Optimal decay of Wannier functions in Chern and Quantum Hall insulators

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based on joint work with D. Monaco, A. Pisante and S. Teufel

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Overview: symmetry, localization, transport, topology

$$\boxed{\text{Time-reversal symmetry}} \overset{\text{Symmetry}}{\longleftrightarrow} \boxed{\text{Broken TR-symmetry}}$$

Overview: symmetry, localization, transport, topology

Time-reversal symmetry	Symmetry $\longleftrightarrow$	**Broken TR-symmetry**
Exponentially localized Wannier functions $\exists \beta > 0 : \mathrm{e}^{\beta\lvert x\rvert} w \in L^2(\mathbb{R}^d)$	Localization $\longleftrightarrow$	**Generically delocalized Wannier functions** $\langle w, \lvert x\rvert^2 w \rangle = +\infty$
Vanishing Hall current	Transport $\longleftrightarrow$	**Generically non-zero Hall current**

Overview: symmetry, localization, transport, topology

Time-reversal symmetry	Symmetry $\longleftrightarrow$	Broken TR-symmetry
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Vanishing Hall current	Transport $\longleftrightarrow$	Generically non-zero Hall current
Trivial Bloch bundle	Topology $\longleftrightarrow$	Generically non trivial Bloch bundle
Classes AI & AII	AZC class $\longleftrightarrow$	Class A Chern insultors, QHI

I.

Introduction to Chern insulators

Kitaev's periodic table of topological insulators

	Symmetry			d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Quantum Hall Effect

He-3 (B phase)

QSHI: HgTe

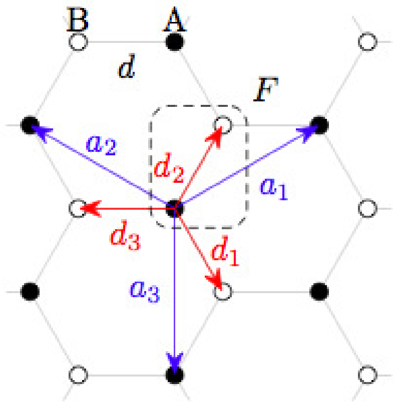
Majorana

$Bi_2\,Se_3$

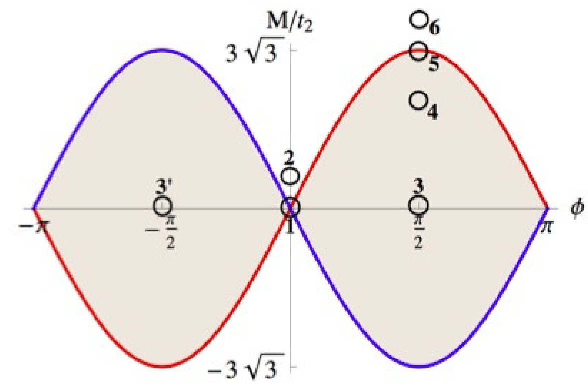
KITAEV, A. : Periodic table for topological insulators and superconductors, *AIP Conf. Proc.* **1134**, 22 (2009).

RYU, S.; SCHNYDER, A. P.; FURUSAKI, A.; LUDWIG, A. W. W. : *New J. Phys.* **12**, 065010 (2010).

Haldanium: the theoretical Chern insulator



The honeycomb structure $\mathcal{C}$



Topological phases of the Haldane model

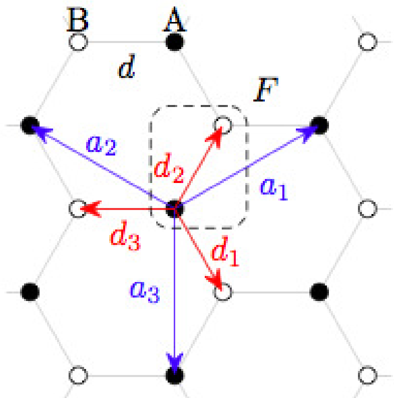
In $\ell^2(\mathcal{C})$ the Haldane Hamiltonian acts, for $\phi = 0$, as

$$\hat{H}_{0,M} = \underbrace{t_2 \sum_j (T_{\mathbf{a}_j} + T_{-\mathbf{a}_j})}_{\text{NNN hopping}} + \underbrace{t_1 \sum_j (T_{\mathbf{d}_j} + T_{-\mathbf{d}_j})}_{\text{NN hopping}} + \underbrace{V_{\mathbf{x}}}_{\text{on-site energy}}$$

where $T_{\mathbf{a}}$ is the translation by the vector $\mathbf{a} \in \mathbb{R}^2$, and $V(\mathbf{x}) := \begin{cases} +M & \text{for } \mathbf{x} \in A, \\ -M & \text{for } \mathbf{x} \in B. \end{cases}$

Haldane, F.D.M. : Model for a Quantum Hall effect without Landau levels: condensed-matter realization of the "parity anomaly". *Phys. Rev. Lett.* **61**, 2017 (1988).

Haldanium: the theoretical Chern insulator



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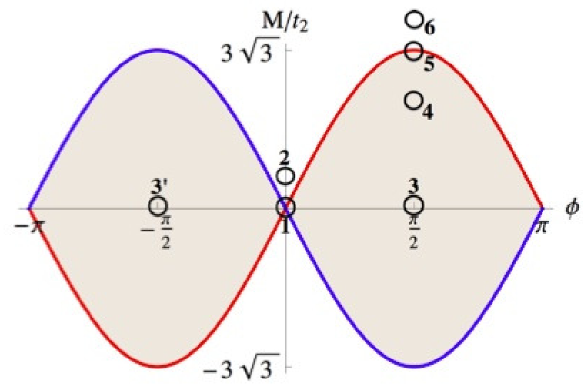


Topological phases of the Haldane model

For $\phi \neq 0$, the Hamiltonian reads instead

$$\hat{H}_{\phi,M} = t_2\, \mathrm{e}^{\mathrm{i}\phi} \sum_j (T_{\mathbf{a}_j} + T_{-\mathbf{a}_j}) \;+\; t_1 \sum_j (T_{\mathbf{d}_j} + T_{-\mathbf{d}_j}) \;+\; V_{\mathbf{x}}$$

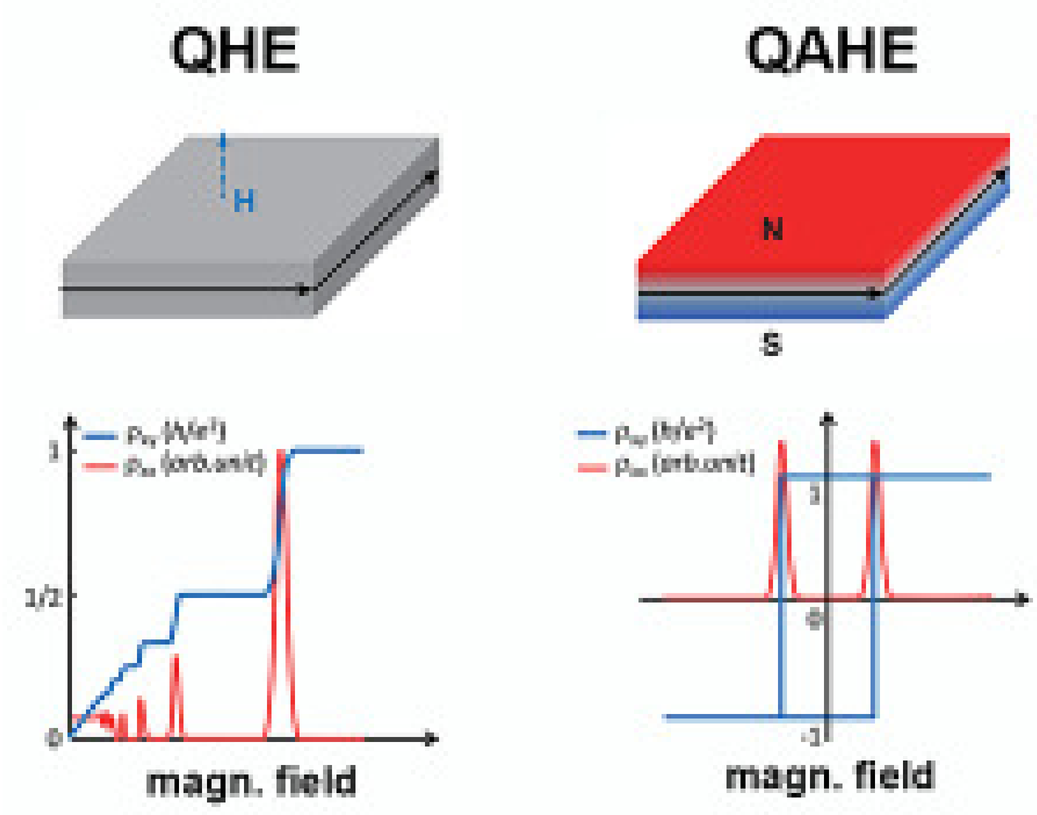
After Fourier transform $\mathcal{F} : \ell^2(\mathcal{C}) \simeq \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2 \longrightarrow L^2(\mathbb{T}^2, \mathbb{C}^2)$, one gets a **decomposed Hamiltonian** $H(\mathbf{k})$ depending on the crystal momentum $\mathbf{k}$.

The relevant object is the **orthogonal projector**

$$P_-(\mathbf{k}) = |u_-(\mathbf{k})\rangle \langle u_-(\mathbf{k})|$$

and the **Chern number** of the vector bundle which is canonically associated.

Experimental discovery of Chern insulators



Thin films of magnetic topological insulators can exhibit a nearly ideal Quantum Hall effect ***without requiring an applied magnetic field***.

(Picture by *Bestwick et al.*)

Bestwick et al. : Precise quantization of the anomalous Hall effect near zero magnetic field. *Phys. Rev. Lett.* **114**, 187201 (2015). Chang et al. : High-precision realization of robust quantum anomalous Hall state in a hard ferromagnetic topological insulator. *Nature Materials* **14**, 473 (2015).

Experimental discovery of Chern insulators

nature materials
LETTERS
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High-precision realization of robust quantum anomalous Hall state in a hard ferromagnetic topological insulator

Cui-Zu Chang[1*], Weiwei Zhao[2*], Duk Y. Kim[2], Haijun Zhang[3], Badih A. Assaf[4], Don Heiman[4], Shou-Cheng Zhang[3], Chaoxing Liu[2], Moses H. W. Chan[2] and Jagadeesh S. Moodera[1,5*]

Selected for a Viewpoint in *Physics*
PRL **114**, 187201 (2015) PHYSICAL REVIEW LETTERS week ending 8 MAY 2015

Precise Quantization of the Anomalous Hall Effect near Zero Magnetic Field

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We report a nearly ideal quantum anomalous Hall effect in a three-dimensional topological insulator thin film with ferromagnetic doping. Near zero applied magnetic field we measure exact quantization in the Hall resistance to within a part per 10 000 and a longitudinal resistivity under 1 Ω per square, with chiral edge transport explicitly confirmed by nonlocal measurements. Deviations from this behavior are found to be caused by thermally activated carriers, as indicated by an Arrhenius law temperature dependence. Using the deviations as a thermometer, we demonstrate an unexpected magnetocaloric effect and use it to reach near-perfect quantization by cooling the sample below the dilution refrigerator base temperature in a process approximating adiabatic demagnetization refrigeration.

A continuous model for Chern insulators

◇ The dynamics of a particle in a crystalline solid subject to an electromagnetic field can be modeled by

$$H_\Gamma = \frac{1}{2}\left(-\mathrm{i}\nabla_x + A_\Gamma(x)\right)^2 + V_\Gamma(x) \qquad \text{acting in } L^2(\mathbb{R}^d).$$

◇ H_Γ should commute with the lattice translation operators

$$(T_\gamma \psi)(x) := \psi(x-\gamma), \quad \gamma \in \Gamma \simeq \mathbb{Z}^d,\ \psi \in L^2(\mathbb{R}^d),$$

thus the potentials $A_\Gamma : \mathbb{R}^d \to \mathbb{R}^d$ and $V_\Gamma : \mathbb{R}^d \to \mathbb{C}$ are **Γ-periodic** functions. Hence, the **magnetic flux per unit cell is zero** ($\Longrightarrow$ no macroscopic magnetic field is measured in the laboratory).

◇ **Assumption:** We assume that the operators $\mathrm{div} A_\Gamma$, $A_\Gamma \cdot \nabla$, A_Γ^2 and V_Γ are infinitesimally Δ-bounded **(Kato small)**.

◇ In view of periodicity, after **(modified) Bloch-Floquet transform** we get a decomposed operator

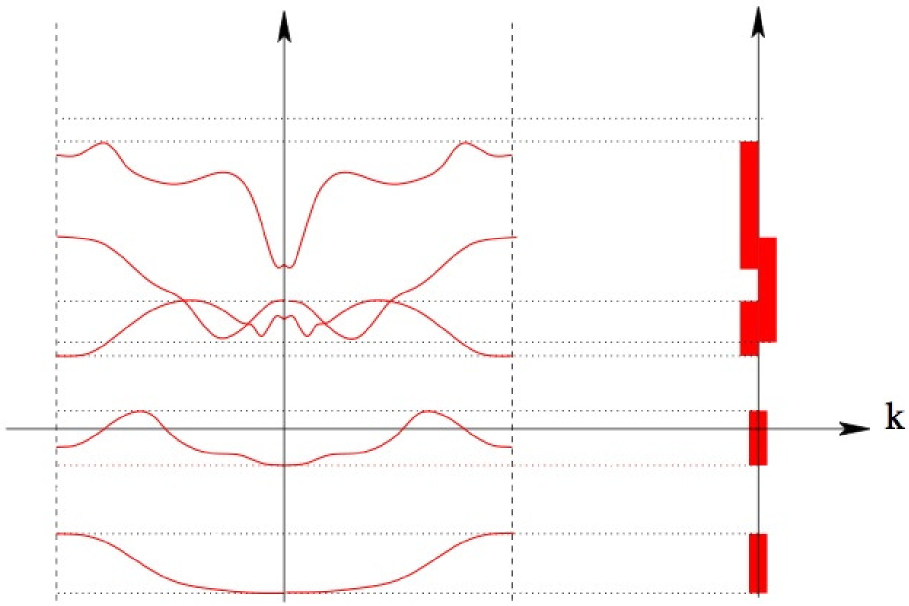
$$H(k) = \frac{1}{2}\left(-\mathrm{i}\nabla_x + A_\Gamma(x) + k\right)^2 + V_\Gamma(x) \qquad \text{acting in } L^2(Y, \mathrm{d}y),$$

where Y is a fundamental domain for the action of Γ on $\mathbb{R}^2$.

$\diamond$ For each fixed $k \in \mathbb{R}^d$, the operator

$$H(k) = \frac{1}{2}\left(-\mathrm{i}\nabla_x + A_\Gamma(x) + k\right)^2 + V_\Gamma(x) \qquad \text{acting in } L^2(Y, \mathrm{d}y),$$

has **compact resolvent**, so pure point spectrum accumulating at $+\infty$.



Spectrum of $H(k)$ as a function of k_j (left).

Question: how can I read from the picture the effect of **broken TR-symmetry**?

Actually, the broken symmetry appears when one considers the orthogonal projection of an **isolated family of Bloch bands**.

◇ For each fixed $k \in \mathbb{R}^d$, the operator

$$H(k) = \frac{1}{2}\left(-\mathrm{i}\nabla_x + A_\Gamma(x) + k\right)^2 + V_\Gamma(x) \qquad \text{acting in } L^2(Y, \mathrm{d}y),$$

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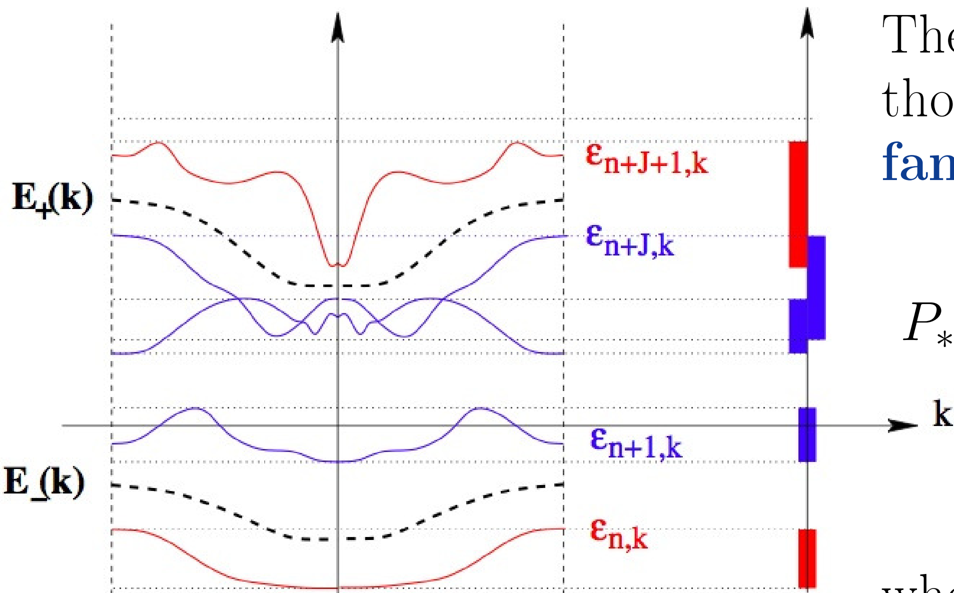

An isolated family of J Bloch bands (blue).

The object of investigation is the orthogonal projection on an **isolated family of bands**

$$P_*(k) = \frac{\mathrm{i}}{2\pi} \oint_{\mathcal{C}_*(k)} \left(H(k) - z\mathbb{1}\right)^{-1} \mathrm{d}z$$

$$= \sum_{n \in \mathcal{I}_*} |u_n(k, \cdot)\rangle \langle u_n(k, \cdot)| \, .$$

where $\mathcal{C}_*(k)$ intersects the real line in $E_-(k)$ and $E_+(k)$.

A continuous model for **Quantum Hall systems**

◇ For $d = 3$, let $A_b(x) = \frac{1}{2c} x \wedge B$ when $d = 3$, where $B = b\,\hat{B}$ is the applied magnetic field. Consider the Hamiltonian

$$H_{\Gamma,b} = \frac{1}{2}\left(-\mathrm{i}\nabla_x + A_b(x)\right)^2 + V_\Gamma(x) \quad \text{ in } L^2(\mathbb{R}^d).$$

◇ The rôle of the ordinary translations is now played by the **magnetic translations**

$$(T_\gamma^{A_b}\psi)(x) := \mathrm{e}^{\mathrm{i}\gamma\cdot A_b(x)}\,\psi(x-\gamma), \quad \gamma \in \Gamma.$$

◇ These commute with $H_{\Gamma,b}$, but satisfy the **pseudo-Weyl relations**

$$T_\gamma^{A_b} T_\mu^{A_b} = \mathrm{e}^{\mathrm{i}\frac{1}{c}B\cdot(\gamma\wedge\mu)}\, T_\mu^{A_b} T_\gamma^{A_b}, \quad \gamma, \mu \in \Gamma.$$

One recognizes the ratio

$$\frac{B\cdot(\gamma\wedge\mu)}{2\pi c} = \frac{\text{magnetic flux}}{hc/e}$$

Zak, J. : Magnetic translation group, *Phys. Review* **134** (1964), A1602.

Notice that, in Hartree units, the quantum of magnetic flux is $\frac{hc}{e} = 2\pi c$.

- ◇ If we assume that $\frac{1}{c} B \cdot (\gamma \wedge \gamma') \in 2\pi\mathbb{Q}$ for all $\gamma, \gamma' \in \Gamma$ then the magnetic translations provide a true **unitary representation of the lattice translation group** Γ on $L^2(\mathbb{R}^d)$.
- ◇ For $d = 2$ it suffices to ask that $B \cdot (a_1 \wedge a_2) = (2\pi c) p/q$ and to define

$$T^{A_b} \colon \Gamma \to \mathcal{U}(L^2(\mathbb{R}^2)), \quad \gamma = \gamma_1 a_1 + \gamma_2 a_2 \mapsto (T^{A_b}_{a_1})^{\gamma_1} (T^{A_b}_{a_2})^{q\gamma_2}.$$

Hereafter, assuming rational flux, we treat **both problems by the same methods:**

(i) **Chern insulators:**
ordinary translations $\{T_\gamma\}_{\gamma \in \Gamma}$ and Bloch Floquet transform $\mathcal{U}_{\mathrm{BF}}$;

(ii) **Quantum Hall systems**:
magnetic translations $\{T^{(b)}_\gamma\}_{\gamma \in \Gamma_b}$, magnetic Bloch-Floquet transform $\mathcal{U}^{(b)}_{\mathrm{mBF}}$.
Notice however that Γ_b, and hence the Brillouin torus $\mathbb{T}^d = \mathbb{R}^d / \Gamma^*_b$, depend on the value of $b \in \mathbb{R}$.

Units : Notice that, in Hartree units, the quantum of magnetic flux is $\frac{hc}{e} = 2\pi c$.

II. Assumptions and main results

Assumptions on the relevant family of projectors

Assumption 1: We consider a family of **orthogonal projections** $\{P(k)\}_{k\in\mathbb{R}^d}$ in $\mathcal{H}$ with the following properties:

(P_1) **smoothness:** the map $k \mapsto P(k)$ is **C^ω-smooth** from $\mathbb{R}^d$ to $\mathcal{B}(\mathcal{H})$ (equipped with the operator norm);

(P_2) **presudo-periodicity** the map $k \mapsto P(k)$ is **τ-covariant**, *i. e.*

$$P(k+\lambda) = \tau(\lambda)\, P(k)\, \tau(\lambda)^{-1} \qquad \forall k \in \mathbb{R}^d, \quad \forall \lambda \in \Gamma^*;$$

where $\Gamma^* \ni \lambda \mapsto \tau(\lambda) \in \mathcal{U}(\mathcal{H})$ is a group homomorphism.

- ◇ For the two relevant examples (ChI & QHI), the projectors $\{P_*(k)\}_{k\in\mathbb{R}^d}$ on an **isolated family of (magnetic) Bloch bands** satisfy the previous assumption:

$$\text{energy gap} \Rightarrow (\mathrm{P}_1), \qquad \text{periodicity} \Rightarrow (\mathrm{P}_2).$$

- ◇ W.l.o.g we may assume **periodicity**, *i. e.* $\tau(\lambda) \equiv \mathbb{1}$: if ($\mathrm{P}_2$) is satisfied, there exists a C^ω-smooth map $k \mapsto U(k) \in \mathcal{U}(\mathcal{H})$, such that

$$\widetilde{P}(k) := U(k)P(k)U(k)^{-1} \qquad \text{is periodic.}$$

Cornean, H.D.; Herbst, I.; Nenciu, G. : to appear in *Annales Henri Poincarè* (2016).

Bloch frames and composite Wannier fucntions

Definition 1: A **Bloch frame** is, by definition, is a map

$$\begin{aligned} \Phi : \ \Omega &\longrightarrow \mathcal{H} \oplus \ldots \oplus \mathcal{H} = \mathcal{H}^m \\ k &\longmapsto (\phi_1(k), \ldots, \phi_m(k)) \end{aligned}$$

such that for a.e. $k \in \Omega$ the set $\{\phi_1(k), \ldots, \phi_m(k)\}$ is an orthonormal basis spanning $\operatorname{Ran} P(k)$.

In this context, a **non-abelian Bloch gauge appears**, since whenever $\{\phi_a\}$ is a Bloch frame, then one obtains another Bloch frame by setting

$$\widetilde{\phi}_a(k) = \sum_{b=1}^{m} \phi_b(k)\, U_{ba}(k) \qquad \text{for some unitary matrix } U(k).$$

Definition 2: The **composite Wannier functions** $\{w_1, \ldots, w_m\} \subset L^2(\mathbb{R}^d)$ associated to a Bloch frame $\{\phi_1, \ldots, \phi_m\} \subset \mathcal{H}_\tau$ are defined as

$$w_a(x) := \left(\mathcal{U}_{\mathrm{mBF}}^{-1} \phi_a\right)(x) = \frac{1}{|\mathbb{B}|^{1/2}} \int_{\mathbb{B}} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}k \cdot x} \phi_a(k, x).$$

Note that the properties of CWFs depend on the choice of the Bloch gauge.

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des Cloizeaux, J. : Analytical properties of n-dimensional energy bands and Wannier functions. *Phys. Rev.* **135**, A698–A707 (1964).

From obstruction to construction

◇ **Question:** There exists a choice of **Bloch gauge** yielding a system of **exponentially localized composite Wannier functions**? This means:

$$\int_{\mathbb{R}^d} \mathrm{e}^{2\beta|x|} |w_a(x)|^2 \mathrm{d}x < +\infty \qquad \text{for all } \beta < \alpha.$$

◇ This is equivalent to ask wether there exists an **analytic** and **periodic** Bloch frame.

◇ It has been shown that the obstruction to the existence of the latter is **purely topological** , and is equivalent to the **triviality of a Hermitian vector bundle**, the so-called **Bloch bundle**.

NENCIU, A.; NENCIU, G. : The existence of generalized Wannier functions for one-dimensional systems. *Commun. Math. Phys.* **190**, 541–548 (1988).

BROUDER ET AL: Exponential localization of Wannier functions in insulators. *Phys. Rev. Lett.* **98** (2007).

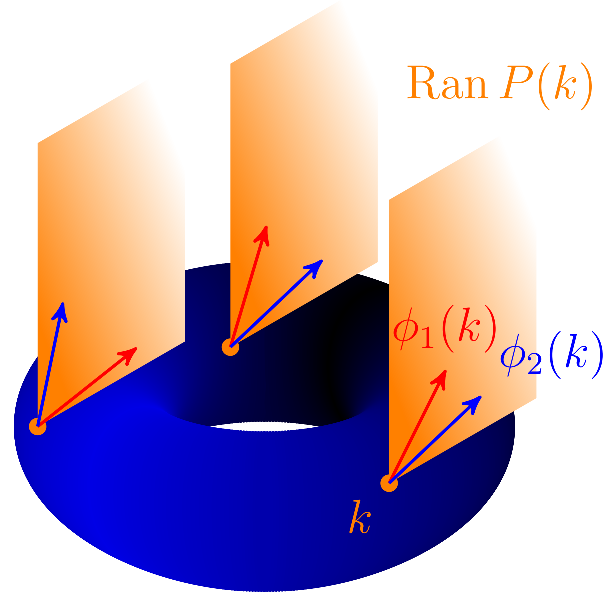
PANATI, G.: Triviality of Bloch and Bloch-Dirac bundles. *Ann. Henri Poincaré* **8**, 995–1011 (2007).

The Bloch bundle and its first Chern numbers

Competition between **regularity** and **periodicity** is encoded by the **Bloch bundle**

$$\mathcal{E} = \left(E \to \mathbb{T}^d_*\right)$$

with fibers $E_k \simeq \operatorname{Ran} P(k)$.



Since $d \leq 3$, this topological obstruction is in the **first Chern numbers**

$$c_1(P)_{ij} := \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{B}_{ij}} \operatorname{Tr}_{\mathcal{H}} \left(P(k) \left[\partial_i P(k), \partial_j P(k)\right]\right) \mathrm{d}k_i \wedge \mathrm{d}k_j, \quad 1 \leq i < j \leq d$$

where $\mathbb{B}_{ij} \subset \mathbb{B}$ is a sub-2-torus of the Brillouin torus parametrized by the coordinates $\{k_i, k_j\}$, $1 \leq i < j \leq d$.

Theorem: if the system is **time-reversal symmetric**, then $c_1(P) = 0$; hence the Bloch bundle is trivial, provided $d \leq 3$.

Avron J.E.; Seiler, R.; Simon, B.: Charge Deficiency, Charge Transport and Comparison of Dimensions. *Commun. Math. Phys.* **159** (1994).

The Chern non-trivial case: what happens?

◇ We focus on the Chern non-trivial case: we assume that at least one of the numbers $c_1(P)_{ij}$ **is non zero**.

◇ **Questions:** What is the **optimal regularity** of a Bloch frame?
[we know that a **continuos** Bloch frame can **not exist**].

◇ Correspondingly, what is the **optimal decay of composite Wannier functions**?

◇ In other words, what is the best $s \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^d} (1 + |x|^2)^s |w_a(x)|^2 \mathrm{d}x < +\infty ?$$

Obviously **true for** $s = 0$, and **false for** $s > d/2$.

For example, is $s = 1$ **allowed?**
For $d = 2$ is Sobolev critical, but for $d = 3$ is much below the threshold of continuity.

Theorem: the localization dichotomy

Theorem: Let $\mathcal{P} = \{P(k)\}_{k \in \mathbb{R}^d}$, for $d \leq 3$, be a family of orthogonal projections in $\mathcal{H}$ satisfying Assumption 1. Then the following statements are equivalent:

(i) **Finite second moment:** There exists a choice of Bloch gauge such that the corresponding CWFs satisfy

$$\langle X^2 \rangle_{w_a} = \int_{\mathbb{R}^d} |x|^2 |w_a(x)|^2 \mathrm{d}x < +\infty;$$

(ii) **Exponential localization:** there exists a choice of Bloch gauge such that the corresponding CWFs satisfy

$$\int_{\mathbb{R}^d} \mathrm{e}^{2\beta|x|} |\tilde{w}_a(x)|^2 \mathrm{d}x < +\infty \qquad \text{for every } \beta < \alpha;$$

(iii) **Trivial Bloch bundle:** the Bloch bundle associated to $\mathcal{P}$ is trivial.

Moreover, it is always possible to construct CWFs such that the **localization estimate holds true for every $s < 1$.** $\diamond$

Monaco, D.; Panati, G; Pisante, A.; Teufel, S. : Optimal decay of Wannier functions in Chern and Quantum Hall insulators, to appear soon in arXive.org.

III. Ideas and methods of the proof

Pars construens: parallel transport

For the sake of simplicity, consider $d = 2$.

- $\diamond$ Construction of a continuous periodic Bloch frame Φ on the 1-skeleton.

- $\diamond$ From the 1-skeleton, **extend to the interior of the square by parallel transport**. This method provides explicit estimates about the behavious of $k \mapsto \phi_a(k)$ and $k \mapsto \nabla_k \phi_a(k)$ **as k approaches the singular point** (say $k = 0$ for convenience).

- $\diamond$ Get the **explicit rate of divergence**, namely
$$|\nabla_k \phi_a(k)| \leq \frac{C}{|k|} \text{ as } k \to 0.$$

- $\diamond$ It follows that $\Phi \in W^{1,p}(\mathbb{T}^d, \mathcal{H}^m)$ for every $p < 2$, hence is in $W^{s,2}$ for every $s < 1$.

Denoting by $\{F^s_{p,q}\}$ the scale of Triebel-Lizorkin spaces, one has that $W^{1,p} = F^1_{p,p} \subseteq F^1_{p,\infty}$ is continuously embedded in $F^s_{2,2} = W^{s,2} = H^s$ for $s = 1 - d(1/p - 1/2)$,

Pars destruens: approximation of Sobolev maps

Theorem: Assume $d \leq 3$. Let $\mathcal{P} = \{P(k)\}_{k \in \mathbb{T}^d}$ be a family of orthogonal projectors satisfying Assumption 1, with finite rank $m \in \mathbb{N}^{\times}$. Suppose that there exists a global periodic Bloch frame Φ for $\mathcal{P}$ in $H^1(\mathbb{T}^d, \mathcal{H}^m)$. Then

(i) **triviality of the Bloch bundle:** for any choice of $i, j \in \{1, \ldots, d\}$ one has

$$c_1(P)_{ij} = 0$$

and, as a consequence, the Bloch bundle associated to $\mathcal{P}$ is trivial;

(ii) **approximation with smooth Bloch frames:** there exists a sequence of global *real-analytic* periodic Bloch frames $\left\{\Psi^{(n)}\right\}_{n \in \mathbb{N}}$ for $\mathcal{P}$, such that $\Psi^{(n)} \longrightarrow \Phi$ in $H^1(\mathbb{T}^d, \mathcal{H}^m)$ as $n \longrightarrow \infty$.

$\diamond$

For the sake of simplicity, consider $d = 2$.

- ◇ Lemma: reduction to a **finite-dimensional Hilbert space** $V \subset \mathcal{H}$.

- ◇ We identify V with $\mathbb{C}^n$ and consider

$$G_m(\mathbb{C}^n) \simeq \left\{ P \in M_n(\mathbb{C}) : P^2 = P = P^*, \operatorname{tr} P = m \right\}$$

$$W_m(\mathbb{C}^n) \simeq \left\{ A \in M_n(\mathbb{C}) : A^* A = \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

- ◇ There is a **natural analytic map** $\pi \colon W_m(\mathbb{C}^n) \to G_m(\mathbb{C}^n)$ sending each m-frame $\Psi = (\psi_1, \ldots, \psi_m)$ into the orthogonal projection on its linear span, namely

$$\Psi \mapsto P_\Psi = \sum_{a=1}^{m} |\psi_a\rangle \langle \psi_a| \,.$$

- ◇ We have a **commutative diagram**

$$\begin{tikzcd} & W_m(\mathbb{C}^n) \arrow[d, "\pi"] \\ \mathbb{T}^d \arrow[ur, "\widetilde{\Phi}"] \arrow[r, "\widetilde{P}"] & G_m(\mathbb{C}^n) \end{tikzcd}$$

  where we consider $\widetilde{P} \in C^\omega(\mathbb{T}^d; G_m(\mathbb{C}^n))$ and $\widetilde{\Phi} \in H^1(\mathbb{T}^d; W_m(\mathbb{C}^n))$.

Key lemma: approximation of Sobolev maps

Lemma: Let $2 \leq d \leq 3$. Consider a compact, boundaryless, smooth submanifold $M \subset \mathbb{R}^\nu$. If $d = 3$, assume moreover that the homotopy group $\pi_2(M)$ is trivial. Then, every Sobolev map $\Psi \in H^1(\mathbb{T}^d, M)$ can be approximated by a sequence

$$\{\Psi^{(\ell)}\}_{\ell \in \mathbb{N}} \subset C^\infty(\mathbb{T}^d, M) \text{ such that } \Psi^{(\ell)} \xrightarrow{H^1} \Psi \text{ as } \ell \to \infty.$$

If, in addition, M is an analytic submanifold, then the approximating sequence can be chosen in $C^\omega(\mathbb{T}^d, M)$. $\diamond$

Reconsider the diagram

$$\begin{tikzcd} & W_m(\mathbb{C}^n) \arrow[d, "\pi"] \\ \mathbb{T}^d \arrow[ur, "\widetilde{\Phi}"] \arrow[r, "\widetilde{P}"] & G_m(\mathbb{C}^n) \end{tikzcd}$$

Define $\widetilde{P}^{(\ell)}(k) := \pi \circ \widetilde{\Phi}^{(\ell)}(k)$. Clearly $\widetilde{P}^{(\ell)} \in C^\infty(\mathbb{T}^d; G_m(\mathbb{C}^n))$, hence the family of projectors $\{\widetilde{P}^{(\ell)}(k)\}_{k \in \mathbb{T}^d}$ defines a smooth rank-m vector bundle over $\mathbb{T}^d$ with smooth trivializing frame $\widetilde{\Phi}^{(\ell)}$.

Hang, F.; Lin, F.H. : Topology of Sobolev mappings. II. *Acta Math.* **191**, 55–107 (2003).

Therefore its first Chern class vanishes.

For $d = 2$, this is equivalent to

$$c_1(\widetilde{P}^{(\ell)}) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^2} \mathrm{Tr}_{\mathbb{C}^n} \left(\widetilde{P}^{(\ell)}(k) \left[\partial_1 \widetilde{P}^{(\ell)}(k), \partial_2 \widetilde{P}^{(\ell)}(k) \right] \right) \, \mathrm{d}k_1 \wedge \mathrm{d}k_2 = 0.$$

Since $\widetilde{\Phi}^{(\ell)} \to \widetilde{\Phi}$ in H^1 as $\ell \to \infty$, we obtain $\widetilde{P}^{(\ell)} \to \widetilde{P}$ in $H^1(\mathbb{T}^2; G_m(\mathbb{C}^n))$.

One concludes the argument by passing the limit inside the integral.

Overview: symmetry, localization, transport, topology

Time-reversal symmetry	$\overset{\text{Symmetry}}{\longleftrightarrow}$	Broken TR-symmetry
$\Downarrow$		
Trivial Bloch bundle	$\overset{\text{Topology}}{\longleftrightarrow}$	Generically non trivial Bloch bundle
$\Uparrow$ $\Downarrow$		$\Uparrow$ $\Downarrow$
Exponentially localized Wannier functions $\exists \beta > 0 : \mathrm{e}^{\beta\|x\|} w \in L^2(\mathbb{R}^d)$	$\overset{\text{Localization}}{\longleftrightarrow}$	Generically delocalized Wannier functions $\langle w, \|x\|^2 w \rangle = +\infty$
$\downarrow$		$\downarrow$
Vanishing Hall current	$\overset{\text{Transport}}{\longleftrightarrow}$	Generically non-zero Hall current

Details, proofs & applications in the preprint:

Optimal decay of Wannier functions in Chern and Quantum Hall insulators

To appear in a few weeks on *arXive.org*

Thank you for your attention!!