijection given earlier between \(q\)-measures \(\mu'\) and \(q\)-harmonic sequences \((\nu'_n)\) to

\[ \mu' = f \mu \iff (\nu'_n = h_n \nu_n) \]

where \((\nu_n)\) its one-dimensional distributions, \(f \in L^\infty(X, \mu)^R\) and \((h_n) \in H(p, \nu_0)\).
Let me conclude by mentioning three developments.

1. Extension of the Connes-Woods construction to an arbitrary Markov chain.
3. Random walks on $P$-graphs, where $P$ is an arbitrary semigroup rather than the semigroup of integers.
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2. **Matrix-valued random walks on a groupoid and extension of the Adams-Elliott-Giordano’s theorem.**

3. **Random walks on $P$-graphs, where $P$ is an arbitrary semigroup rather than the semigroup of integers.**
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The End

Thank you for your attention!