Splitting characterizations of point processes

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joint work with Hans Zessin and Benjamin Nehring

Yerevan, September 10, 2016
Warm-up for splitting

Direct problem

$N$ balls

- compute $\mathcal{L}(N_q, N_q^*)$
- compute $\mathcal{L}(N_q^*|N_q) =: \Upsilon(N_q, \cdot)$
Warm-up for splitting
Direct problem

- compute $\mathcal{L}(N_q, N_q^*)$
- compute $\mathcal{L}(N_q^*|N_q) =: \gamma(N_q, \cdot)$
Warm-up for splitting

Direct problem

- compute $\mathcal{L}(N_q, N_q^*)$
- compute $\mathcal{L}(N_q^*|N_q) = \Upsilon(N_q, \cdot)$
Warm-up for splitting
Indirect problem

Now $\mathcal{L}(N)$ unknown

$N_q$ balls
Warm-up for splitting

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$\gamma(N_q, \cdot)$

$N_q$ balls

$\mathcal{L}(N_q^* | N_q) = \gamma(N_q, \cdot)$
Warm-up for splitting
Indirect problem

Now $\mathcal{L}(N)$ unknown

$\gamma(N_q, \cdot)$

$N_q$ balls

$\mathcal{L}(N_q^*|N_q) = \gamma(N_q, \cdot)$

Which $N$ satisfy the splitting equation

$Ef(N_q, N_q^*) = E\left[ E[f(N_q, N_q^*)|N_q] \right] = \int\int f(k, l)\gamma(k, d\ell)\mathbb{P}_q(d\ell)$
Warm-up for splitting
Indirect problem

Now $\mathcal{L}(N)$ unknown

$\Upsilon(N_q, \cdot )$

Which $N$ satisfy the (dependent) convolution equation

$$\mathbb{E}g(N) = \mathbb{E} \left[ \mathbb{E}[g(N_q + N_q^*)|N_q] \right] = \int \int g(k + l) \Upsilon(k, dl) \mathbb{P}_q( dk)$$
$N_q$ is observed, conditional law of $N_q^*$ is . . .

**Example 1** \( \Upsilon(k, \cdot ) = \text{Poi}(1 - q); \)
then \( N \sim \text{Poi}(1) \) and this is the only choice!

**Example 2** \( \Upsilon(k, \cdot ) = \text{Bin} \left( n - k, \frac{p(1-q)}{1-pq} \right); \)
then \( N \sim \text{Bin}(n, p) \)

**Example 3** \( \Upsilon(k, \cdot ) = \text{NegBin} \left( n + k, p(1 - q) \right); \)
then \( N \sim \text{NegBin}(n, p) \)
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Integration by parts formula

\( N \) satisfies IBPF for some function \( \pi : \mathbb{N}_0 \rightarrow \mathbb{R}_+ \), if for bounded \( f \),
\[
E[Nf(N)] = E[\pi(N)f(N + 1)].
\]

Problem
Given \( \pi \), what is the distribution of \( N \)?

Examples

1. \( \pi(k) = 1 \) for all \( k \in \mathbb{N}_0 \), then \( N \sim \text{Poi}(1) \)
2. \( \pi(k) = z(n - k) \) for \( k = 0, 1, \ldots, n \), then \( N \sim \text{Bin} \left( n, \frac{z}{1+z} \right) \);
3. \( \pi(k) = z(n + k) \) for \( k \in \mathbb{N}_0 \), then \( N \sim \text{NegBin}(n, z) \)
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Given $\pi$, what is the distribution of $N$?

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2. $\pi(k) = z(n - k)$ for $k = 0, 1, \ldots, n$, then $N \sim \text{Bin} \left(n, \frac{z}{1+z}\right)$;
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Integration by parts
Distributions

Integration by parts formula

$N$ satisfies IBPF for some function $\pi : \mathbb{N}_0 \to \mathbb{R}_+$, if for bounded $f$, $E[Nf(N)] = E[\pi(N)f(N + 1)]$.

How to determine the law of $N$?

1. choose $f = 1\{k\}$, then $kP(N = k) = \pi(k)P(N = k - 1)$, $k = 1, 2, \ldots$

2. $P(N = k) = \frac{\pi(k) \cdots \pi(1)}{k!} P(N = 0)$

3. $P(N = k) = \exp(-\pi) \frac{\pi[k]}{k!}$
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Splitting and integration by parts

Connection

$q$-Splitting kernel
If $N$ satisfies IBPF($\pi$), then $\Upsilon(k, \cdot)$ satisfies IBPF($((1 - q)\pi(k + \cdot))$).

$N_q$
$N_q$ satisfies an IBPF. If $N$ satisfies IBPF($\pi$), then that function is the “average” $q \sum_j \pi(k + j) \Upsilon(k, j)$.

Equivalent statements

1. $N$ satisfies IBPF($\pi$)
2. $N$ satisfies the splitting equation
3. $N$ satisfies the (dependent) convolution equation
Splitting and integration by parts

Connection

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If $N$ satisfies $\text{IBPF}(\pi)$, then $\gamma(k, \cdot)$ satisfies $\text{IBPF}((1 - q)\pi(k + \cdot))$.

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$N_q$ satisfies an IBPF. If $N$ satisfies $\text{IBPF}(\pi)$, then that function is the “average” $q \sum_j \pi(k + j)\gamma(k,j)$.

Equivalent statements

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Point processes

A point process is a random point measure (r.v. \( N \) is now \( \{N_\Lambda\}_\Lambda \)).
**Point processes**

A point process is a random point measure (r.v. $N$ is now $\{N_\Lambda\}_\Lambda$).

**Poisson process**

- $N_\Lambda \sim \text{Poi}(m(\Lambda))$
- given $N_\Lambda$, points are distributed iid
- $\Lambda \cap \Lambda' = \emptyset$, then $N_\Lambda$ and $N_{\Lambda'}$ independent
Spatial picture

Point processes

A point process is a random point measure (r.v. $N$ is now $\{N_\Lambda\}_\Lambda$).

Gibbs process

- defined locally by
  $$G(\cdot | \hat{\mathcal{F}}_\Lambda)(\mu) := \frac{e^{-\mathcal{V}(\cdot | \mu^\Lambda)}}{Z_{\Lambda, \mu}} P_\Lambda$$
- existence? uniqueness?
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Gibbs process  

Nguyen, Zessin 79

DLR equations equivalent to IBPF

\[
\int \int h(x, \mu) \mu(dx) G(d\mu) \\
= \int \int h(x, \mu + \delta_x) e^{-V(x, \mu)} m(dx) G(d\mu)
\]
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Papangelou process

replace \(e^{-V(\cdot, \mu)} \, d\mu\) by \(\pi(\mu, \cdot)\)

\[
\int \int h(x, \mu) \mu(dx) P(d\mu) = \int \int h(x, \mu + \delta x) \pi(\mu, dx) P(d\mu)
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Papangelou process, examples

- $\pi(\mu, \cdot) = m$
- $\pi(\mu, \cdot) = z(m - \mu)$
- $\pi(\mu, \cdot) = z(m + \mu)$

Each $N_\Lambda$ satisfies an IBPF.
\textbf{Spatial picture}
\textit{Point processes}

\textbf{$q$-splittings and thinnings}

- choose colour for each “ball” independently, e.g. blue with probability $q$
- joint law of \textcolor{red}{red} and \textcolor{blue}{blue} point configurations is $q$-splitting $S^q$
- marginals are thinnings
- conditional law of \textcolor{red}{red} point configuration given \textcolor{blue}{blue} point configuration is splitting kernel
Examples

1. Poisson process $P_m$:
   \[ P^q = P_{qm}, \quad S^q = P_{qm} \otimes P_{(1-q)m} \]

2. Difference process $D_{z,m}$:
   \[ D^q_{z,m} = D_{\frac{qz}{1+(1-q)z},m}, \]
   \[ \Upsilon(\nu, \cdot) = D_{(1-q)z,m-\nu} \]

3. Sum process $S_{z,m}$:
   \[ S^q_{z,m} = S_{\frac{qz}{1-(1-q)z},m}, \]
   \[ \Upsilon(\nu, \cdot) = S_{(1-q)z,m+\nu} \]
Spatial picture
Properties of Splittings and Thinnings

Splitting kernel \((1)\) Karr; \((2)\) Nehring, R, Zessin\)

1. If \(P\) is finite, then \(\Upsilon(\nu, \cdot) \sim (1 - q)^N P^I_\nu\).
2. If \(P\) satisfies IBPF for \(\pi\), then \(\Upsilon(\nu, \cdot)\) satisfies IBPF for \((1 - q)\pi(\nu + \cdot, \cdot)\).

Thinnings \((Nehring, R, Zessin)\)

If \(P\) satisfies IBPF for \(\pi\), then also \(P^q\) does for \(q \int \pi(\mu + \nu, \cdot) \Upsilon(\mu, d\nu)\).
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If \(P\) satisfies IBPF for \(\pi\), then also \(P^q\) does for
\[q \int \pi(\mu + \nu, \cdot) \Upsilon(\mu, d\nu).\]
Characterization (Nehring, R, Zessin)

The following statements are equivalent

1. $P$ solves IBPF for $\pi$;
2. $P$ satisfies the splitting equation

$$S_P(h) = \int\int h(\mu, \nu) \Upsilon(\mu, d\nu) P^q(d\mu)$$

3. $P$ satisfies the (dependent) convolution equation

$$P(\phi) = \int\int \phi(\mu + \nu) \Upsilon(\mu, d\nu) P^q(d\mu)$$
Uniqueness of solutions of splitting and convolution equation

Uniqueness of solutions of IBPF implies uniqueness for splitting and convolution equation.

$\alpha$-condensability (Ambartzumian)

$P$ is $\alpha$-condensable if there exists $Q$ such that $Q^{1/\alpha} = P$.

- if $P$ solves IBPF for $\sigma$, condensability “reduces” to solving

$$\sigma(\nu, \cdot) = q \int \pi(\nu + \mu, \cdot) \Upsilon(\nu, d\mu)$$
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  $\sigma(\nu, \cdot) = q \int \pi(\nu + \mu, \cdot) \Upsilon(\nu, d\mu)$
Spatial picture

Consequences

Spatial birth processes

Let $P$ solve IBPF for $\pi$, $(N_q)_q$ (point measure valued) process such that transition kernel

$$p_{q,q'}(\mu, \cdot) = \Upsilon_{q,q'}(\mu, \cdot)$$

solves an IBPF for $(q' - q) \int \pi(\mu + \kappa, \cdot) \Upsilon'(\mu, d\kappa)$.

- law of $N_q$ is $P^q$
- $q \mapsto N_q$ increasing

Cox processes and condensability

$P$ is a Cox process iff $q \mapsto N^q$ extends to $\mathbb{R}_+$.

- (otherwise only on $[0, T]$ for some $T \geq 1$)
- exit space of pure birth process given by mixtures of Poisson pure birth
Spatial picture
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Further examples

log-Gauss Cox process

(Coles, Jones 91; Møller, Syversveen, Waagepetersen 98)

\( P \sim \text{IGC}(\mu, c) \) if \( P \) is a Cox process driven by \( e^Y \), where \( Y \) is Gaussian with mean \( \mu \) and covariance \( c \).

Reduced Palm measures of log-Gauss Cox processes

(Cœurjolly, Møller, Waagepetersen 15)

If \( P \sim \text{IGC}(\mu, c) \), then its reduced Palm measure \( P^l_\nu \) for a simple and finite point measure \( \nu \) is log-Gauss Cox with parameters

\[
\mu + \int c_x \cdot \nu(dx), \quad c.
\]
Further examples

log-Gauss Cox process

**Thinning**
If $P \sim \text{lgGC}(\mu, c)$, then its $q$-thinning is log-Gauss Cox $P \sim (\mu + \ln q, c)$.

**Splitting**
If $P \sim \text{lgGC}(\mu, c)$ a finite process, then its $q$-splitting kernel is

$$\gamma(\nu, \cdot) = \frac{(1 - q)^n}{Z_\nu} P^!_\nu,$$

i.e. is log-Gauss Cox process with parameters

$$\mu + \int c_x \cdot \nu(dx) + \ln(1 - q), \quad c.$$
Further examples

Gauss Poisson process

Gauss-Poisson process (Newman 70; Milne, Westcott 72; Macchi 72)

$P \sim GP(\lambda, H)$ if $P$ has Laplace transform

$$L(f) = \exp \left( - \int 1 - e^{-f(x)} \lambda(dx) + \frac{1}{2} \iint [1 - e^{-f(x)}][1 - e^{-f(y)}] H(dx, dy) \right).$$

Thinning (Milne, Westcott 72)

If $P \sim GP(\lambda, H)$, then its $q$-thinning is Gauss Poisson

$P \sim GP(q\lambda, q^2 H)$. 
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