Limiting Absorption Principle, Generalized Eigenfunctions and Scattering Matrix for Laplace Operators with Boundary conditions on Hypersurfaces

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Introduction

Given $\Omega \subset \mathbb{R}^n$ open and bounded with boundary Γ, let Δ_Γ° be the symmetric, not positive, operator in $L^2(\mathbb{R}^n)$ defined by $\Delta_\Gamma^\circ := \Delta|\mathcal{C}^\infty_{\rm comp}(\mathbb{R}^n\backslash\Gamma)$. We are interested in determining the whole family of its self-adjoint extensions, characterize them by boundary conditions on Γ, express their resolvents in term of the resolvent $(-\Delta+z)^{-1}$ of the self-adjoint, free Laplacian

$$\Delta : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

and study the scattering system $\{\widehat\Delta, \Delta\}$, where $\widehat\Delta$ belongs to a large sub-family of extensions of Δ_Γ°.

Let $\Sigma \subset \Gamma$ relatively open. Then

$$\Delta_\Gamma^\circ \subset \Delta_\Sigma^\circ := \Delta|\mathcal{C}^\infty_{\rm comp}(\mathbb{R}^n\backslash\overline{\Sigma}) \;\Rightarrow\; (\Delta_\Sigma^\circ)^* \subset (\Delta_\Gamma^\circ)^*.$$

Thus $\widehat\Delta$ could be a self-adjoint realizations of the Laplacian with boundary conditions supported only on Σ.

Trace maps and boundary-layer operators

Given $\Omega \subset \mathbb{R}^n$ open and bounded with smooth boundary Γ, we adopt the notation $\Omega_{\text{in}} = \Omega$, $\Omega_{\text{ex}} = \mathbb{R}^n \backslash \overline{\Omega}$. $H^s(\mathbb{R}^n)$, $H^s(\Omega_{\text{in}})$, $H^s(\Omega_{\text{ex}})$, $H^s(\Gamma)$ $s \in \mathbb{R}$, denote the usual scales of Sobolev-Hilbert spaces of function on $\mathbb{R}^n$, Ω_{in}, Ω_{ex} and Γ respectively.
We recall the basic definitions and some properties of traces and layer operators related to the self-adjoint operator $\Delta : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and the hypersurface Γ. The zero and first-order traces on Γ are defined on smooth functions as

$$\gamma_0 u = \, u|_\Gamma \; , \qquad \gamma_1 u = \nu \cdot \nabla u | \Gamma \, ,$$

ν the unit normal to Γ, and extend to the bounded, surjective linear operators

$$\gamma_0 \in \mathsf{B}(H^2(\mathbb{R}^n), H^{\frac{3}{2}}(\Gamma)) \, , \qquad \gamma_1 \in \mathsf{B}(H^2(\mathbb{R}^n), H^{\frac{1}{2}}(\Gamma)) \, .$$

Such trace maps can be equivalently defined as the means

$$\gamma_0 = \frac{1}{2}(\gamma_0^{\mathsf{ex}} + \gamma_0^{\mathsf{in}})\,, \quad \gamma_1 = \frac{1}{2}(\gamma_1^{\mathsf{ex}} + \gamma_1^{\mathsf{in}})\,,$$

of the one-sided traces

$$\gamma_j^{\#} \in \mathsf{B}(H^2(\Omega_{\#}), H^{\frac{3}{2}-j}(\Gamma))\,, \quad \# = \mathsf{in}, \mathsf{ex}\,, \quad j = 0, 1\,.$$

By Green's formula and duality, such lateral traces extend to

$$\hat{\gamma}_j^{\#} \in \mathsf{B}(H_{\Delta}(\Omega_{\#}), H^{-\frac{1}{2}-j}(\Gamma))\,,$$

where

$$H_{\Delta}(\Omega_{\#}) := \{u_{\#} \in L^2(\Omega_{\#}) : \Delta u_{\#} \in L^2(\Omega_{\#})\}\,.$$

Setting $H_\Delta(\mathbb{R}^n \backslash \Gamma) := H_\Delta(\Omega_{\mathsf{in}}) \oplus H_\Delta(\Omega_{\mathsf{ex}})$, one has the bounded maps

$$\hat{\gamma}_j \in \mathsf{B}(H_\Delta(\mathbb{R}^n \backslash \Gamma), H^{-\frac{1}{2}-j}(\Gamma)),$$

$$[\hat{\gamma}_j] \in \mathsf{B}(H_\Delta(\mathbb{R}^n \backslash \Gamma), H^{-\frac{1}{2}-j}(\Gamma))$$

where

$$\hat{\gamma}_j u := \frac{1}{2}(\hat{\gamma}_j^{\mathsf{ex}}(u|\Omega_{\mathsf{in}}) + \hat{\gamma}_j^{\mathsf{in}}(u|\Omega_{\mathsf{ex}})), \quad j = 0, 1,$$

and

$$[\hat{\gamma}_j]u = \hat{\gamma}_j^{\mathsf{ex}}(u|\Omega_{\mathsf{in}}) - \hat{\gamma}_j^{\mathsf{in}}(u|\Omega_{\mathsf{ex}}), \quad j = 0, 1.$$

Notice that $\hat{\gamma}_j | H^2(\mathbb{R}^n) = \gamma_j$ and that $u \in H_\Delta(\mathbb{R}^n \backslash \Gamma)$ belongs to $H^2(\mathbb{R}^n)$ if and only if $[\hat{\gamma}_0]u = [\hat{\gamma}_1]u = 0$.

Finally, we define the bounded maps

$$\hat{\gamma} \in \mathsf{B}(H_\Delta(\mathbb{R}^n \backslash \Gamma), H^{-\frac{1}{2}}(\Gamma) \oplus H^{-\frac{3}{2}}(\Gamma))\,,$$

$$\hat{\gamma} u := (\hat{\gamma}_0 u) \oplus (\hat{\gamma}_1 u)\,,$$

and

$$[\hat{\gamma}] \in \mathsf{B}(H_\Delta(\mathbb{R}^n \backslash \Gamma), H^{-\frac{3}{2}}(\Gamma) \oplus H^{-\frac{1}{2}}(\Gamma))\,,$$

$$[\hat{\gamma}] u = (-[\hat{\gamma}_1] u) \oplus ([\hat{\gamma}_0] u)\,.$$

For $z \in \rho(\Delta) = \mathbb{C}\backslash(-\infty, 0]$, the single and double layer operators

$$SL_z \in \mathsf{B}(H^{-\frac{3}{2}}(\Gamma), L^2(\mathbb{R}^n))\,, \quad DL_z \in \mathsf{B}(H^{-\frac{1}{2}}(\Gamma), L^2(\mathbb{R}^n))$$

are defined by

$$SL_z = (\gamma_0(-\Delta + \bar{z})^{-1})^*\,, \qquad DL_z = (\gamma_1(-\Delta + \bar{z})^{-1})^*\,.$$

Considering the integral kernel of $(-\Delta + z)^{-1}$ given by

$$\mathscr{G}_z(x, y) := \frac{1}{2\pi}\left(\frac{\sqrt{z}}{2\pi\|x - y\|}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(\sqrt{z}\|x-y\|), \quad \mathrm{Re}(\sqrt{z}) > 0,$$

one has

$$SL_z\phi(x) = \int_\Gamma \mathscr{G}_z(x, y)\phi(y)\, d\sigma(y)\,, \quad x \notin \Gamma\,,\ \phi \in L^2(\Gamma)\,,$$

$$DL_z\varphi(x) = \int_\Gamma \nu\cdot\nabla\mathscr{G}_z(x, y)\varphi(y)\, d\sigma(y)\,, \quad x \notin \Gamma\,,\ \varphi \in L^2(\Gamma)\,,$$

Then, given $z \in \mathbb{C}\backslash(-\infty, 0]$, we define

$$G_z \in \mathsf{B}(H^{-\frac{3}{2}}(\Gamma) \oplus H^{-\frac{1}{2}}(\Gamma), L^2(\mathbb{R}^n)), \quad G_z(\phi \oplus \varphi) := SL_z\phi + DL_z\varphi .$$

By the definition of G_z, it results

$$G_z \in \mathsf{B}(H^{-\frac{3}{2}}(\Gamma) \oplus H^{-\frac{1}{2}}(\Gamma), H_\Delta(\mathbb{R}^n\backslash\Gamma))$$

and, by the well known jumps relations for the layer operators,

$$[\hat{\gamma}]G_z = 1_{H^{-\frac{3}{2}}(\Gamma)\oplus H^{-\frac{1}{2}}(\Gamma)} .$$

Laplacians with boundary conditions on hypersurfaces

Now we introduce the densely defined, closed and symmetric operator in $L^2(\mathbb{R}^n)$ given by $\Delta_\Gamma := \Delta|\ker(\gamma)$ and we look for the whole family of its self-adjoint extensions. Such a family coincides with the one describing the self-adjoint extensions of the symmetric operator $\Delta^\circ_\Gamma := \Delta|\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}^n\backslash\Gamma)$ and contains self-adjoint realizations of the Laplacian with boundary conditions on Γ and on any $\Sigma \subset \Gamma$. The adjoint operator Δ^*_Γ identifies with

$$\mathrm{dom}(\Delta^*_\Gamma) = H_\Delta(\mathbb{R}^n\backslash\Gamma) \quad \Delta^*_\Gamma u = \Delta(u|\Omega_{\mathrm{in}}) \oplus \Delta(u|\Omega_{\mathrm{ex}})\,.$$

Setting $G := G_1$, an equivalent representation of Δ^*_Γ is given by

$$\begin{aligned}&\mathrm{dom}(\Delta^*_\Gamma)\\ =&\left\{u = u_0 + G(\phi \oplus \varphi) : u_0 \in H^2(\mathbb{R}^n)\,,\ \phi \oplus \varphi \in H^{-\frac{3}{2}}(\Gamma) \oplus H^{-\frac{1}{2}}(\Gamma)\right\}\,,\end{aligned}$$

$$\Delta^*_\Gamma u = \Delta u_0 + G(\phi \oplus \varphi) = \Delta u - [\hat\gamma_1]u\,\delta_\Gamma - [\hat\gamma_0]u\,\nu\cdot\nabla\delta_\Gamma.$$

Given an orthogonal projection Π in $H^{\frac{3}{2}}(\Gamma) \oplus H^{\frac{1}{2}}(\Gamma)$, we denote the dual orthogonal projector in $H^{-\frac{3}{2}}(\Gamma) \oplus H^{-\frac{1}{2}}(\Gamma)$ by Π', so that $\mathrm{ran}(\Pi') = \mathrm{ran}(\Pi)'$. We say that the densely defined linear operator

$$\Theta : \mathrm{dom}(\Theta) \subseteq \mathrm{ran}(\Pi)' \to \mathrm{ran}(\Pi)$$

is self-adjoint whenever $\Theta = \Theta'$, equivalently whenever the operator

$$\tilde{\Theta} = \Theta(\Lambda^3 \oplus \Lambda)\,, \quad \mathrm{dom}(\tilde{\Theta}) = (\Lambda^3 \oplus \Lambda)^{-1}\mathrm{dom}(\Theta)\,,$$

is a self-adjoint operator in the Hilbert space $\mathrm{ran}(\Pi)$, where the unitary maps $\Lambda^s = (-\Delta_{LB} + 1)^s$ represent the duality mappings on $H^s(\Gamma)$ onto $H^{-s}(\Gamma)$.

Then we define the operator-valued Weyl function

$$\mathbb{C}\backslash(-\infty,0] \ni z \mapsto M_z \in \mathsf{B}(H^{-\frac{3}{2}}(\Gamma)\oplus H^{-\frac{1}{2}}(\Gamma), H^{\frac{3}{2}}(\Gamma)\oplus H^{\frac{1}{2}}(\Gamma))$$

${M_z}' = M_{\bar{z}}$, by

$$M_z := \gamma(G - G_z) \equiv \begin{bmatrix} \gamma_0(SL - SL_z) & \gamma_0(DL - DL_z) \\ \gamma_1(SL - SL_z) & \gamma_1(DL - DL_z) \end{bmatrix},$$

and, given $\Theta : \mathrm{dom}(\Theta) \subseteq \mathrm{ran}(\Pi)' \to \mathrm{ran}(\Pi)$ self-adjoint, the set

$$Z_{\Pi,\Theta} := \{z \in \mathbb{C}\backslash(-\infty,0] : (\Theta + \Pi M_z \Pi')^{-1} \in \mathsf{B}(\mathrm{ran}(\Pi), \mathrm{ran}(\Pi)')\}.$$

Theorem

Any self-adjoint extension of $\Delta_\Gamma = \Delta| \ker(\gamma)$ *is of the kind* $\Delta_{\Pi,\Theta} = \Delta^*_\Gamma|\mathrm{dom}(\Delta_{\Pi,\Theta})$*, where* Π *is an orthogonal projection in* $H^{\frac{3}{2}}(\Gamma) \oplus H^{\frac{1}{2}}(\Gamma)$*,* $\Theta : dom(\Theta) \subseteq ran(\Pi)' \to ran(\Pi)$ *is self-adjoint and*

$$\mathrm{dom}(\Delta_{\Pi,\Theta}) = \{u = u_0 + G(\phi \oplus \varphi) : u_0 \in H^2(\mathbb{R}^n)\,,\\ \phi \oplus \varphi \in \mathrm{dom}(\Theta)\,,\ \Pi\gamma u_0 = \Theta(\phi \oplus \varphi)\}\,.$$

Moreover, $\mathbb{C}\backslash\mathbb{R} \subseteq Z_{\Pi,\Theta} \subseteq \rho(\Delta_{\Pi,\Theta})$*, and, for any* $z \in Z_{\Pi,\Theta}$ *the resolvent of the self-adjoint extension* $\Delta_{\Pi,\Theta}$ *is given by the Krĕın type formula*

$$(-\Delta_{\Pi,\Theta}+z)^{-1} = (-\Delta+z)^{-1}+G_z\Pi'(\Theta+\Pi M_z\Pi')^{-1}\Pi\gamma(-\Delta+z)^{-1}\,.$$

Corollary

If $\mathrm{dom}(\Theta) \subseteq H^{\frac{1}{2}}(\Gamma) \oplus H^{\frac{3}{2}}(\Gamma)$ *then one has an alternative description of* $\Delta_{\Pi,\Theta}$*:*

$$\mathrm{dom}(\Delta_{\Pi,\Theta}) = \{u \in H^2(\mathbb{R}^n \backslash \Gamma) : [\gamma]u \in \mathrm{dom}(\Theta)\,,\ \Pi\gamma u = B_\Theta[\gamma]u\}\,,$$

and

$$(-\Delta_{\Pi,\Theta}+z)^{-1} - (-\Delta+z)^{-1} = G_z\Pi'(B_\Theta - \Pi\gamma G_z\Pi')^{-1}\Pi\gamma(-\Delta+z)^{-1}\,,$$

where the boundary operator B_Θ *is related to* Θ *by*

$$B_\Theta = \Theta + \Pi\gamma G\Pi' : \mathrm{dom}(\Theta) \subseteq \mathrm{ran}(\Pi)' \to \mathrm{ran}(\Pi)\,.$$

Examples.

1) Dirichlet boundary conditions on Γ:

$$\Pi = \mathbb{1}\oplus 0, \quad B_\Theta = 0\,,\ i.e. \quad \Theta = -\gamma_0 SL : H^{\frac{1}{2}}(\Gamma) \subseteq H^{-\frac{3}{2}}(\Gamma) \to H^{\frac{3}{2}}(\Gamma)$$

2) Neumann boundary conditions on Γ:

$$\Pi = 0\oplus\mathbb{1}, \quad B_\Theta = 0\,,\ i.e. \quad \Theta = -\gamma_1 DL : H^{\frac{3}{2}}(\Gamma) \subseteq H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$$

3) Robin boundary conditions on Γ, $\gamma_1^{\#}u = b_{\#}\gamma_0^{\#}u$, $\# = \mathrm{in}, \mathrm{ex}$:
$\Pi = \mathbb{1} \oplus \mathbb{1}$,

$$B_\Theta : H^{\frac{3}{2}}(\Gamma) \oplus H^{\frac{3}{2}}(\Gamma) \subseteq H^{-\frac{1}{2}}(\Gamma) \oplus H^{-\frac{3}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \oplus H^{\frac{3}{2}}(\Gamma)\,,$$

$$B_\Theta := -\frac{1}{[b]}\begin{bmatrix} \mathbb{1} & \langle b\rangle \\ \langle b\rangle & b_{\mathrm{ex}} b_{\mathrm{in}} \end{bmatrix}, \quad \langle b\rangle := \frac{b_{\mathrm{ex}} + b_{\mathrm{in}}}{2}\,,\ [b] := b_{\mathrm{ex}} - b_{\mathrm{in}}\,.$$

4) δ-interactions on Γ, with strength α, i.e. $[\gamma_0]u = 0$ and $\alpha\gamma_0 = [\gamma_1]u$: $\Pi = \mathbb{1} \oplus 0$, $B_\Theta = -\frac{1}{\alpha}$, i.e.

$$\Theta = -\left(\frac{1}{\alpha} + \gamma_0 SL\right) : H^{\frac{3}{2}}(\Gamma) \subseteq H^{-\frac{3}{2}}(\Gamma) \to H^{\frac{3}{2}}(\Gamma)$$

5) δ'-interactions on Γ, with strength β, i.e. $[\gamma_1]u = 0$ and $\beta\gamma_1 = [\gamma_0]u$: $\Pi = 0 \oplus \mathbb{1}$, $B_\Theta = -\frac{1}{\beta}$, i.e.

$$\Theta = -\left(\frac{1}{\beta} + \gamma_1 DL\right) : H^{\frac{3}{2}}(\Gamma) \subseteq H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)\,.$$

To obtain the same kind of boundary conditions on a (relatively open) piece $\Sigma \subset \Gamma$ with Lipschitz boundary, one need to compress the previous parametrizing self-adjoint operators Θ onto subspaces of $H^s(\Gamma)$. To do that one introduces the orthogonal projectors

$$\Pi^s_\Sigma : H^s(\Gamma) \to H^s(\Gamma)\,, \quad \mathrm{ran}(\Pi^s_\Sigma) = H^s_{\Sigma^c}(\Gamma)^\perp \quad s = \frac{1}{2}, \frac{3}{2}\,.$$

One has identifications

$$H^s_{\Sigma^c}(\Gamma)^\perp \simeq H^s(\Sigma)\,, \qquad \mathrm{ran}(\Pi^s_\Sigma{}') \simeq H^{-s}_{\overline{\Sigma}}(\Gamma) \simeq H^s(\Sigma)'$$

and the orthogonal projection Π^s_Σ can be identified with the restriction map

$$R_\Sigma : H^s(\Gamma) \to H^s(\Sigma)\,, \qquad R_\Sigma \phi := \phi|\Sigma\,.$$

6) Dirichlet boundary conditions on $\Sigma \subset \Gamma$:

$$\Pi = \Pi_{\Sigma}^{\frac{3}{2}} \oplus 0, \quad \Theta : \mathrm{dom}(\Theta) \subseteq H_{\overline{\Sigma}}^{-\frac{3}{2}}(\Gamma) \to H^{\frac{3}{2}}(\Sigma)\,, \Theta\phi = -(\gamma_0 SL\phi)|\Sigma$$

$$\mathrm{dom}(\Theta) = \{\phi \in H_{\overline{\Sigma}}^{-\frac{3}{2}}(\Gamma) : (\gamma_0 SL\phi)|\Sigma \in H^{\frac{3}{2}}(\Sigma)\}$$

7) Neumann boundary conditions on $\Sigma \subset \Gamma$:

$$\Pi = 0 \oplus \Pi_{\Sigma}^{\frac{1}{2}}, \quad \Theta : \mathrm{dom}(\Theta) \subseteq H_{\overline{\Sigma}}^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Sigma)\,, \Theta\phi = -(\gamma_1 DL\phi)|\Sigma$$

$$\mathrm{dom}(\Theta) = \{\phi \in H_{\overline{\Sigma}}^{-\frac{1}{2}}(\Gamma) : (\gamma_1 DL\phi)|\Sigma \in H^{\frac{1}{2}}(\Sigma)\}$$

8) Robin boundary boundary conditions on $\Sigma \subset \Gamma$:

$$\Pi = \Pi_{\Sigma}^{\frac{3}{2}} \oplus \Pi_{\Sigma}^{\frac{1}{2}} \quad \text{and so on ....}$$

Theorem

Suppose

$$\mathrm{dom}(\Theta) \subseteq H^{s_1}(\Gamma) \oplus H^{s_2}(\Gamma)$$

and define

$$s := \min\left\{ s_1 + \frac{3}{2}, s_2 + \frac{1}{2} \right\}.$$

1) If $s > 0$ *then*

$$(-\Delta_{\Pi,\Theta} + z)^{-1} - (-\Delta + z)^{-1} \in \mathfrak{S}_\infty(L^2(\mathbb{R}^n)).$$

2) If $s \geq 2$*, then, for any integer* $k \geq 1$ *and for any* $p > \frac{n-1}{2(k-1)+s}$*,*

$$(-\Delta_{\Pi,\Theta} + z)^{-k} - (-\Delta + z)^{-k} \in \mathfrak{S}_p(L^2(\mathbb{R}^n)).$$

Theorem

Let $f_{\tilde{\Theta}}$ *be the sesquilinear form in the Hilbert space* $\operatorname{ran}(\Pi)$ *associated with the self-adjoint operator* $\tilde{\Theta} := \Theta(\Lambda^3 \oplus \Lambda)$, $\Lambda := (-\Delta_{LB} + 1)^{1/2}$. *Define*

$$\tilde{s} := \min\left\{ s_1 - \frac{3}{2}, s_2 - \frac{1}{2} \right\} .$$

If $\tilde{s} \geq 1$, *then, for any integer* $k \geq 1$ *and for any* $p > \frac{n-1}{2(k-1)+2\tilde{s}}$,

$$(-\Delta_{\Pi,\Theta} + z)^{-k} - (-\Delta + z)^{-k} \in \mathfrak{S}_p(L^2(\mathbb{R}^n)) .$$

Corollary

1) Suppose

$$\mathrm{dom}(\Theta) \subseteq H^{s_1}(\Gamma) \oplus H^{s_2}(\Gamma)\,, \qquad s_1 > -\frac{3}{2}\,, \quad s_2 > -\frac{1}{2}\,.$$

Then

$$\sigma_{ess}(\Delta_{\Pi,\Theta}) = (-\infty, 0]\,.$$

Suppose either

$$\mathrm{dom}(\Theta) \subseteq H^{\frac{1}{2}}(\Gamma) \oplus H^{\frac{3}{2}}(\Gamma)\,.$$

$$\Big(\ \textit{equivalently}\ \mathrm{dom}(\tilde{\Theta}) \subseteq H^{\frac{7}{2}}(\Gamma) \oplus H^{\frac{5}{2}}(\Gamma)\ \Big)$$

or

$$\mathrm{dom}(f_{\tilde{\Theta}}) \subseteq H^{\frac{5}{2}}(\Gamma) \oplus H^{\frac{3}{2}}(\Gamma)\,,$$

hold, then

the wave operators

$$W_{\pm} := \text{s-}\lim_{t\to\pm\infty} e^{-it\Delta_{\Pi,\Theta}} e^{it\Delta}\,,$$

$$\hat{W}_{\pm} := \text{s-}\lim_{t\to\pm\infty} e^{-it\Delta} e^{it\Delta_{\Pi,\Theta}} P_{ac}$$

exist and are complete, i.e. the limits exists everywhere w.r.t. strong convergence,

$$\operatorname{ran}(W_{\pm}) = L^2(\mathbb{R}^n)_{ac}\,, \quad \operatorname{ran}(\hat{W}_{\pm}) = L^2(\mathbb{R}^n)\,, \quad W_{\pm}^{*} = \hat{W}_{\pm}\,,$$

where $L^2(\mathbb{R}^n)_{ac}$ denotes the absolutely continuous subspace of $L^2(\mathbb{R}^n)$ with respect to $\Delta_{\Pi,\Theta}$ and P_{ac} is the corresponding orthogonal projector. That implies

$$\sigma_{ac}(\Delta_{\Pi,\Theta}) = (-\infty, 0]\,.$$

The previous results do not exclude the presence of negative eigenvalues embedded in the essential spectrum, an information that is relevant for the issues to be treated later. Let us pose

$$E^{-}_{\Pi,\Theta} := \{\lambda \in (-\infty, 0) : \lambda \notin \sigma_p(\Delta_{\Pi,\Theta})\}\,,$$

so that absence of negative eigenvalues is equivalent to

$$E^{-}_{\Pi,\Theta} = (-\infty, 0)\,.$$

Theorem

Let $\Gamma_0 \subseteq \Gamma$ be a closed set such that $\mathrm{supp}(\phi) \cup \mathrm{supp}(\varphi) \subseteq \Gamma_0$ for any $\phi \oplus \varphi \in \mathrm{dom}(\Theta) \subseteq \mathrm{ran}(\Pi)'$. If the open set $\mathbb{R}^n \backslash \Gamma_0$ is connected, then $E^{-}_{\Pi,\Theta} = (-\infty, 0)$.

Obviously, in the case $\Gamma_0 = \Gamma$, one has that $\mathbb{R}^n \backslash \Gamma = \Omega_{\mathsf{in}} \cup \Omega_{\mathsf{ex}}$ is not connected. However, if Ω_{ex} is connected then, by similar reasonings, one gets $u_\lambda | \Omega_{\mathsf{ex}} = 0$. Thus, if the boundary conditions appearing in $\mathrm{dom}(\Delta_{\Pi,\Theta})$ are such that

$$u|\Omega_{\mathsf{ex}} = 0\,,\ (\Delta u - \lambda u)|\Omega_{\mathsf{in}} = 0\,,\ u \in \mathrm{dom}(\Delta_{\Pi,\Theta}) \quad \Longrightarrow \quad u|\Omega_{\mathsf{in}} = 0\,,$$

then $E^-_{\Pi,\Theta} = (-\infty, 0)$.

For example, two cases where that hypotheses hold are the δ- and δ'-interactions on Γ.

The limiting absorption principle

We introduce the family of weighted spaces $L^2_\sigma(\mathbb{R}^n)$ and $H^2_\sigma(\mathbb{R}^n)$, defined, for any $\sigma \in \mathbb{R}$, by

$$L^2_\sigma(\mathbb{R}^n) := \left\{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n) : \|u\|_{L^2_\sigma(\mathbb{R}^n)} < +\infty\right\},$$

$$H^2_\sigma(\mathbb{R}^n) := \left\{u \in H^2_{\mathrm{loc}}(\mathbb{R}^n) : \|u\|_{H^2_\sigma(\mathbb{R}^n)} < +\infty\right\},$$

$$\|u\|^2_{L^2_\sigma(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |u(x)|^2 \, \left(1 + \|x\|^2\right)^\sigma dx$$

$$\|u\|^2_{H^2_\sigma(\mathbb{R}^n)} := \|u\|^2_{L^2_\sigma(\mathbb{R}^n)} + \sum_{1\le i\le n} \|\partial_{x_i} u\|^2_{L^2_\sigma(\mathbb{R}^n)} + \sum_{1\le i,j\le n} \|\partial^2_{x_i x_j} u\|^2_{L^2_\sigma(\mathbb{R}^n)}.$$

The spaces $L^2_\sigma(\Omega_\#)$ and $H^2_\sigma(\Omega_\#)$, $\# = \mathrm{in}, \mathrm{ex}$, are defined in a similar way. Since Ω is bounded, one has

$$L^2_\sigma(\Omega_{\mathrm{in}}) = L^2(\Omega_{\mathrm{in}})\,, \quad H^2_\sigma(\Omega_{\mathrm{in}}) = H^2(\Omega_{\mathrm{in}})$$

and so

$$L^2_\sigma(\mathbb{R}^n) = L^2(\Omega_{\mathrm{in}}) \oplus L^2_\sigma(\Omega_{\mathrm{ex}})$$

and

$$H^2_\sigma(\mathbb{R}^n \backslash \Gamma) := H^2_\sigma(\Omega_{\mathrm{in}}) \oplus H^2_\sigma(\Omega_{\mathrm{ex}}) = H^2(\Omega_{\mathrm{in}}) \oplus H^2_\sigma(\Omega_{\mathrm{ex}})\,.$$

The trace operators are extended to $H^2_\sigma(\mathbb{R}^n \backslash \Gamma)$, $\sigma < 0$, by

$$\gamma_0^{\mathrm{ex}} u_{\mathrm{ex}} := \gamma_0^{\mathrm{ex}}(\chi u_{\mathrm{ex}}), \quad \gamma_1^{\mathrm{ex}} u_{\mathrm{ex}} := \gamma_1^{\mathrm{ex}}(\chi u_{\mathrm{ex}}),$$

where $\chi \in \mathcal{C}^\infty_{\mathrm{comp}}(\Omega^c)$, $\chi = 1$ on a neighborhood of Γ.

From now on we assume that $-\Delta_{\Pi,\Theta} \geq c_\circ > -\infty$ and

$$(-\Delta_{\Pi,\Theta} + z)^{-1} - (-\Delta + z)^{-1} \in \mathfrak{S}_\infty(L^2(\mathbb{R}^n)) .$$

Theorem

The limits

$$R^{\pm}_{\Pi,\Theta,-k^2} := \lim_{\epsilon \downarrow 0} (-\Delta_{(\Pi,\Theta)} - (k^2 \pm i\epsilon))^{-1}$$

exist in $\mathsf{B}(L^2_\alpha(\mathbb{R}^n), L^2_{-\alpha}(\mathbb{R}^n))$ *for all* $\alpha > \frac{1}{2}$ *and for all real k such that* $-k^2 \in E^-_{\Pi,\Theta}$.

Corollary

$\Delta_{\Pi,\Theta}$ *has empty singular continuous spectrum:*

$$L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)_{pp} \oplus L^2(\mathbb{R}^n)_{ac} .$$

The previous theorem give no answer to the obvious question: "does Kreĭn's formula survive in the limit $\epsilon \downarrow 0$?" This is provided by the next

Theorem

1) For any $k \in \mathbb{R}\backslash\{0\}$ *and for any* $\alpha > \frac{1}{2}$, *the limits*

$$G^{\pm}_{-k^2} := \lim_{\epsilon\downarrow 0} G_{-k^2\pm i\epsilon}\,, \qquad M^{\pm}_{-k^2} := \lim_{\epsilon\downarrow 0} M_{-k^2\pm i\epsilon}$$

exist in $\mathsf{B}(H^{-\frac{3}{2}}(\Gamma)\oplus H^{-\frac{1}{2}}(\Gamma), L^2_{-\alpha}(\mathbb{R}^n))$ *and* $\mathsf{B}(H^{-\frac{3}{2}}(\Gamma)\oplus H^{-\frac{1}{2}}(\Gamma), H^{\frac{3}{2}}(\Gamma)\oplus H^{\frac{1}{2}}(\Gamma))$ *respectively.*
Moreover

$$G^{\pm}_{-k^2} = G_z + (z+k^2)R^{\pm}_{-k^2}G_z\,, \quad z\in\mathbb{C}\backslash(-\infty,0]\,,$$

$$(G^{\pm}_{-k^2})' = \gamma R^{\pm}_{-k^2}\,,$$

$$M^{\pm}_{-k^2} = M_z - (z+k^2)\gamma R^{\pm}_{-k^2}G_z\,, \quad z\in\mathbb{C}\backslash(-\infty,0]\,,$$

2) For any $k \in \mathbb{R}\backslash\{0\}$ such that $-k^2 \in E^-_{\Pi,\Theta}$, the limits

$$L^{\pm}_{\Pi,\Theta,-k^2} := \lim_{\epsilon\downarrow 0}(\Theta + \Pi M_{-k^2 \pm i\epsilon}\Pi')^{-1}$$

exist in $\mathsf{B}(\mathrm{ran}(\Pi), \mathrm{ran}(\Pi)')$. Moreover

$$L^{\pm}_{\Pi,\Theta,-k^2} = (\Theta + \Pi M^{\pm}_{-k^2}\Pi')^{-1}$$

and so

$$\begin{aligned} R^{\pm}_{\Pi,\Theta,-k^2} - R^{\pm}_{-k^2} =& G^{\pm}_{-k^2}\Pi'(\Theta + \Pi M^{\pm}_{-k^2}\Pi')^{-1}\Pi\gamma R^{\pm}_{-k^2} \\ & \left(= G^{\pm}_{-k^2}\Pi'(B_\Theta - \Pi\gamma G^{\pm}_{-k^2}\Pi')^{-1}\Pi\gamma R^{\pm}_{-k^2}\right). \end{aligned}$$

Generalized eigenfunction and scattering

We introduce the following extension of $\Delta_{\Pi,\Theta}$ to the larger space $L^2_{-\alpha}(\mathbb{R}^n)$, $\alpha > 0$:

$$\tilde{\Delta}_{\Pi,\Theta} : \mathrm{dom}(\tilde{\Delta}_{\Pi,\Theta}) \subseteq L^2_{-\alpha}(\mathbb{R}^n) \to L^2_{-\alpha}(\mathbb{R}^n)\,,$$

$$\begin{aligned}\mathrm{dom}(\tilde{\Delta}_{\Pi,\Theta}) := \{u = u_0 + G(\phi\oplus\varphi) : u_0 \in H^2_{-\alpha}(\mathbb{R}^n)\,,\\ \phi\oplus\varphi \in \mathrm{dom}(\Theta)\,,\ \Pi\gamma u_0 = \Theta(\phi\oplus\varphi)\}\end{aligned}$$

$$\left(= \{u \in H^2_{-\alpha}(\mathbb{R}^n\backslash\Gamma) : [\gamma]u \in \mathrm{dom}(\Theta)\,,\ \Pi\gamma u = B_\Theta[\gamma]u\}\right),$$

$$\tilde{\Delta}_{\Pi,\Theta} u := \Delta u_0 + G(\phi\oplus\varphi) = \Delta u - [\hat{\gamma}_1]u\,\delta_\Gamma - [\hat{\gamma}_0]u\,\nu\cdot\nabla\delta_\Gamma\,.$$

By definitions one has

$$\mathrm{graph}(\tilde{\Delta}_{\Pi,\Theta}) \cap \left(L^2(\mathbb{R}^n)\oplus L^2(\mathbb{R}^n)\right) = \mathrm{graph}(\Delta_{\Pi,\Theta})\,.$$

Theorem

Let $u \neq 0$ be a generalized eigenfunction of $\Delta_{\Pi,\Theta}$ with eigenvalue $-k^2 \in E^{-}_{\Pi,\Theta}$, i.e. u belongs to $\mathrm{dom}(\tilde{\Delta}_{\Pi,\Theta})$ and solves the equation

$$(\tilde{\Delta}_{\Pi,\Theta} + k^2)u = 0\,.$$

Then either $u = u_k^+$ or $u = u_k^-$, where

$$u_k^{\pm} := u_k + G^{\pm}_{-k^2}\Pi'(\Theta + \Pi M^{\pm}_{-k^2}\Pi')^{-1}\Pi\gamma u_k\,.$$

and $u_k \in H^2_{-\alpha}(\mathbb{R}^n)$ is a generalized eigenfunction of the free Laplacian with the same eigenvalue $-k^2$.

proof. Let us set $u = u_0 + G(\phi \oplus \varphi)$, with $u_0 \in H^2_{-\alpha}(\mathbb{R}^n)$ and $\phi \oplus \varphi \in \mathrm{dom}(\Theta)$ such that $\Pi\gamma u_0 = \Theta(\phi \oplus \varphi)$. Then

$$(\tilde{\Delta}_{\Pi,\Theta} + k^2)u = 0 \iff (\Delta + k^2)u_0 = (1 + k^2)G(\phi \oplus \varphi)\,.$$

By applying $R^{\pm}_{-k^2}$ to both sides of the relation on the right one gets

$$u_0 = u_k + (1 + k^2)R^{\pm}_{-k^2}G(\phi \oplus \varphi)\,,$$

where $u_k \in H^2_{-\alpha}(\mathbb{R}^n)$ is any solution of the equation $(\Delta + k^2)u_k = 0$. Imposing the boundary conditions one then obtains

$$\begin{aligned} \Pi\gamma u_0 &= \gamma u_k + (1 + k^2)\gamma R^{\pm}_{-k^2}G(\phi \oplus \varphi) \\ =&\Pi\gamma u_k - \Pi M^{\pm}_{-k^2}(\phi \oplus \varphi) = \Theta(\phi \oplus \varphi)\,, \end{aligned}$$

i.e.

$$\phi \oplus \varphi = (\Theta + \Pi M^{\pm}_{-k^2}\Pi')^{-1}\Pi\gamma u_k\,.$$

□

We recall the following definition: we say that a solution u of the Helmholtz equation $(\Delta + k^2)u(x) = 0$, $x \in \Omega_{ex}$, satisfies the $(\pm)$ Sommerfeld radiation condition whenever

$$\lim_{\|x\| \to +\infty} \|x\|^{(n-1)/2} (\hat{x} \cdot \nabla \pm ik) u(x) = 0$$

holds uniformly in $\hat{x} = x/\|x\|$. The plus sign corresponds to an inward wave and the minus one corresponds to a outward wave.

Lemma

1) $G^{\pm}_{-k^2}(\phi \oplus \varphi)$ *satisfies the* $(\pm)$ *Sommerfeld radiation condition.*

2) If $u \in \ker(\tilde{\Delta}_{\Pi,\Theta} + k^2)$, *satisfies the Sommerfeld radiation condition, then* $u = 0$.

Considering the usual family of generalized eigenfunctions $u_\xi \in H^2_{-\alpha}(\mathbb{R}^n)$, $\alpha > \frac{n}{2}$, of $\Delta : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ given by $u^0_\xi(x) := e^{i\,\xi\cdot x}$, one obtains the family of generalized eigenfunctions of $\Delta_{\Pi,\Theta}$ defined by

$$u^\pm_\xi := u^0_\xi + G^\mp_{-k^2}\Pi'(\Theta + \Pi M^\mp_{-k^2}\Pi')^{-1}\Pi\gamma u^0_\xi\,,\ k = \|\xi\|\,,\ -k^2 \in E^-_{\Pi,\Theta}\,.$$

Since

$$[\hat\gamma]G^\pm_{-k^2} = \mathbb{1}_{H^{-\frac{3}{2}}(\Gamma)\oplus H^{-\frac{1}{2}}(\Gamma)}\,.$$

and $[\gamma]u^0_\xi = 0$, one gets

$$[\hat\gamma]u^\pm_\xi = (\Theta + \Pi M^\mp_{-k^2}\Pi')^{-1}\Pi\gamma u_\xi\,,$$

and so the functions $u^\pm_\xi \in \mathrm{dom}(\tilde\Delta_{\Pi,\Theta})$ solve the Lippmann-Schwinger type equation

$$u^\pm_\xi = u^0_\xi + G^\mp_{-k^2}[\hat\gamma]u^\pm_\xi\,.$$

Let us now define, for any $u \in L^2_\alpha(\mathbb{R}^n)$,

$$F_\pm u(\xi) := \frac{1}{(2\pi)^{n/2}} \langle u_\xi^\pm, u\rangle = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \overline{u_\xi^\pm}(x) u(x)\, dx$$

Theorem

1) $F_\pm$ extend to isometries $F_\pm : L^2(\mathbb{R}^n)_{ac} \to L^2(\mathbb{R}^n)$.

2) $\forall u \in \mathrm{dom}(\Delta_{\Pi,\Theta})\,, \quad (F_\pm P_{ac}\, \Delta_{\Pi,\Theta} u)(\xi) = -\|\xi\|^2 F_\pm u(\xi)\,.$

3) Suppose $W_\pm$ exist. Then

$$W_\pm = F_\pm^* F\,,$$

where F denotes Fourier transform.

sketch of the proof. 3) We equivalently show that $F_\pm W_\pm u = Fu$ for any u in the Schwartz space of rapidly decreasing functions. We define $W_\pm(t) := P_{ac}\, e^{-it\Delta_{\Pi,\Theta}} e^{it\Delta}$.

$$
\begin{aligned}
F_\pm W_\pm u &= \lim_{t\to\pm\infty} F_\pm W_\pm(t) u = \lim_{\epsilon\to 0\pm} \epsilon \int_0^{\pm\infty} e^{-\epsilon t} F_\pm W_\pm(t) u \, dt \\
&= \lim_{\epsilon\to 0\pm} \int_0^{\pm\infty} e^{-\epsilon t}\, \frac{d}{dt} F_\pm W_\pm(t) u \, dt + F_\pm u \, .
\end{aligned}
$$

Then, since $F_\pm$ diagonalize $P_{ac}\, \Delta_{\Pi,\Theta}$,

$$
(F_\pm W_\pm(t) u)(\xi) = \frac{1}{(2\pi)^{n/2}} \left\langle u_\xi^\pm, e^{it(\Delta+\|\xi\|^2)} u \right\rangle
$$

and, since $(\Delta + \|\xi\|^2) u_\xi^0 = 0$ and $u_\xi^\pm = u_\xi^0 + G^\mp_{-\|\xi\|^2}[\gamma] u_\xi^\pm$,

$$
\begin{aligned}
e^{-\epsilon t}\, \frac{d}{dt} (F_\pm W_\pm(t) u)(\xi) &= \frac{i e^{-\epsilon t}}{(2\pi)^{n/2}} \left\langle u_\xi^\pm, (\Delta + \|\xi\|^2) e^{it(\Delta+\|\xi\|^2)} u \right\rangle \\
= &\frac{i}{(2\pi)^{n/2}} \left\langle G^\mp_{-\|\xi\|^2}[\gamma] u_\xi^\pm, (\Delta + \|\xi\|^2) e^{it(\Delta+\|\xi\|^2+i\epsilon)} u \right\rangle
\end{aligned}
$$

$$= \frac{1}{(2\pi)^{n/2}} \frac{d}{dt} \left\langle G^{\mp}_{-\|\xi\|^2}[\gamma] u^{\pm}_{\xi}, (\Delta+\|\xi\|^2)(\Delta+\|\xi\|^2+i\epsilon)^{-1} e^{it(\Delta+\|\xi\|^2+i\epsilon)} u \right\rangle .$$

Therefore

$$\begin{aligned} & (F_{\pm} W_{\pm} u)(\xi) \\ = & \lim_{\epsilon \to 0\pm} \int_0^{\pm\infty} e^{-\epsilon t} \frac{d}{dt} (F_{\pm} W_{\pm}(t) u)(\xi)\, dt + F_{\pm} u(\xi) \\ = & \frac{-1}{(2\pi)^{n/2}} \lim_{\epsilon \to 0\pm} \left\langle G^{\mp}_{-\|\xi\|^2}[\gamma] u^{\pm}_{\xi}, (\Delta + \|\xi\|^2)(\Delta + \|\xi\|^2 + i\epsilon)^{-1} u \right\rangle + F_{\pm} u(\xi) \\ = & - \frac{1}{(2\pi)^{n/2}} \left\langle G^{\mp}_{-\|\xi\|^2}[\gamma] u^{\pm}_{\xi}, u \right\rangle + F_{\pm} u(\xi) \\ = & F u(\xi) . \end{aligned}$$

Let us now introduce the scattering operator $S := W_+^* W_-$, so that, by $W_\pm = F_\pm^* F$, one gets

$$FSF^* = F_+ F_-^* \,.$$

Using the isomorphism
$L^2(\mathbb{R}^n) \simeq L^2(\mathbb{R}_+) \otimes L^2(\mathbb{S}^{n-1}) \simeq L^2(\mathbb{R}_+; L^2(\mathbb{S}^{n-1}))$, $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$, given by $u_k(\hat{\xi}) := u(k\hat{\xi})$, the on-shell scattering operator (scattering matrix)

$$S_k : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1}) \,, \quad k > 0,$$

is then defined by the relation $S_k(Fu)_k = (FSu)_k$, i.e.

$$S_k(F_- u)_k = (F_+ u)_k \,.$$

Theorem

For any $k > 0$ such that $-k^2 \in E^-_{\Pi,\Theta}$,

$$S_k f(\hat{\xi}) = f(\hat{\xi}) - \int_{\mathbb{S}^{n-1}} s_k(\hat{\xi}, \hat{\xi}')\, f(\hat{\xi}')\, d\mu(\hat{\xi}')\,,$$

where

$$\begin{aligned} s_k(\hat{\xi}, \hat{\xi}') :=& \frac{i}{4\pi} \left(\frac{k}{2\pi}\right)^{n-2} \langle \Pi\gamma u^0_{k\hat{\xi}'}, (\Theta + \Pi M^-_{-k^2}\Pi')^{-1}\Pi\gamma u^0_{k\hat{\xi}} \rangle \\ =& \frac{i}{4\pi} \left(\frac{k}{2\pi}\right)^{n-2} \langle (\Theta + \Pi M^+_{-k^2}\Pi')^{-1}\Pi\gamma u^0_{k\hat{\xi}'}, \Pi\gamma u^0_{k\hat{\xi}} \rangle \\ &\left(= \frac{i}{4\pi} \left(\frac{k}{2\pi}\right)^{n-2} \langle \Pi\gamma u^0_{k\hat{\xi}'}, (B_\Theta - \Pi\gamma G^-_{-k^2}\Pi')^{-1}\Pi\gamma u^0_{k\hat{\xi}} \rangle \right) \\ &\left(= \frac{i}{4\pi} \left(\frac{k}{2\pi}\right)^{n-2} \langle (B_\Theta - \Pi\gamma G^+_{-k^2}\Pi')^{-1}\Pi\gamma u^0_{k\hat{\xi}'}, \Pi\gamma u^0_{k\hat{\xi}} \rangle \right). \end{aligned}$$

Proof.

By the definition of S_k we only need to show that

$$(F_+ u)_k(\hat{\xi}) = (F_- u)_k(\hat{\xi}) - \int_{\mathbb{S}^{n-1}} s_k(\hat{\xi}, \hat{\xi}')\,(F_- u)_k(\hat{\xi}')\,d\mu(\hat{\xi}')\,, \forall u \in L^2_\alpha(\mathbb{R}^n)\,.$$

Let

$$v_{k\hat{\xi}} := u^-_{k\hat{\xi}} - u^+_{k\hat{\xi}} - \int_{\mathbb{S}^{n-1}} s_k(\hat{\xi}, \hat{\xi}')\,u^-_{k\hat{\xi}'}\,d\mu(\hat{\xi}')\,.$$

By Lippman-Schwinger,

$$\begin{aligned} v_{k\hat{\xi}} =& G^+_{-k^2}[\gamma]u^-_{k\hat{\xi}} - G^-_{-k^2}[\gamma]u^+_{k\hat{\xi}} - \int_{\mathbb{S}^{n-1}} s_k(\hat{\xi}, \hat{\xi}')\,G^+_{-k^2}[\gamma]u^-_{k\hat{\xi}'}\,d\mu(\hat{\xi}') \\ & -\frac{i}{4\pi}\,\left(\frac{k}{2\pi}\right)^{n-2} \int_{\mathbb{S}^{n-1}} \langle \gamma u^0_{k\hat{\xi}'}, [\gamma]u^+_{k\hat{\xi}}\rangle u^0_{k\hat{\xi}'}\,d\mu(\hat{\xi}')\,. \end{aligned}$$

By integrating the plane waves u^0_ξ over $\mathbb{S}^{n-1}$, one gets

$$\int_{\mathbb{S}^{n-1}} \bar{u}^0_{k\hat{\xi}}(x)\,u^0_{k\hat{\xi}}(y)\,d\mu(\hat{\xi}) = 4\pi i\left(\frac{2\pi}{k}\right)^{n-2}\Big(\mathscr{G}^-_{-k^2}(x-y) - \mathscr{G}^+_{-k^2}(x-y)\Big)\,.$$

This gives

$$-\frac{i}{4\pi}\left(\frac{k}{2\pi}\right)^{n-2}\int_{\mathbb{S}^{n-1}}\langle\gamma u^0_{k\hat{\xi}'},[\gamma]u^+_{k\hat{\xi}}\rangle\, u^0_{k\hat{\xi}'}\,d\mu(\hat{\xi}')=\left(G^-_{-k^2}-G^+_{-k^2}\right)[\gamma]u^+_{k\hat{\xi}}$$

and so

$$v_{k\hat{\xi}}=G^+_{-k^2}[\gamma]v_{k\hat{\xi}}\,.$$

Therefore $v_{k\hat{\xi}}$ satisfies the Sommerfeld radiation condition. Since $u^\pm_{k\hat{\xi}}\in\ker(\tilde{\Delta}_{\Pi,\Theta}+k^2)$, one has $v_{k\hat{\xi}}\in\ker(\tilde{\Delta}_{\Pi,\Theta}+k^2)$. Thus $v_{k\hat{\xi}}=0$ and so

$$u^+_{k\hat{\xi}}=u^-_{k\hat{\xi}}-\int_{\mathbb{S}^{n-1}}s_k(\hat{\xi},\hat{\xi}')\,u^-_{k\hat{\xi}'}\,d\mu(\hat{\xi}')\,.$$

Integrating both the left and right sides with respect to $u\in L^2_\alpha(\mathbb{R}^n)$ one gets the result.

Dirichlet boundary conditions on Γ.

Let us consider the self-adjoint extension $\Delta_D = \Delta_D^{\rm in} \oplus \Delta_D^{\rm ex}$ corresponding to the direct sum of Dirichlet Laplacian in $L^2(\Omega_{\rm in})$ and $L^2(\Omega_{\rm ex})$. We know that such a self-adjoint extension correspond to $\Pi = \mathbb{1} \oplus 0$ and $\Theta = -\gamma_0 SL$, i.e. $B_\Theta = 0$. Since

$$(\gamma_0 SL_z)^{-1} = P_z^{\rm in} - P_z^{\rm ex}\,,$$

where $P_z^{\rm in}$ and $P_z^{\rm ex}$ denote the Dirichlet-to-Neumann operators for $\Omega_{\rm in}$ and $\Omega_{\rm in}$ respectively, one has

$$(-(\Delta_D^{\rm in} \oplus \Delta_D^{\rm ex}) + z)^{-1} = (-\Delta + z)^{-1} + SL_z(P_z^{\rm ex} - P_z^{\rm in})\gamma_0(-\Delta + z)^{-1}\,.$$

and, for any $k > 0$ such that $-k^2 \notin \sigma(\Delta_D^{\rm in})$,

$$s_k(\hat\xi, \hat\xi') = \frac{i}{4\pi}\left(\frac{k}{2\pi}\right)^{n-2}\langle (P^{\rm ex}_{-k^2} - P^{\rm in}_{-k^2})^+ \gamma_0 u^\circ_{k\hat\xi'}, \gamma_0 u^\circ_{k\hat\xi}\rangle\,,$$

Dirichlet boundary conditions on $\Sigma \subset \Gamma$.

By taking $\Pi = \Pi_\Sigma^{3/2} \oplus 0$ and $\Theta = \Theta_{D,\Sigma}$, $\Theta_{D,\Sigma}\phi := -(\gamma_0 SL\phi)|\Sigma$, one gets the self-adjoint operator

$$\Delta_{D,\Sigma} u = \Delta u - [\hat{\gamma}_1]u\,\delta_{\overline{\Sigma}}\,,$$

$$\begin{aligned}&\mathrm{dom}(\Delta_{D,\Sigma}) = \\ =&\{u \in H^1(\mathbb{R}^n) \cap H_\Delta(\mathbb{R}^n\backslash\Gamma) : [\hat{\gamma}_1]u \in \mathrm{dom}(\Theta_{D,\Sigma})\,,\ (\gamma_0 u)|\Sigma = 0\}\end{aligned}$$

$$\begin{aligned}&(-\Delta_{D,\Sigma} + z)^{-1} \\ =&(-\Delta + z)^{-1} - SL_z\Pi'_\Sigma(R_\Sigma\gamma_0 SL_z\Pi'_\Sigma)^{-1}R_\Sigma\gamma_0(-\Delta + z)^{-1}\end{aligned}$$

and, for any $k > 0$,

$$s_k(\hat{\xi}, \hat{\xi}') = -\frac{i}{4\pi}\left(\frac{k}{2\pi}\right)^{n-2}\langle(R_\Sigma\gamma_0 SL^+_{-k^2}\Pi'_\Sigma)^{-1}R_\Sigma\gamma_0 u^\circ_{k\hat{\xi}'}, R_\Sigma\gamma_0 u^\circ_{k\hat{\xi}}\rangle\,.$$