Stochastic and Analytic Methods in Mathematical Physics

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On the justification of Gibbs formula

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Background

In the theory of random processes different classes of processes are usually characterized by some properties of their finite-dimensional or conditional distributions. However for the application it is important to have some representation theorem expressing processes in terms of simple and convenient objects, such as transition matrices for Markov chains, characteristic functions for processes with independent increments, spectral functions for stationary processes, and so on. Due to representation theorems one can construct various models of random processes.

The situation is quite different for the class of Gibbs random fields. Historically, instead of being characterized by some properties of their finite-dimensional or conditional distributions, Gibbs random fields have been defined directly by the well-known representation of their conditional distributions in terms of potentials.

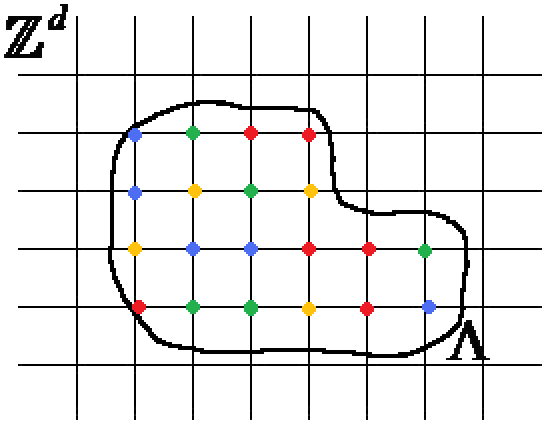
So in order for the theory of Gibbs random fields to be reconsidered as a theory of full value it is necessary to have a general pure probabilistic definition of a Gibbs random field (without notion of potential) and corresponding representation theorem.

Goals of the talk

- to give a mathematical (without notion of potential) definition of potential energy;
  • on the base of this definition to give some justification of the Gibbs formula;

- to give a pure probabilistic (without notion of potential) definition of Gibbs random field;
  • on the base of this definition to introduce elements of the general theory of Gibbs random field.

Gibbs formula



Let $\mathbb{Z}^d$ be an integer lattice, $d \geq 1$;
$W = \{J \subset \mathbb{Z}^d : |J| < \infty\}$;
X be a finite set (spin space), $X \subset \mathbb{R}$;
$X^\Lambda = \{(x_t, t \in \Lambda)\}$, $x_t \in X$, be a set of configurations on $\Lambda \in W$.

The potential energy H and the probability P are functions on X^Λ. Since Boltzmann it is assumed that the connection between the potential energy and the probability is logarithmical

$$H \sim \ln P.$$

The probability P is determined by its properties

$$P(x) \geq 0, \quad x \in X^\Lambda \qquad \text{and} \qquad \sum_{x \in X^\Lambda} P(x) = 1.$$

The potential energy $H = H(\Phi)$ is defined through the interaction potential Φ, and its properties are induced by properties of potential.

$$P_\Lambda(x) = \frac{\exp\{H_\Lambda(x)\}}{\sum\limits_{z\in X^\Lambda} \exp\{H_\Lambda(z)\}},$$

where the Hamiltonian H_Λ is defined by the following way

$$H_\Lambda(x) = H_\Lambda^\Phi(x) = \sum_{J\subset\Lambda} \Phi_J(x_J), \qquad x \in X^\Lambda.$$

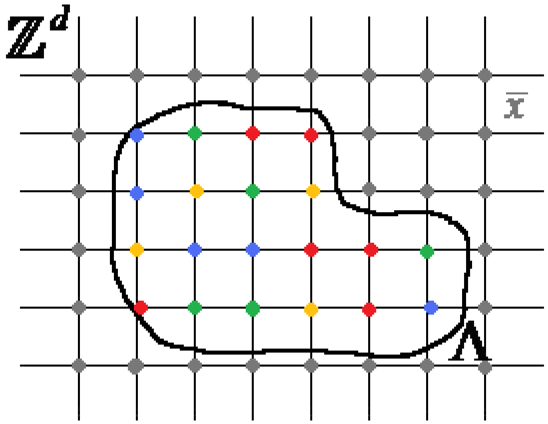
Φ — interaction potential.

Gibbs formula explicitly includes the Hamiltonian, i.e. potential energy, which is defined up to a constant, and therefore its calculation is not possible. The following questions arise immediately.

1. *How the calculations of probabilities by Gibbs formula are performed?*
2. *Whence it follows that the potential energy is the sum of local interactions?*

Physical systems in the infinite volumes (Gibbs random fields) Dobrushin–Lanford–Ruelle theory

A random field is a probabilistic measure on $(X^{\mathbb{Z}^d}, \Im^{\mathbb{Z}^d})$, where $\Im^{\mathbb{Z}^d}$ is σ-algebra generated by cylindric subsets of $X^{\mathbb{Z}^d}$.



R.Dobrushin introduced the Hamiltonian $H_\Lambda^\Phi(\cdot/\bar{x})$ with boundary conditions $\bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}$

$$H_\Lambda^\Phi(x/\bar{x}) = H_\Lambda^\Phi(x) + \sum_{\emptyset \neq J \subset \Lambda} \sum_{\emptyset \neq \tilde{J} \subset \mathbb{Z}^d \setminus \Lambda} \Phi(x_J \bar{x}_{\tilde{J}}),$$

and a system of distributions $\mathcal{Q} = \left\{ q_\Lambda^{\bar{x}}, \Lambda \in W, \bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda} \right\}$, where

$$q_\Lambda^{\bar{x}}(x) = \frac{\exp\{H_\Lambda^\Phi(x/\bar{x})\}}{\sum_{z \in X^\Lambda} \exp\{H_\Lambda^\Phi(z/\bar{x})\}}, \qquad x \in X^\Lambda,$$

which he called a *Gibbs specification corresponding to the potential* Φ.

The fundamental property of the Gibbs specification is the following

$$\frac{q^{\bar{x}}_{\Lambda\cup I}(xy)}{\left(q^{\bar{x}}_{\Lambda\cup I}\right)_I(y)} = q^{\bar{x}y}_{\Lambda}(x),$$

$x \in X^{\Lambda}$, $y \in X^{I}$, $\Lambda, I \in W$, $\bar{x} \in X^{\mathbb{Z}^d\setminus\Lambda\setminus I}$, which specifies the connection between its elements (*consistency condition*).

The equivalent form of consistency condition is: for all $\Lambda, I \in W$, $x \in X^{\Lambda}$, $y, v \in X^{I}$ and $\bar{x} \in X^{\mathbb{Z}^d\setminus\Lambda\setminus I}$

$$\frac{q^{\bar{x}}_{\Lambda\cup I}(xy)}{q^{\bar{x}}_{\Lambda\cup I}(xv)} = \frac{q^{\bar{x}x}_{I}(y)}{q^{\bar{x}x}_{I}(v)}.$$

Specification

Everywhere defined system of probability distributions parameterized by boundary conditions $Q = \left\{ q_\Lambda^{\bar{x}}, \Lambda \in W, \bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda} \right\}$, which elements satisfy Dobrushin's consistency condition: for all $\Lambda, I \in W$, $x \in X^\Lambda$, $y \in X^I$ и $\bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda \setminus I}$

$$\frac{q_{\Lambda \cup I}^{\bar{x}}(xy)}{q_{\Lambda \cup I}^{\bar{x}}(xv)} = \frac{q_I^{\bar{x}x}(y)}{q_I^{\bar{x}x}(v)}.$$

is called *specification*.

Dobrushin considered the specification as a possible version of conditional distribution of a random field.

If there exists a random field P for which given specification Q is a version of its conditional distribution, then we call P compatible with specification Q.

Random field compatible with Gibbs specification $\mathcal{Q}$ constructed by potential Φ is called ***Gibbs random field with potential*** Φ.

Dobrushin's Theorems

Theorem (**Existence**). *Let specification Q be quasilocal, i.e. for any $x \in X^{\Lambda}$*

$$\sup_{\bar{x},\tilde{x}\in X^{\mathbb{Z}^d\setminus\Lambda}:\bar{x}_V=\tilde{x}_V} \left| q_{\Lambda}^{\bar{x}}(x) - q_{\Lambda}^{\tilde{x}}(x) \right| \xrightarrow[V\uparrow\mathbb{Z}^d\setminus\Lambda]{} 0.$$

Then there exists a random field P *conditional distribution of which coincides with Q almost everywhere.*

For any pair of points $s, t \in \mathbb{Z}^d$, $s \neq t$, denote

$$\rho_{s,t} = \sup \frac{1}{2} \sum_{x\in X} \left| q_t^{\bar{x}}(x) - q_t^{\bar{y}}(x) \right|.$$

where supremum is taken over all configurations $\bar{x}, \bar{y} \in X^{\mathbb{Z}^d\setminus\{t\}}$ which are coincide on $\mathbb{Z}^d\setminus\{t, s\}$.

Theorem (**Uniqueness**). *Let Q be a quasilocal specification such that for any $t \in \mathbb{Z}^d$*

$$\sum_{s\in\mathbb{Z}^d\setminus\{t\}} \rho_{s,t} \leq \alpha < 1$$

Then the random field compatible with specification Q is unique.

Dobrushin's problem

Let $Q^{(1)} = \left\{ q_t^{\bar{x}}, \bar{x} \in X^{\mathbb{Z}^d \setminus t}, t \in \mathbb{Z}^d \right\}$ be the system of one-point probability distributions indexed by infinite boundary conditions.

1. Under which consistency conditions the system $Q^{(1)}$ will be the subsystem of some specification Q?

2. If for a consistent system $Q^{(1)}$ the important properties (for example, quasilocality, positivity, homogeneity and etc.) are valid, is it true that the same properties are valid for whole specification too?

3. Let $\mathscr{P}_1$ be the set of random fields which is compatible with specification Q and let $\mathscr{P}_2$ be the set of random fields which is compatible with $Q^{(1)}$. Is the equality $\mathscr{P}_1 = \mathscr{P}_2$ true?

The solution of Dobrushin's problem

The system of one–point probability distributions parameterized by boundary conditions

$$Q^{(1)} = \left\{ q_t^{\bar{x}}, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}} \right\},$$

which elements satisfy consistency condition: for all $t, s \in \mathbb{Z}^d$, $x, u \in X^{\{t\}}$, $y, v \in X^{\{s\}}$ и $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t,s\}}$

$$q_t^{\bar{x}u}(x) q_s^{\bar{x}x}(v) q_t^{\bar{x}v}(y) q_s^{\bar{x}y}(u) = q_t^{\bar{x}u}(y) q_s^{\bar{x}y}(v) q_t^{\bar{x}v}(x) q_s^{\bar{x}x}(u) \qquad (1)$$

is called ***1–specification***.

$$P(A/B)P(B/C)P(C/D)P(D/A) = P(A/D)P(D/C)P(C/B)P(B/A)$$

Theorem. *The system* $Q^{(1)}$ *will be a subsystem of some specification* Q *if and only if the consistency conditions* (1) *are fulfilled. The corresponding specification* Q *is restored by* $Q^{(1)}$ *uniquely.*

Theorem. *Let* $Q^{(1)}$ *be a positive quasilocal 1–specification. Then there exists a random field* P *one–point conditional distribution of which coincides with* $Q^{(1)}$ *almost everywhere. If* $Q^{(1)}$ *is such that for any* $t \in \mathbb{Z}^d$

$$\sum_{s \in \mathbb{Z}^d \setminus \{t\}} \rho_{s,t} \leq \alpha < 1,$$

then the random field P *is unique.*

The elements of specification Q containing the given 1-specification $Q^{(1)}$ have the following form: for all $\Lambda \in W$ and $\bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}$

$$q_{\Lambda}^{\bar{x}}(x) = \frac{q_{t_1}^{\bar{x}u_{\{t_2,...,t_n\}}}\left(x_{t_1}\right) q_{t_2}^{\bar{x}x_{t_1}u_{\{t_3,...,t_n\}}}\left(x_{t_2}\right) ... q_{t_n}^{\bar{x}x_{\left\{t_1,t_2,...,t_{n-1}\right\}}}\left(x_{t_n}\right)}{q_{t_1}^{\bar{x}u_{\{t_2,...,t_n\}}}\left(u_{t_1}\right) q_{t_2}^{\bar{x}x_{t_1}u_{\{t_3,...,t_n\}}}\left(u_{t_2}\right) ... q_{t_n}^{\bar{x}x_{\left\{t_1,t_2,...,t_{n-1}\right\}}}\left(u_{t_n}\right)} \times C,$$

where C is the normalizing factor, $u \in X^{\Lambda}$ some fixed configuration and $\Lambda = \{t_1, t_2, ..., t_n\}$.

New approach to the theory of Gibbs random fields

Transition Energy
(finite volume)

Let finite volume $\Lambda \subset \mathbb{Z}^d$ be fixed.

Denote by $\Delta_\Lambda(x, y)$ the energy which is necessary to change the state of the system from x to y (*transition energy*), $x, y \in X^\Lambda$.

The function Δ_Λ must satisfy the following relation

$$\Delta_\Lambda(x, y) = \Delta_\Lambda(x, z) + \Delta_\Lambda(z, y)$$

for all $x, y, z \in X^\Lambda$.

It is well known that each function satisfying the condition above has the following form

$$\Delta_\Lambda(x, y) = H_\Lambda(y) - H_\Lambda(x),$$

where H_Λ is some function defined on X^Λ, which is naturally interpreted as a potential energy.

Let P_Λ be a probability distribution on X^Λ. Put

$$\Delta_\Lambda(x,y) = \ln \frac{P_\Lambda(x)}{P_\Lambda(y)}, \qquad x, y \in X^\Lambda.$$

It is clear that the function $\Delta_\Lambda(x,y)$ is a transition energy.

Then

$$P_\Lambda(x) = \frac{\exp\{\Delta_\Lambda(x,y)\}}{\sum\limits_{z\in X^\Lambda} \exp\{\Delta_\Lambda(z,y)\}} = \frac{\exp\{H_\Lambda(x)\}}{\sum\limits_{z\in X^\Lambda} \exp\{H_\Lambda(z)\}}$$

for all $x \in X^\Lambda$.

Transition Energy
(infinite volume)

Let T be a set of all configurations which differ on the finite set of points of $\mathbb{Z}^d$

$$T = \Big\{(x,y) : x,y \in X^{\mathbb{Z}^d} \text{ and } \Big|\{t \in \mathbb{Z}^d : x_t \neq y_t\}\Big| < \infty\Big\}.$$

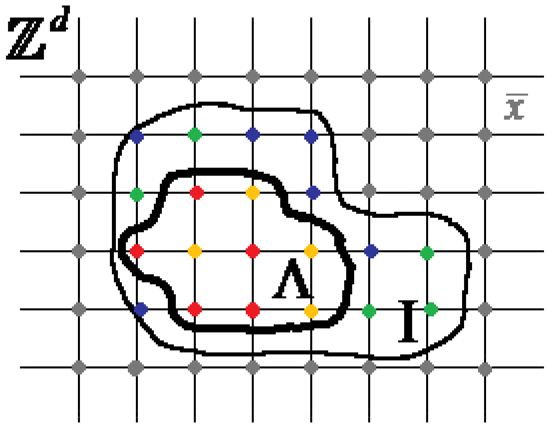
Let function Δ on T satisfy the following condition

$$\Delta(x,y) = \Delta(x,z) + \Delta(z,y), \quad (x,y),(x,z),(z,y) \in T.$$

For any $\Lambda \in W$ and $\bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}$ put

$$\Delta_\Lambda^{\bar{x}}(x,y) = \Delta(x\bar{x}, y\bar{x}).$$

Elements of the set of functions $\left\{\Delta_\Lambda^{\bar{x}}, \Lambda \in W, \bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}\right\}$, constructed by Δ, obviously satisfy the following relations



$$\Delta_\Lambda^{\bar{x}}(x,y) = \Delta_\Lambda^{\bar{x}}(x,z) + \Delta_\Lambda^{\bar{x}}(z,y), \; x,y,z \in X^\Lambda,$$

$$\Delta_{\Lambda \cup I}^{\bar{x}}(xu,yu) = \Delta_\Lambda^{\bar{x}u}(x,y), \qquad u \in X^I.$$

The opposite is also true:
If the set $\left\{\Delta_\Lambda^{\bar{x}}, \Lambda \in W, \bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}\right\}$ satisfies the conditions mentioned above, then there exists a function Δ on T satisfying

$$\Delta(x,y) = \Delta(x,z) + \Delta(z,y) \quad \text{and} \quad \Delta(x\bar{x}, y\bar{x}) = \Delta_\Lambda^{\bar{x}}(x,y).$$

The function $\Delta_\Lambda^{\bar{x}}$ satisfying the following conditions:

(1) for all $x, y, z \in X^\Lambda$, $\bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}$, $\Lambda \in W$

$$\Delta_\Lambda^{\bar{x}}(x, y) = \Delta_\Lambda^{\bar{x}}(x, z) + \Delta_\Lambda^{\bar{x}}(z, y);$$

(2) for all $x, y \in X^\Lambda$, $u \in X^I$, $\bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}$, $\Lambda, I \in W$, $\Lambda \cap I = \varnothing$

$$\Delta_{\Lambda \cup I}^{\bar{x}}(xu, yu) = \Delta_\Lambda^{\bar{x}u}(x, y),$$

is called a *transition energy at the finite volume Λ from the state x to the state y.*

Let $Q = \left\{ q_{\Lambda}^{\bar{x}}, \Lambda \in W, \bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda} \right\}$ be a system of positive probability distributions parameterized by boundary conditions. Then

$$\Delta_{\Lambda}^{\bar{x}}(x, y) = \ln \frac{q_{\Lambda}^{\bar{x}}(x)}{q_{\Lambda}^{\bar{x}}(y)}, \qquad x, y \in X^{\Lambda},$$

is a transition energy.

Indeed, the Condition (1) holds since for all $x, y, z \in X^{\Lambda}$

$$\ln \frac{q_{\Lambda}^{\bar{x}}(x)}{q_{\Lambda}^{\bar{x}}(y)} = \ln \frac{q_{\Lambda}^{\bar{x}}(x) q_{\Lambda}^{\bar{x}}(z)}{q_{\Lambda}^{\bar{x}}(z) q_{\Lambda}^{\bar{x}}(y)} = \ln \frac{q_{\Lambda}^{\bar{x}}(x)}{q_{\Lambda}^{\bar{x}}(z)} + \ln \frac{q_{\Lambda}^{\bar{x}}(z)}{q_{\Lambda}^{\bar{x}}(y)},$$

and for the fulfilment of the Condition (2) it is necessary that for for any $u \in X^{I}$

$$\ln \frac{q_{\Lambda \cup I}^{\bar{x}}(xu)}{q_{\Lambda \cup I}^{\bar{x}}(yu)} = \Delta_{\Lambda \cup I}^{\bar{x}}(xu, yu) = \Delta_{\Lambda}^{\bar{x}u}(x, y) = \ln \frac{q_{\Lambda}^{\bar{x}u}(x)}{q_{\Lambda}^{\bar{x}u}(y)},$$

or, equivalently,

$$\frac{q_{\Lambda \cup I}^{\bar{x}}(xu)}{q_{\Lambda \cup I}^{\bar{x}}(yu)} = \frac{q_{\Lambda}^{\bar{x}u}(x)}{q_{\Lambda}^{\bar{x}u}(y)} \text{ (Dobrushin's consistency condition).}$$

Any specification can be represented in a Gibbsian form.

Theorem. *Let* $Q = \left\{q_\Lambda^{\bar{x}}, \Lambda \in W, \bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}\right\}$ *be a set of positive probability distributions parameterized by boundary conditions. For* Q *to be a specification it is necessary and sufficient that its elements have the form*

$$q_\Lambda^{\bar{x}}(x) = \frac{\exp\{\Delta_\Lambda^{\bar{x}}(x,y)\}}{\sum\limits_{z \in X^\Lambda} \exp\{\Delta_\Lambda^{\bar{x}}(z,y)\}}, \qquad x, y \in X^\Lambda,$$

where $\Delta_\Lambda^{\bar{x}}(x,y)$ *is the transition energy,* $\bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}$*,* $\Lambda \in W$*.*

It is easy to see that the condition

$$\Delta^{\bar{x}}_{\Lambda \cup I}(xu, yu) = \Delta^{\bar{x}u}_{\Lambda}(x, y)$$

is equivalent to

$$\Delta^{\bar{x}}_{\Lambda \cup I}(xu, yv) = \Delta^{\bar{x}u}_{\Lambda}(x, y) + \Delta^{\bar{x}y}_{I}(u, v).$$

From here it follows that

$$\Delta^{\bar{x}}_{\Lambda}(x, y) = \Delta^{\bar{x}x_{t_2}x_{t_3}\ldots x_{t_n}}_{t_1}(x_{t_1}, y_{t_1}) +$$

$$+ \Delta^{\bar{x}y_{t_1}x_{t_3}\ldots x_{t_n}}_{t_2}(x_{t_2}, y_{t_2}) + \ldots + \Delta^{\bar{x}y_{t_1}y_{t_2}\ldots y_{t_{n-1}}}_{t_n}(x_{t_n}, y_{t_n}),$$

where $\Lambda = \{t_1, t_2, \ldots, t_n\}$.

From here it follows that it is sufficient to consider the transition energy $\Delta^{\bar{x}}_{t}(x, y)$ necessary to change the state of a particle located at the point t from x to y, which satisfies suitable consistency condition.

Transition Energy (for one particle)

The function $\Delta_t^{\bar{x}}(x,y)$ satisfying the following conditions

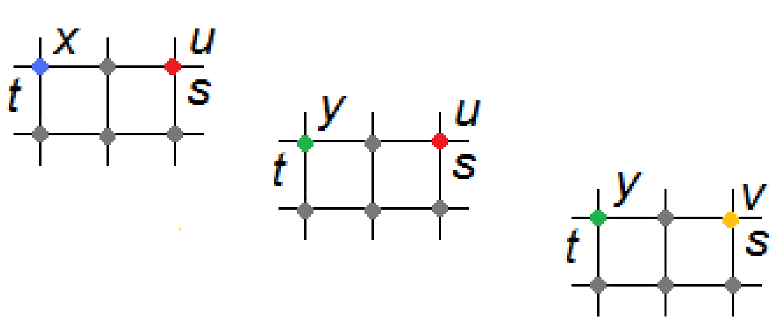
(1) for all $x, y, z \in X$, $t \in \mathbb{Z}^d$ and $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$

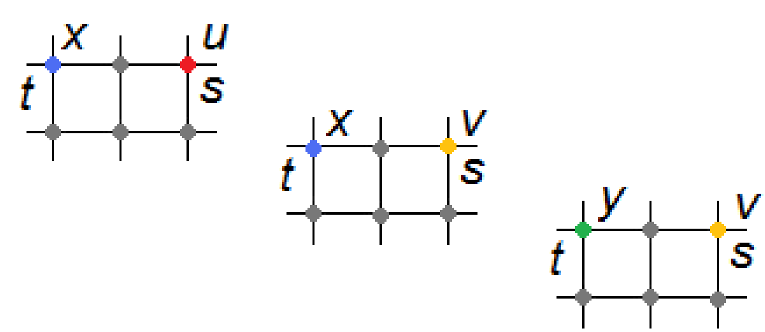
$$\Delta_t^{\bar{x}}(x,y) = \Delta_t^{\bar{x}}(x,z) + \Delta_t^{\bar{x}}(z,y)$$

(2) for all $x, y \in X^{\{t\}}$, $u, v \in X^{\{s\}}$, $t, s \in \mathbb{Z}^d$ and $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t,s\}}$

$$\Delta_t^{\bar{x}u}(x,y) + \Delta_s^{\bar{x}y}(u,v) = \Delta_s^{\bar{x}x}(u,v) + \Delta_t^{\bar{x}v}(x,y)$$

is called a *(one-point) transition energy at the point t from the state x to the state y.*





Let $Q^{(1)} = \left\{q_t^{\bar{x}}, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}\right\}$ be a set of positive one-point probability distributions parameterized by boundary conditions.

Put

$$\Delta_t^{\bar{x}}(x,y) = \ln \frac{q_t^{\bar{x}}(x)}{q_t^{\bar{x}}(y)}, \qquad x, y \in X.$$

Then the condition (1) holds automatically, and for the fulfilment of the condition (2) the set $Q^{(1)}$ must satisfy the following relation

$$\frac{q_t^{\bar{x}u}(x)}{q_t^{\bar{x}u}(y)} \cdot \frac{q_s^{\bar{x}y}(u)}{q_s^{\bar{x}y}(v)} = \frac{q_s^{\bar{x}x}(u)}{q_s^{\bar{x}x}(v)} \cdot \frac{q_t^{\bar{x}v}(x)}{q_t^{\bar{x}v}(y)},$$

which is equivalent to the consistency condition of 1–specification

$$q_t^{\bar{x}u}(x) q_s^{\bar{x}x}(v) q_t^{\bar{x}v}(y) q_s^{\bar{x}y}(u) = q_t^{\bar{x}u}(y) q_s^{\bar{x}y}(v) q_t^{\bar{x}v}(x) q_s^{\bar{x}x}(u)$$

Any 1–specification can be represented in a Gibbsian form.

Theorem. *Let* $Q^{(1)} = \left\{ q_t^{\bar{x}}, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}} \right\}$ *be a set of positive one-point probability distributions parameterized by boundary conditions. For* $Q^{(1)}$ *to be a 1–specification it is necessary and sufficient that its elements have the form*

$$q_t^{\bar{x}}(x) = \frac{\exp\{\Delta_t^{\bar{x}}(x,y)\}}{\sum_{z \in X} \exp\{\Delta_t^{\bar{x}}(z,y)\}}, \qquad x, y \in X,$$

where $\Delta_t^{\bar{x}}(x,y)$ *is the transition energy,* $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$*,* $t \in \mathbb{Z}^d$*.*

Let

$$\Delta_t^{\bar{x}}(x,y) = H_t^{\bar{x}}(x) - H_t^{\bar{x}}(y).$$

Then the Condition (1) holds automatically, and the Condition (2) can be written as follows: for all $t, s \in \mathbb{Z}^d$, $x, y \in X^{\{t\}}$, $u, v \in X^{\{s\}}$ and $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t,s\}}$

$$H_t^{\bar{x}u}(x) - H_t^{\bar{x}u}(y) + H_s^{\bar{x}y}(u) - H_s^{\bar{x}y}(v) =$$

$$= H_s^{\bar{x}x}(u) - H_s^{\bar{x}x}(v) + H_t^{\bar{x}v}(x) - H_t^{\bar{x}v}(y).$$

Potential energy

Potential energy is a set $H = \left\{ H_t^{\bar{x}}, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}, t \in \mathbb{Z}^d \right\}$ of functions such that for any $t, s \in \mathbb{Z}^d$, $x, y \in X^{\{t\}}$, $u, v \in X^{\{s\}}$ and $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t,s\}}$

$$H_t^{\bar{x}u}(x) - H_t^{\bar{x}u}(y) + H_s^{\bar{x}y}(u) - H_s^{\bar{x}y}(v) = \\ = H_s^{\bar{x}x}(u) - H_s^{\bar{x}x}(v) + H_t^{\bar{x}v}(x) - H_t^{\bar{x}v}(y). \tag{2}$$

For each $t \in \mathbb{Z}^d$, $x \in X^{\{t\}}$ and $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$ the function $H_t^{\bar{x}}(x)$ is a potential energy of the system being in state x at point t with boundary conditions $\bar{x}$ outside.

Functions

$$H_t^{\bar{x}}(x) = \sum_{J \subset W(\mathbb{Z}^d \setminus \{t\})} \Phi_{\{t\} \cup J}(x \bar{x}_J)$$

satisfy the Condition (2) and hence define a potential energy.

Theorem. *For $Q^{(1)}$ to be a 1–specification it is necessary and sufficient that its elements have the form*

$$q_t^{\bar{x}}(x) = \frac{\exp\{H_t^{\bar{x}}(x)\}}{\sum\limits_{z \in X} \exp\{H_t^{\bar{x}}(z)\}}, \qquad x \in X,$$

where $H = \left\{ H_t^{\bar{x}}, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}, t \in \mathbb{Z}^d \right\}$ *is a potential energy.*

Theorem. *Specification $Q^{(1)}$ is a Gibbs specification if and only if $H_t^{\bar{x}}$ is quasilocal, i.e. for any $x \in X$*

$$\sup_{\bar{x}, \bar{y} \in X^{\mathbb{Z}^d \setminus \{t\}} : \bar{x}_V = \bar{y}_V} \left| H_t^{\bar{x}}(x) - H_t^{\bar{y}}(x) \right| \xrightarrow[V \uparrow \mathbb{Z}^d \setminus \{t\}]{} 0.$$

Gibbs random fields

A random field $\mathsf{P} = \{\mathsf{P}_\Lambda, \Lambda \in W\}$ is called **Gibbs random field** if

1. for all $\Lambda \in W$ and $x \in X^\Lambda$

$$\mathsf{P}_\Lambda(x) > 0;$$

2. limits

$$q_t^{\bar{x}}(x) = \lim_{\Lambda \uparrow \mathbb{Z}^d \setminus \{t\}} \frac{\mathsf{P}_{\{t\} \cup \Lambda}(x\bar{x}_\Lambda)}{\mathsf{P}_\Lambda(\bar{x}_\Lambda)},$$

$t \in \mathbb{Z}^d$, $x \in X$, $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$, exist, strictly positive, and the convergence is uniform with respect to $\bar{x}$.

If P is a Gibbs random field then the limits $q_t^{\bar{x}}$ define a system of one–point probability distributions

$$\mathcal{Q}^{(1)} = \left\{ q_t^{\bar{x}}, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}} \right\},$$

which is the unique quasilocal 1–specification (**canonical specification**)

Theorem of representation of Gibbs random fields

Theorem. *If* P *is a Gibbs random filed then its canonical 1–specification admits the Gibbs representation with the aid of uniformly convergent potential* Φ*. Conversely, if a random field* P *has a version of conditional distribution admitting the Gibbs representation with the uniformly convergent potential* Φ*, then* P *is a Gibbs random field.*

Indeed, let canonical specification $\mathcal{Q}^{(1)}$ be presented in a Gibbsian form by means of potential energy H. Let $\theta \in X$ be fixed. For any $\Lambda \in W$ and $x \in X^{\Lambda}$ put

$$\Phi_{\Lambda}(x) = (-1)^{|\Lambda|-1} \sum_{J \subset \Lambda \setminus \{t\}} (-1)^{|J|} H_t^{\bar{\theta}_{\mathbb{Z}^d \setminus J \setminus \{t\}} x_J}(x_t),$$

where t is a point in Λ. The value of $\Phi_{\Lambda}(x)$ does not depend on the choice of the point t. Since H is quasilocal the potential $\Phi = \{\Phi_{\Lambda}, \Lambda \in W\}$ is uniformly convergent potential.

The second station of the theorem is a direct consequences from the DLR equation.

• The set $\mathscr{G}$ of all Gibbs random fields is not empty since, as it follows immediately from the above definition, it contains the set $\mathscr{M}$ of all strictly positive Markov random fields.

• Not all strictly positive random fields are Gibbsian.

<u>Example 1.</u> Let $X = \{0{,}1\}$ and consider the random field P with finite dimensional distribution

$$\mathsf{P}_\Lambda(x) = \frac{1}{(|\Lambda| + 1)\ C_{|\Lambda|}^{|x|}}, \qquad x \in X^\Lambda,\ \Lambda \in W,$$

where $|x| = |\{t \in \Lambda : x_t = 1\}|$.

For all $t \in \mathbb{Z}^d$, $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$ and $\Lambda \in W(\mathbb{Z}^d \setminus \{t\})$ we have

$$q_t^{\bar{x}_\Lambda}(1) = \frac{\mathsf{P}_{\{t\} \cup \Lambda}(1\bar{x}_\Lambda)}{\mathsf{P}_\Lambda(\bar{x}_\Lambda)} = \frac{|\bar{x}_\Lambda| + 1}{|\Lambda| + 2}.$$

Example 2. Let $\alpha, p_1, p_2 \in (0,1)$ be such that $p_1 \neq p_2$. Consider a random field P which is the mixture of Bernoulli random fields $\mathbf{B}^{p_1}$ and $\mathbf{B}^{p_2}$ with the coefficients α and $\beta = 1 - \alpha$, that is

$$\mathsf{P}_\Lambda(x) = \alpha\, p_1^{|x|}(1-p_1)^{|\Lambda|-|x|} + \beta\, p_2^{|x|}(1-p_2)^{|\Lambda|-|x|}, \quad x \in X^\Lambda,\ \Lambda \in W.$$

For all $t \in \mathbb{Z}^d$, $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$ and $\Lambda \in W\left(\mathbb{Z}^d \setminus \{t\}\right)$ put

$$q_t^{\bar{x}_\Lambda}(1) = \frac{\mathsf{P}_{\{t\}\cup\Lambda}(1\bar{x}_\Lambda)}{\mathsf{P}_\Lambda(\bar{x}_\Lambda)} = \frac{\alpha p_1 + \beta p_2 \exp\Big\{|\Lambda|\, H_\Lambda(\bar{x}_\Lambda)\Big\}}{\alpha + \beta \exp\Big\{|\Lambda|\, H_\Lambda(\bar{x}_\Lambda)\Big\}},$$

where

$$H_\Lambda(\bar{x}_\Lambda) = \frac{|\bar{x}_\Lambda|}{|\Lambda|} \ln \frac{p_2}{p_1} + \left(1 - \frac{|\bar{x}_\Lambda|}{|\Lambda|}\right) \ln \frac{1-p_2}{1-p_1}.$$

• A mixture of Gibbs random fields is not always a Gibbs random field.

Example 3. Let $\alpha \in (0,1)$ and $\mathsf{P}^{(p_1)}$, $\mathsf{P}^{(p_2)}$ be Bernoulli random fields with parameters p_1 and p_2 correspondingly, and $p_1 \neq p_2$. The random field P finite dimensional distributions of which are of the form

$$\mathsf{P}_V(x) = \alpha \mathsf{P}_V^{(p_1)}(x) + (1-\alpha)\mathsf{P}_V^{(p_2)}(x), \qquad x \in X^V,\, V \in W,$$

is not Gibbsian.

• Different Gibbs random fields can correspond to the same canonical 1–specification (existence of phase transition).

Example 4. Let $d = 1$ and let numbers c_i, $0 < c_i < 1$, $i \in \mathbb{N}$ be such that $\prod_{i=1}^{\infty} c_i > 0$. Let finite dimensional distributions of random fields P^+ and P^- with the same phase space $X = \{-1, 1\}$ be defined as follows

$$\mathsf{P}_V^{\pm}(x) = \left(\prod_{i=1}^{|V|-1} \frac{1 + c_i x_{t_i} x_{t_{i+1}}}{2}\right) \cdot \frac{1}{2}\left(1 \pm x_{t_{|V|}} \prod_{i=|V|}^{\infty} c_i\right),$$

$x \in X^V$, $V = \{t_1, t_2, ..., t_{|V|}\} \in W$, $t_1 < t_2 < ... < t_{|V|}$. Then random fields P^+ and P^- correspond to the same canonical 1–specification.

• The convex mixture of Gibbs random fields with the same canonical specification is a Gibbs random field.

• The set of Gibbs random fields is dense in the space of all random fields with respect to the topology of weak convergence.

Theorem. *The set of Gibbs random fields corresponding to the canonical 1–specification is convex and closed.*

Conclusions

• The proposed mathematical definition of potential energy gives some justifications of the Gibbs formula and, perhaps, will be useful in various problems of mathematical statistical physics.

• The proposed purely probabilistic definition of Gibbs random field opens up new possibilities for the development of the general theory of Gibbs random fields, especially in the study of such questions as uniqueness, decreasing of correlation, as well as in proofs of limit theorems.

• At the same time, the classical definition of Gibbs random field based on the concept of the potential, is convenient for the construction of specific models and for study the problem of phase transitions in such models.

Thank you

for your attention

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