One-dimensional mean-field games with generic nonlinearity

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Outline

Outline

MFG models

- Introduced in 2006/07 by J. M. Lasry and P. L. Lions in the Mathematics community and P. Caines et. al. in Engineering community.
- Statistical physics: modeling of systems with a very large number of particles.
- Game theory: Nash equilibrium with a very large number of players.
- Economics: population dynamics according to their preferences.

Outline

Stationary MFGs

Given $H : \mathbb{T}^d \times \mathbb{R}^d \times X \to \mathbb{R}$ and $\sigma \geq 0$ find $u : \mathbb{T}^d \to \mathbb{R}$, $m \in \mathcal{P}(\mathbb{T}^d)$ and $\overline{H} \in \mathbb{R}$ such that the triplet $(u, m, \overline{H})$ solves the system

$$\begin{cases} -\sigma\Delta u + H(x, Du, m) = \overline{H}, \\ -\sigma\Delta m - \operatorname{div}(mD_pH(x, Du, m)) = 0. \end{cases} \tag{1}$$

- H - Hamiltonian of the system. Models cost function and interaction. Dependence in m is often called non-linearity.
- σ - diffusion parameter, $\sigma > 0$ stochastic MFGs, $\sigma = 0$ deterministic MFGs.
- u - value function.
- m - distribution of the agents.
- $\overline{H}$ - effective Hamiltonian.
- $E = \mathbb{R}$ (local interaction) or $E =$ functional space (global interaction).

Non-stationary MFGs

Given $H : \mathbb{T}^d \times \mathbb{R}^d \times E \to \mathbb{R}$, $u_T : \mathbb{T}^d \to \mathbb{R}$, $m_0 \in \mathcal{P}(\mathbb{T}^d)$ and $\sigma \geq 0$ find $u : \mathbb{T}^d \times [0, T] \to \mathbb{R}$, $m : \mathbb{T}^d \times [0, T] \to \mathbb{R}^+$ such that the pair (u, m) solves the system

$$\begin{cases} -u_t - \sigma \Delta u + H(x, Du, m) = 0, \\ m_t - \sigma \Delta m - \operatorname{div}(m D_p H(x, Du, m)) = 0, \\ \int\limits_{\mathbb{T}^d} m(x,t)dx = 1, \text{ for all } t \in [0, T], \\ m(x, 0) = m_0(x), \ u(x, T) = u_T(x), \ x \in \mathbb{T}^d. \end{cases} \tag{2}$$

- u_T - terminal cost function.
- m_0 - initial distribution of agents.

Mathematical structure of MFGs

- Hamilton-Jacobi (HJ) equation for u.
- Fokker-Planck (FP) equation for m.
- FP equation is the adjoint of the linearized Hamilton-Jacobi equation.

Interpretation of the structure

- HJ equation: individual agent aims to minimize the action

$$u(x,t) = E_{xt} \int_t^T L(x, \dot{x}, m) ds + u_T(x),$$

where L is the Lagrangian given by the Legendre transform

$$L(x, v, m) = \sup_p \left(-v \cdot p - H(x, p, m) \right),$$

so u solves corresponding HJ equation as a value function.

- FP equation: optimal drift of an agent is given by

$$\dot{x}^* = -D_p H(x, Du, m),$$

so the distribution evolves according to corresponding FP equation.

Outline

Standard assumptions on Hamiltonian

Assumptions on Hamiltonian

- $H(x, p, m)$ is convex and coercive in p.
- $H(x, p, m)$ is "non-increasing" in m.
- Additional technical assumptions.

Dual assumptions on Lagrangian

- $L(x, v, m)$ is convex and coercive in p.
- $L(x, v, m)$ is "non-decreasing" in m.
- Additional technical assumptions.

Interpretation of standard assumptions and consequences

Interpretation

- Convexity is essential in minimization problems. It guarantees existence, uniqueness and regularity of minimizers.
- $L(x, v, m)$ "non-decreasing" in m means that agents prefer sparsely populated areas.

Consequences

- Existence and uniqueness of solutions.
- Sparse areas attract agents, so $m > 0$.
- Construction of weak solutions via gradient type flow (D. Gomes, R. Ferreira).

Outline

Generic non-linearity

In general, we are interested in systems of the form

$$\begin{cases} -\sigma\Delta u + H(x, Du) = g(m) + \overline{H}, \\ -\sigma\Delta m - \operatorname{div}(mD_pH(x, Du)) = 0, \end{cases} \tag{3}$$

and

$$\begin{cases} -u_t - \sigma\Delta u + H(x, Du) = g(m), \\ m_t - \sigma\Delta m - \operatorname{div}(mD_pH(x, Du)) = 0, \end{cases} \tag{4}$$

where g is not "non-decreasing" as it is usually assumed.

- g non-increasing means that agents prefer densely populated areas.
- g decreasing and then increasing means that agents prefer that are not too dense.

Fundamental difficulty with generic non-linearity

By monotonicity and convexity one has that

$$\int_{\mathbb{T}^d} (g(m_2) - g(m_1))(m_2 - m_1)dx$$
$$+ \int_{\mathbb{T}^d} m_1(H(x, Dv_2) - H(x, Dv_1) - D_pH(x, Dv_1)(v_2 - v_1))dx$$
$$+ \int_{\mathbb{T}^d} m_2(H(x, Dv_1) - H(x, Dv_2) - D_pH(x, Dv_2)(v_1 - v_2))dx \geq 0,$$

for arbitrary (u_i, m_i), $i = 1, 2$.
If g is not "non-decreasing" the above inequality is not valid.

Natural questions that arise

- Do solutions exist in general? Are they unique?
- Are the solutions non-degenerate ($m > 0$) and how smooth are they?
- Is there any general mechanism to construct solutions?

First steps towards the general theory: explicit solutions

Consider 1-dimensional stationary deterministic MFG

$$\begin{cases} \frac{(u_x+p)^2}{2} + V(x) = g(m) + \overline{H}, \\ -(m(u_x+p))_x = 0. \end{cases} \tag{5}$$

Current formulation, $j > 0$

From (5) we have $j = m(u_x + p) = \text{const}$, so for $j \neq 0$ (5) is equivalent to

$$\begin{cases} \frac{j^2}{2m^2} - g(m) = \overline{H} - V(x), \\ m > 0, \ \int\limits_{\mathbb{T}} m dx = 1, \\ \int\limits_{\mathbb{T}} \frac{1}{m} dx = \frac{p}{j}. \end{cases} \tag{6}$$

First steps towards the general theory: explicit solutions

Current formulation, $j = 0$

For $j = 0$, (5) is equivalent to

$$\begin{cases} \frac{(u_x+p)^2}{2} - g(m) = \overline{H} - V(x); \\ m \geq 0, \ \int\limits_{\mathbb{T}} m dx = 1; \\ m(u_x + p) = 0, \ x \in \mathbb{T}. \end{cases} \tag{7}$$

Explicit solutions for $g(m) = m, \ j > 0$

We begin with the standard monotone g as a reference case.

Proposition

For every $j > 0$, (5) has a unique smooth solution, $(u_j, m_j, \overline{H}_j)$, with current j. This solution is given by

$$m_j(x) = F_j^{-1}(\overline{H}_j - V(x)), \quad u_j(x) = \int_0^x \frac{j}{m_j(y)} dy - p_j x,$$

where $p_j = \int_{\mathbb{T}} \frac{j}{m_j(y)} dy, \ F_j(t) = \frac{j^2}{2t^2} - t$, and $\overline{H}_j$ is such that $\int_{\mathbb{T}} m_j(x) dx = 1$.

Explicit solutions for $g(m) = m,\ j = 0$

Proposition

Define $m(x) = (V(x) - \overline{H})^+$, where $\overline{H}$ is such that $\int_{\mathbb{T}} m = 1$.

Furthermore, let

$$u^{\pm}(x) = \pm \int_0^x \sqrt{2(\overline{H} - V(y))^+} dy - px,$$

where $p = \pm \int_{\mathbb{T}} \sqrt{2(\overline{H} - V(y))^+} dy$. Then triplets $(u^{\pm}, m, \overline{H})$ are solution of (5) with current $j = 0$.

Note

m can vanish at some sites. $m > 0$ if and only if $\int_{\mathbb{T}} V(x)dx \leq 1 + \min_{\mathbb{T}} V$, that is V is a small perturbation.

Explicit solutions for $g(m) = m,\ j = 0$

Let $m = (V(x) - \overline{H})^+$ be as before. Let x_0 be such that $V(x_0) < \overline{H}$. Such a point exists if and only if $\int\limits_{\mathbb{T}} V(x)dx - 1 > \min\limits_{\mathbb{T}} V$. Let

$$(u^{x_0}(x))_x = \sqrt{2(\overline{H} - V(x))^+} \cdot \chi_{x<x_0} - \sqrt{2(\overline{H} - V(x))^+} \cdot \chi_{x>x_0} - p^{x_0},$$

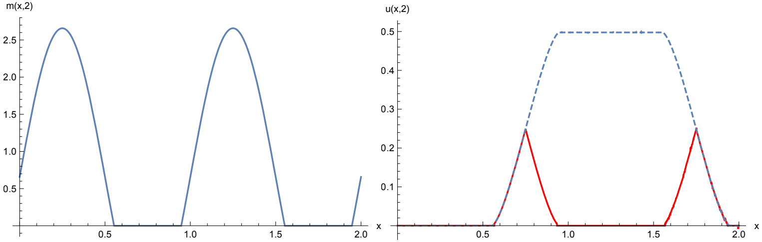
where $p^{x_0} = \int\limits_{y<x_0} \sqrt{2(\overline{H} - V(y))^+} dy - \int\limits_{y>x_0} \sqrt{2(\overline{H} - V(y))^+} dy.$

Then the triplet $(u^{x_0}, m, \overline{H})$ is a solution of (5) with current $j = 0$.

Note

u is no more a C^1 function.

Explicit solutions for $g(m) = m$, $j = 0$



$m(x, 2)$ (left) and two distinct solutions $u(x, 2)$ (right).

Conclusions and interpretation

- If g is increasing, (5) has unique smooth solution for nonzero current.
- If g is increasing, (5) has degenerate solutions ($m = 0$) only with current 0, and only when $\int_{\mathbb{T}} V(x)dx - 1 > \min_{\mathbb{T}} V$.
- If g is increasing, (5) has multiple solutions u only with current 0, and only when $\int_{\mathbb{T}} V(x)dx - 1 > \min_{\mathbb{T}} V$.
- If g is increasing, (5) has singular solutions u only with current 0, and only when $\int_{\mathbb{T}} V(x)dx - 1 > \min_{\mathbb{T}} V$.
- Hence, if g is increasing (5) degenerates in all directions at once!

Explicit solutions for $g(m) = -m$

Current formulation, $j > 0$

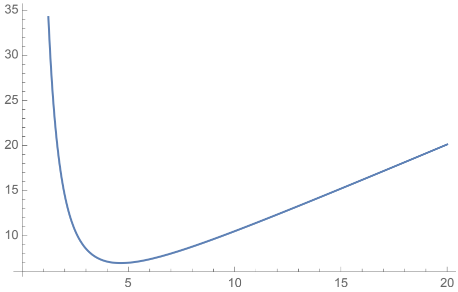
$$\begin{cases} \frac{j^2}{2m^2} + m = \overline{H} - V(x); \\ m > 0, \ \int\limits_{\mathbb{T}} m dx = 1; \\ \int\limits_{\mathbb{T}} \frac{1}{m} dx = \frac{p}{j}. \end{cases} \tag{8}$$

Current formulation, $j = 0$

$$\begin{cases} \frac{(u_x+p)^2}{2} + m = \overline{H} - V(x); \\ m \geq 0, \ \int\limits_{\mathbb{T}} m dx = 1; \\ m(u_x + p) = 0, \ x \in \mathbb{T}. \end{cases} \tag{9}$$

Explicit solutions for $g(m) = -m, \ j > 0$

The minimum of $F_j(t) = t \mapsto j^2/2t^2 + t$ is attained at $t_{min} = j^{2/3}$. Thus, $j^2/2t^2 + t \geq 3j^{2/3}/2$ for $t > 0$. Furthermore, $F_j(t)$ is decreasing on the interval $(0, t_{min})$ and increasing on the interval $(t_{min}, +\infty)$.



$$F_j(t) = \frac{j^2}{2t^2} + t, \ t_{min} = j^{2/3}$$

Explicit solutions for $g(m) = -m, \ j > 0$

Therefore, a lower bound for $\overline{H}$ is

$$\overline{H} \geq \overline{H}_j^{cr} = \max_{\mathbb{T}} V + \frac{3j^{2/3}}{2}, \qquad (10)$$

where the superscript ^{cr} stands for critical. For any $\overline{H}$ satisfying (10), let $m_{\overline{H}}^{-}$ and $m_{\overline{H}}^{+}$ be the solutions of

$$\frac{j^2}{2(m_{\overline{H}}^{\pm}(x))^2} + m_{\overline{H}}^{\pm}(x) = \overline{H} - V(x),$$

with $0 \leq m_{\overline{H}}^{-}(x) \leq t_{min} \leq m_{\overline{H}}^{+}(x)$.

Explicit solutions for $g(m) = -m,\ j > 0$

Let $m_j^- := m^-_{\overline{H}_j^{cr}}$ and $m_j^+ := m^+_{\overline{H}_j^{cr}}$. Note that $m_j^-(x) \leq m_j^+(x)$ for all $x \in \mathbb{T}$, and the equality holds only at the maximum points of V.

The two fundamental quantities for our analysis are

$$\begin{cases} \alpha^+(j) = \int\limits_0^1 m_j^+(x)dx, \\ \alpha^-(j) = \int\limits_0^1 m_j^-(x)dx,\ j > 0. \end{cases} \tag{11}$$

If V is not constant, we have

$$\alpha^-(j) < \alpha^+(j), \quad j > 0.$$

Explicit solutions for $g(m) = -m, \ j > 0$

Suppose that $x = 0$ is the single of maximum of V. Then, for every $j > 0$, there exists a unique number, p_j, such that (5) has a unique solution with a current level j. Moreover, the solution, $(u_j, m_j, \overline{H}_j)$, is given as follows.
If $\alpha^+(j) \leq 1$,

$$m_j(x) = m^{+}_{\overline{H}_j}(x), \quad u_j(x) = \int_0^x \frac{j dy}{m_j(y)} - p_j x, \tag{12}$$

where $p_j = \int_{\mathbb{T}} \frac{j dy}{m_j(y)}$ and $\overline{H}_j$ is such that $\int_{\mathbb{T}} m_j(x) dx = 1$.

Explicit solutions for $g(m) = -m,\ j > 0$

If $\alpha^-(j) \geq 1$,

$$m_j(x) = m^-_{\overline{H}_j}(x), \quad u_j(x) = \int_0^x \frac{j dy}{m_j(y)} - p_j x, \tag{13}$$

where $p_j = \int_{\mathbb{T}} \frac{j dy}{m_j(y)}$ and $\overline{H}_j$ is such that $\int_{\mathbb{T}} m_j(x) dx = 1$.

Explicit solutions for $g(m) = -m,\ j > 0$

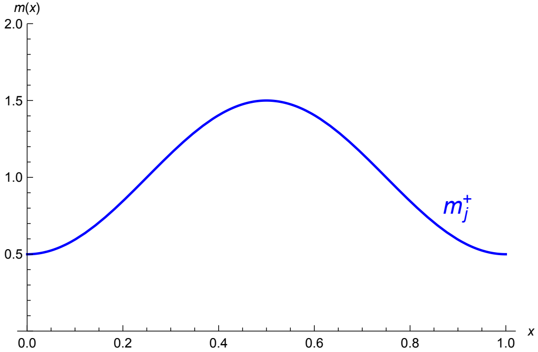
If $\alpha^-(j) < 1 < \alpha^+(j)$, we have that $\overline{H}_j = \overline{H}_j^{cr}$, and

$$m_j(x) = m_j^-(x)\chi_{[0,d_j)} + m_j^+(x)\chi_{[d_j,1)},\ u_j(x) = \int_0^x \frac{jdy}{m_j(y)} - p_j x, \tag{14}$$

where $p_j = \int_{\mathbb{T}} \frac{jdy}{m_j(y)}$ and d_j is such that

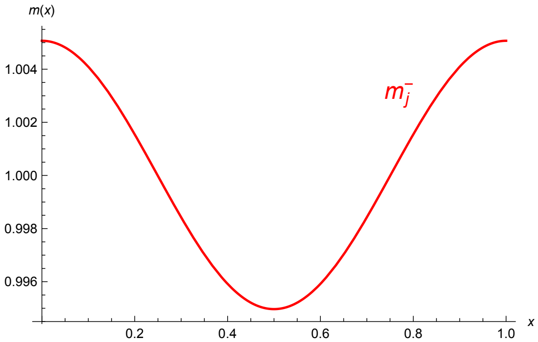
$$\int_0^1 m_j(x)dx = \int_0^{d_j} m_j^-(x)dx + \int_{d_j}^1 m_j^+(x)dx = 1.$$

Explicit solutions for $g(m) = -m,\ j > 0$



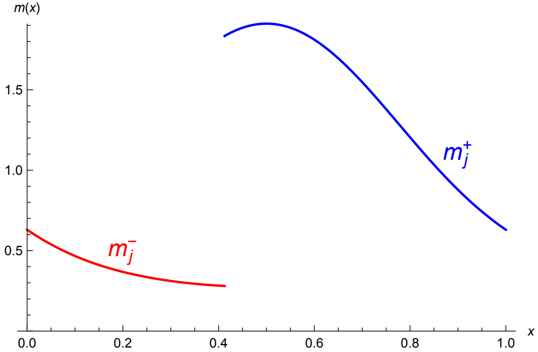
Solution m for $j = 0.001$ and $V(x) = \frac{1}{2}\sin(2\pi(x + 1/4))$.

Explicit solutions for $g(m) = -m,\ j > 0$



Solution m for $j = 10$ and $V(x) = \frac{1}{2}\sin(2\pi(x + 1/4))$.

Explicit solutions for $g(m) = -m,\ j > 0$



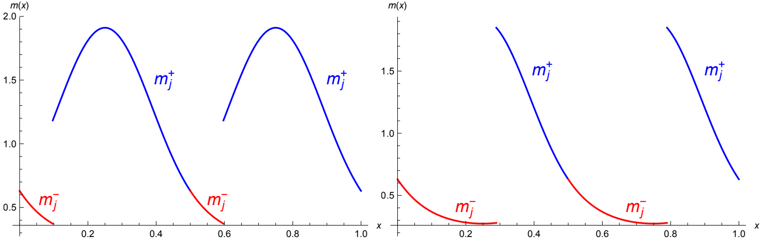
Solution m_j for $j = 0.5$ and $V(x) = \frac{1}{2}\sin(2\pi(x + 1/4))$.

Explicit solutions for $g(m) = -m, \ j > 0$

Non-uniqueness of solutions for V with multiple maxima

Suppose that V attains a maximum at $x = 0$ and at $x = x_0 \in (0, 1)$. Let j be such that $\alpha^-(j) < 1 < \alpha^+(j)$. Then, there exist infinitely many numbers, p, and pairs, (u, m), such that $(u, m, \overline{H}_j^{cr})$ solves (8).

Explicit solutions for $g(m) = -m,\ j > 0$



Two distinct solutions for $j = 0.5$ and $V(x) = \frac{1}{2}\sin(4\pi(x + 1/8))$.

Explicit solutions for $g(m) = -m, \ j = 0$

If $1 + \int_{\mathbb{T}} V \geq \max_{\mathbb{T}} V$, then the triplet $(u_0, m_0, \overline{H}_0)$ with

$$m_0(x) = \overline{H}_0 - V(x), \ u_0(x) = 0, \tag{15}$$

solves (9) in the classical sense for $p_0 = 0$.

Explicit solutions for $g(m) = -m, \ j = 0$

If $\max\limits_{\mathbb{T}} V > 1 + \int\limits_{\mathbb{T}} V$, define

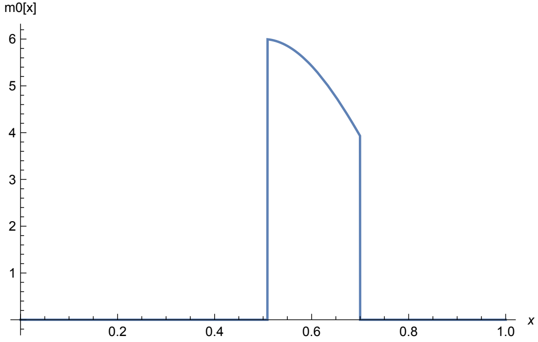
$$m_0^{d_1,d_2}(x) = \begin{cases} \overline{H}_0 - V(x), \ x \in [d_1, d_2], \\ 0, \ x \in \mathbb{T} \setminus [d_1, d_2], \end{cases} \tag{16}$$

and

$$(u_0^{d_1,d_2})_x(x) = \begin{cases} \sqrt{2(\overline{H}_0 - V(x))} - p_0^{d_1,d_2}, \ x \in [0, d_1), \\ -p_0^{d_1,d_2}, \ x \in [d_1, d_2], \\ -\sqrt{2(\overline{H}_0 - V(x))} - p_0^{d_1,d_2}, \ x \in (d_2, 1], \end{cases}$$

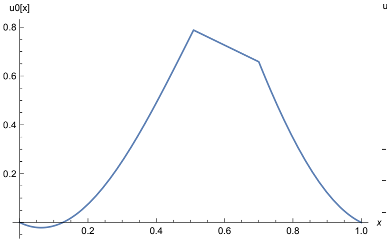
where $p_0^{d_1,d_2}$ and (d_1, d_2) are such that u is periodic and m is probability. Then the triplet $(u_0^{d_1,d_2}, m_0^{d_1,d_2}, \overline{H}_0)$ solves (9)

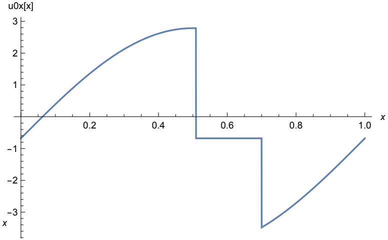
Explicit solutions for $g(m) = -m,\ j = 0$



m_0 as defined in (16) for $V(x) = 3\cos(2\pi x)$ with $d_2 = 0.7$ and d_1 such that m_0 is a probability measure.

Explicit solutions for $g(m) = -m$, $j = 0$





u_0 (left) and $(u_0)_x$ (right)as defined in (16) for $V(x) = 3\cos(2\pi x)$ with $d_2 = 0.7$ and d_1 such that m_0 is a probability measure.

"Unhappiness traps"

- Our solutions suggest that when $g(m) = -m$ agents prefer to stick together, rather than be at better place.
- It is not the case for $g(m) = m$.
- Results are coherent with the intuition that g models the crowd preference of the agents.

Regularity regimes

Next, we define

$$j_{lower} = \inf\{j > 0 \text{ s.t. } \alpha^{+}(j) > 1\}, \tag{17}$$

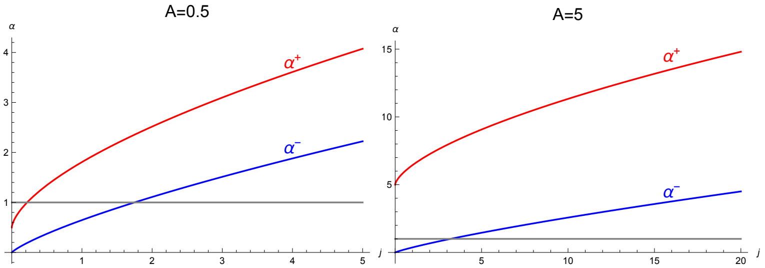
and

$$j_{upper} = \inf\{j > 0 \text{ s.t. } \alpha^{-}(j) > 1\}. \tag{18}$$

These two numbers characterize the regularity regimes of (8). We have

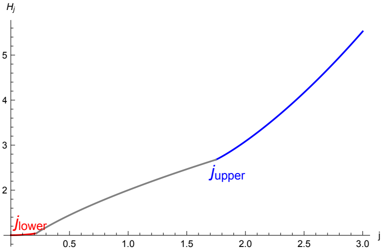
i. $0 \leq j_{lower} < j_{upper} < \infty$;
ii. for $j \geq j_{upper}$ the system (5) has smooth solutions;
iii. for $j_{lower} < j < j_{upper}$ the system (5) has only discontinuous solutions;
iv. if $j_{lower} > 0$, the system (5) has smooth solutions for $0 < j \leq j_{lower}$.

Regularity regimes



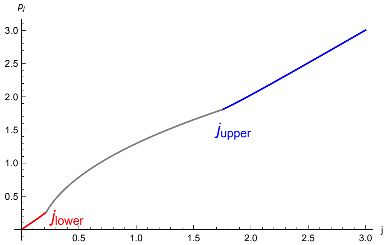
α^+ and α^- for $V(x) = A\sin(2\pi(x+1/4))$. $j_{lower} = 0.218$, $j_{upper} = 1.750$ $(A = 0.5)$; $j_{lower} = 0$, $j_{upper} = 3.203$ $(A = 5)$.

$\overline{H}_j$ for $g(m) = -m$



$\bar{H}_j$ for $V(x) = \frac{1}{2}\sin(2\pi(x + \frac{1}{4}))$.

p_j for $g(m) = -m$



p_j for $V(x) = \frac{1}{2}\sin(2\pi(x + \frac{1}{4}))$.

Outline

Conclusion and further extensions

- Qualitative properties of the solutions are dramatically different.
- Agents prefer densely populated areas even if they are not happy with these areas on the individual level.
- What happens in the time dependent case?
- What happens in the stochastic case?
- What about higher dimensions?

Thank you!