New ideas in the non-equilibrium statistical physics and the micro approach to transportation flows

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Three topics

1. Boltzmann hypothesis
2. From N-body problem directly (without Boltzmann equation) to hydrodynamic equations
3. Transportation flow without collisions

Boltzmann hypothesis

For mathematicians ergodicity means that Cesaro sums converge to the unique limit, which, for Hamiltonian systems of N particles in finite volume, is the Liouville measure on the energy surface.

Or simpler, there is the only invariant measure

ANY linear system is not ergodic (due to invariant tori).
"Non-linear" is the sacred word for some mathematicians and they believe that sufficiently non-linear dynamical systems are ergodic.
In spite of the fact that no N-particle non-linear example is shown to be ergodic.
So, one can try
Unsolved problem - prove that "almost all" non-linear systems are NOT ergodic

More correct formulation

"Almost any" (non-linear or linear) system is ergodic, if at least ONE fixed particle (of N) is subjected to random external perturbation.

Moreover

1. If the external perturbation conserves energy then it converges to Liouville,

2. If the external perturbation is Gaussian without memory (white noise) then it converges to classical Gibbs, otherwise also converges but to something different.

This is proved for almost any linear system (with quadratic hamiltonian), papers below.

For almost any non-linear the proof is in progress.

Convergence to GIBBS

Lykov A. A., Malyshev V. A. Harmonic chain with weak dissipation. Markov Processes and Related Fields, 2012, 18, 4, pp. 721-729.

Lykov A. A., Malyshev V. A.., Muzychka S. A. *Linear hamiltonian systems under microscopic random influence*. Theory Probability Appl., 2013, 57, 4, pp. 684-688.

Lykov A. A., Malyshev V. A. *Role of the memory in convergence to invariant Gibbs measure*", Doklady Mathematics, 2013, v. 88, 2, 513-515.

Lykov A.A., Malyshev V.A. Convergence to Gibbs equilibrium - unveiling the mystery. Markov Processes and Related Fields, 2013, v. 19, 4

Convergence to LIOUVILLE

Lykov A.A., Malyshev V.A. A new approach to Boltzmann's ergodic hypothesis. Doklady Mathematics, 2015, 92, 2, pp. 1-3.

Lykov A.A., Malyshev V.A. *Liouville ergodicity of linear multi-particle hamiltonian systems with one marked particle velocity flips*. Markov Processes and Related Fields, 2015, 2, 381 - 412

From N-body problem to Euler equations

Rigorous mathematical example of deduction of the system of Euler hydrodynamic equations from hamiltonian equations for N particle system as $N \to \infty$, moreover without use of stochastic dynamics, thermodynamics

Continuum media (d-dimensional) - a bounded open subset $\Lambda \subset R^d$

Dynamics of continuum media - system of such domains $\Lambda_t, t \in [0, T), 0 < T \leq \infty$, together with the system of diffeomorphisms $U^t : \Lambda = \Lambda_0 \to \Lambda_t, t \in [0, T)$, smooth also in t.

Trajectory of the point (particle) $x \in \Lambda_0$ of continuum media - the function $y(t, x) = U^t x$.

The main unknown variable in the Euler equations is the **velocity** $u(t,y)$ of the particle, which, at time t, was at point y. For this definition had sense it is necessary that the trajectories $y(t,x)$ did not collide for different x at the same t. This property obviously should be related to the similar property for N particle system, if we want to get continuum media trajectories in the limit $N \to \infty$.

We say that the N-particle system **has no collisions**, if for all $1 \leq j < k \leq N$ and all $t \in [0, \infty)$

$$x_j(t) \neq x_k(t),$$

and has strong property of absence of collisions if

$$\inf_{t \geqslant 0} \inf_{j,k:j \neq k} |(x_j(t) - x_k(t))| > 0 \tag{1}$$

We consider N-particle system on the real line with quadratic Hamiltonian and prove that its particle trajectories, for $N \to \infty$, converge, in the sense defined below, to the trajectories of the continuum particle system.
We define for the latter 3 functions: $u(t, x)$ - the velocity, $p(t, x)$ - the pressure and $\rho(t, x)$ - the density. And prove that they satisfy the system of 3 equations of the Euler type

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{2}$$

$$u_t + uu_x = -\frac{p_x}{\rho}, \tag{3}$$

$$p = p(\rho), \tag{4}$$

In continuum mechanics these equations correspond to the conservation laws of mass, momentum and to the thermodynamic equation of state. In physics the first two equations are quite general, but the third one depends on the matter type and thermodynamic situation and should be given separately. In our deduction all these equations and functions get very simple intuitive mechanical sense for some N-particle system. In particular, the pressure appears to be the limit of the forces on the separate particles which is most close to the particle of the continuum media. We stress that we do not use neither probability theory nor thermodynamics.

Model

Hamiltonian system of N particles (of unit mass) with coordinates $x_1, ..., x_N$ on R with the Hamiltonian

$$H = \sum_{k=1}^{N} \frac{v_k^2}{2} + \sum_{k=1}^{N-1} \frac{\omega^2}{2}(x_{k+1} - x_k - a)^2$$

where $a > 0$. The corresponding system of linear differential equations is

$$\ddot{x}_1 = \omega^2(x_2 - x_1 - a), \quad (5)$$

$$\ddot{x}_k = \omega^2(x_{k+1} - x_k - a) - \omega^2(x_k - x_{k-1} - a),\ k = 2, 3, \ldots, N-1, \quad (6)$$

$$\ddot{x}_N = -\omega^2(x_N - x_{N-1} - a), \quad (7)$$

with initial conditions

$$x_1(0) = 0, \; \dot{x}_1(0) = v, \tag{8}$$

for some $v \in \mathbb{R}$, and some functions $X, V \in C^4([0,1])$, where $X > 0$,

$$x_{k+1}(0) - x_k(0) = \frac{1}{N} X\left(\frac{k}{N}\right) > 0, \quad \dot{x}_{k+1}(0) - \dot{x}_k(0) = \frac{1}{N} V\left(\frac{k}{N}\right), \tag{9}$$

$k = 1, \ldots, N-1$
The functions X and V define smooth profile of initial conditions. It is convenient to assume also that following condition

$$X(0) = X(1) = 1, \; V(0) = V(1) = 0, \tag{10}$$

Scaling

Our system contains two parameters ω and a. Scaling of the mass and/or time can be reduced to the scaling the parameter ω. We are interested in the limiting case when $N \to \infty$, and when the particles are situated initially on some fixed interval $\Lambda_0 = [0, L_0]$. That is why in our (one-dimensional) case we put $a = \frac{1}{N}$. This gives equilibrium (zero forces), when $L_0 = 1$. Scaling of ω is

$$\omega = \omega' N \tag{11}$$

for some $\omega' > 0$, not depending on N. Note that the scaling of ω creates repulsion necessary for the particles did not collide. Then the second condition (10) means that two leftmost (two rightmost) particles initially have almost (up to $O(N^{-2})$) equal velocities, and the first condition (10) means that both boundary particles are subjected to almost zero force.

Absence of collisions for N particle case

Theorem

Let the conditions (8)-(10) hold. Then for any $t \geqslant 0,\ k = 1, 2, \ldots, N-1$

$$\frac{1-\gamma}{N} \leqslant x_{k+1}(t) - x_k(t) \leqslant \frac{1+\gamma}{N}$$

where

$$\gamma = 2\alpha + \frac{\beta N}{\omega},\ \ \alpha = \int_0^1 |X''(y)|\, dy\ \ \beta = \int_0^1 |V''(y)|\, dy.$$

It follows that if (11) holds and if

$$\gamma = 2\alpha + \frac{\beta}{\omega'} < 1 \tag{12}$$

then the particles never collide in the strong sense (1).

To understand the importance of the choice of α note that $X(y) - 1$ characterizes the deviation of the chain from the equilibrium, $X'(y)$ - the speed of change of this equilibrium. Thus the following simple statement is useful to get such estimate at initial time moment.

Lemma

For any $y \in [0, 1]$

$$1 - \alpha \leqslant X(y) \leqslant 1 + \alpha$$

Convergence to continuous chain dynamics
Denote $q(t,x)$ the solution of the wave equation with fixed boundary conditions (below the lower indices define the derivatives in correponding variables)

$$q_{tt} = (\omega')^2 q_{xx}, \quad q(t,0) = q(t,1) = 0. \tag{13}$$

and initial conditions:

$$q(0,x) = X(x) - 1,\ q_t(0,x) = V(x). \tag{14}$$

Let $x_k^{(N)}(t)$ be the solution of the main system (5)-(7) for given N.
Define two algorithms of how for any point of the continuous media one could relate one of the particles $x_k^{(N)}(t)$ of the N-particle approximation. To do this we will use two coordinate systems on the real line: x and z, where $z(x)$ for $x \in (0, L), L = L(0) = \int_0^1 X(y)dy$, is uniquely defined from the equation:

$$\int_0^{z(x)} X(x')dx' = x. \tag{15}$$

First algorithm to any $z \in (0, 1]$ we relate the number $[zN]$ then $1 < [zN] \leq N$ starting from some N

Second algorithm - to any $x \in [0, L]$ we relate the number $k(x, N) = k(x, N, 0)$ so that

$$x^{(N)}_{k(x,N)}(0) \leqslant x < x^{(N)}_{k(x,N)+1}(0) \qquad (16)$$

Due to positivity of $X(y)$ such number is uniquely defined. Then it is natural to call the function $x^{(N)}_{k(x,N)}(t)$ N-particle approximation of the trajectory of the particle $x \in [0, L]$ of continuum media. By definition we put $x^{(N)}_{N+1} = \infty$.

Theorem

Let the conditions of the Th 1 hold and also assume (11). Then

1) for any $T > 0$ *uniformly in* $x \in [0, L]$ *and* $t \in [0, T]$

$$\lim_{N\to\infty} x^{(N)}_{k(x,N)}(t) = y(t,x) = G(t, z(x)), \quad (17)$$

2) the function $G(t, z)$ *satisfies to the wave equation*

$$\frac{d^2 G(t,z)}{dt^2} = (\omega')^2 \frac{d^2 G(t,z)}{dz^2} \quad (18)$$

with boundary conditions $G_z(t,0) = G_z(t,1) = 1$ *and initial conditions*

$$G(0,z) = \int_0^z X(x')dx', \quad G_t(0,z) = v + \int_0^z V(x')dx'.$$

Continuiuty equation (**mass conservation law**)
Further on the function $y(t,x)$ will be called the trajectory of the particle $x \in [0,L]$, and will assume that (12) holds. Then the particles do not collide and one can uniquely define the function $u(t,y)$ as the speed of the (unique) particle situated at time t point y, that is

$$u(t,y(t,x)) = \frac{dy(t,x)}{dt}. \tag{19}$$

Also we will need the notation:

$$Y_0(t) = y(t,0), \quad Y_L(t) = y(t,L).$$

For given N we define the distribution function at time t:

$$F^{(N)}(t,y) = \frac{1}{N}|\{k \in \{1,2,\ldots,N\} : \ x_k^{(N)}(t) \leqslant y\}|, \ y \in \mathbb{R}$$

where $|\cdot|$ is the number of particle in the set.

Lemma

Denote $x(t, y) \in [0, L]$ *the particle, which got to the point* y *at time* t*, that is*

$$y(t, x(t, y)) = y. \tag{20}$$

Then uniformly in $y \in [Y_0(t), Y_L(t)]$ *and in* $t \in [0, T]$*, for any* $T < \infty$*, we have*

$$\lim_{N \to \infty} F^{(N)}(t, y) = z(x(t, y)) = F(t, y),$$

where the (smooth) function $z(x)$ *is the same as the one introduced in (15).*

In connection with Lemma 4 define the density by the formula

$$\rho(t,y) = \frac{dF(t,y)}{dy} = \frac{dz(x(t,y))}{dy} \tag{21}$$

Theorem

For any $t \geqslant 0,\ y \in [Y_0(t), Y_L(t)]$

$$\frac{\partial \rho(t,y)}{\partial t} + \frac{d}{dy}\left(u(t,y)\rho(t,y)\right) = 0. \tag{22}$$

Euler equation (momentum conservation) and equation of state

Theorem

For any $t \geqslant 0, \;\; y \in [Y_0(t), Y_L(t)]$ *we have:*

$$\frac{\partial u(t,y)}{\partial t} + u(t,y)\frac{\partial u(t,y)}{\partial y} = -(\omega')^2 \frac{1}{\rho(t,y)} \frac{d}{dy} \frac{1}{\rho(t,y)} =$$

$$= -\frac{1}{\rho(t,y)} \frac{dp(t,y)}{dy}, \tag{23}$$

if we put

$$p(t,y) = \frac{(\omega')^2}{\rho(t,y)} + C. \tag{24}$$

Constant C can be chosen as $C = -(\omega')^2$, so that at equilibrium (when $\rho = 1$) the pressure were zero.

Right side of the Euler equation as the limit of interaction forces

For given y and t define the number $k(y,N,t)$ so that

$$x^{(N)}_{k(y,N,t)}(t) \leqslant y < x^{(N)}_{k(y,N,t)+1}(t) \tag{25}$$

Consider the point $y \in [Y_0(t), Y_L(t)]$ and the force acting on the particle with the number $k(y,N,t)$:

$$R^{(N)}(t,y) = \omega^2 \left(x^{(N)}_{k(y,N,t)+1} - x^{(N)}_{k(y,N,t)} - \frac{1}{N} \right) -$$

$$-\omega^2 \left(x^{(N)}_{k(y,N,t)} - x^{(N)}_{k(y,N,t)-1} - \frac{1}{N} \right)$$

Theorem

Let the conditions of the theorem 3 hold. Then for any $0 < T < \infty$, *uniformly in* $y \in [Y_0(t), Y_L(t)]$ *and in* $t \in [0, T]$ *the following equality holds:*

$$\lim_{N\to\infty} R^{(N)}(t,y) = R(t,y) = -\frac{1}{\rho(t,y)}\frac{dp(y)}{dy},$$

where functions p, ρ *are the same as in theorem 6.*

Thus, the pressure is just an analog of interaction potential for continuum media.

1. $N = 30,\ \omega = N\ , V(x) = 0.$

$$X(x) = 1 + \epsilon \sum_{k=1}^{n} (\sin(2\pi x) + \sin(8\pi x)),\ \ \epsilon = \frac{7}{16\pi}$$

2. $N = 10,\ \omega = N$, $X(x)$ is the same as in (1), $V(x) = X(x) - 1.$

3.

$$X(x) = 1 + \epsilon \sum_{k=1}^{n} \frac{s_k}{k^2} \sin(\pi k x).$$

$V(x) = 0, N = 1000, \quad \omega = N$, where s_k distributed uniformly on $[0, 1]$, $n = 100$. ϵ is chosen so that the trajectories do not collide

$$\epsilon < \frac{1}{2\pi^2 n}.$$

4. Another realizations of s_k and ϵ is greater
5. All as in (3) but $V(x) = X(x)$, s_k deffers and ϵ is less.

Transportation flow without collisions

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Lykov A.A., Malyshev V.A., Melikian M.V. Phase diagram for one-way traffic flow with local control/arXiv.org

At any time $t \geq 0$ there are many (even infinite) number of point particles (may be called also cars, units etc.) with coordinates $z_k(t)$ on the real line, enumerated as follows:

$$... < z_{k+1}(t) < z_k(t) < ... < z_0(t)$$

$$r_k(t) = z_{k-1}(t) - z_k(t), \ k = 1, 2, \ldots .$$

The rightmost car 0 is the leader. This unit moves "as it wants but not TOO much". The trajectory $z_0(t)$ is assumed to be sufficiently smooth with velocity $v_0(t) = \dot{z}_0(t)$ and natural upper bounds on the velocity and acceleration

$$\sup_{t>0} |v_0(t)| \leqslant v_{max}, \quad \sup_{t>0} |\ddot{z}_0(t)| \leqslant a_{max} \tag{26}$$

Locality (of the control) means that the “driver” of the k-th car, at any time t, knows only its own velocity $v_k(t)$ and the distance $r_k(t)$ from the previous car. Thus, for any $k \geq 1$ the trajectory $z_k(t)$, being deterministic, is uniquely defined by the trajectory $z_{k-1}(t)$ of the previous particle. The utmost simplicity - the acceleration of the car k is defined by the virtual force

$$F_k(t) = \omega^2(z_{k-1}(t) - z_k(t) - d) \qquad (27)$$

we get that the trajectories are uniquely defined by the system of equations for $k \geq 1$

$$\frac{d^2 z_k}{dt^2} = F_k(t) - \alpha \frac{dz_k}{dt} = \omega^2(z_{k-1}(t) - z_k(t) - d) - \alpha \frac{dz_k}{dt} \qquad (28)$$

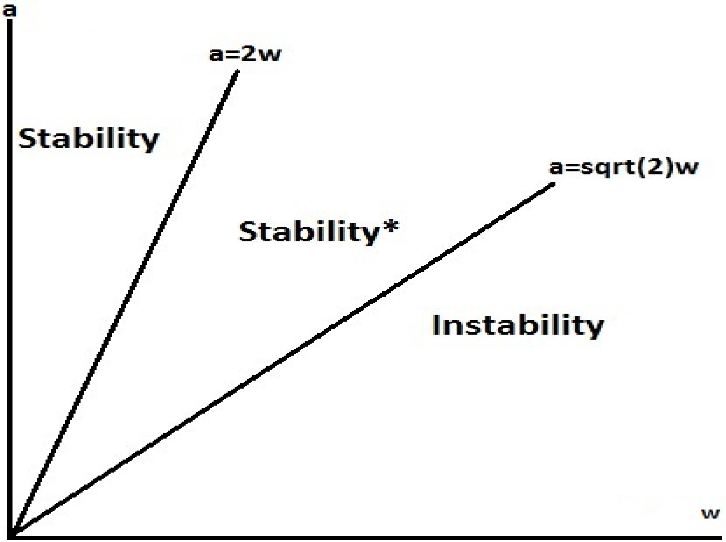
We want to propose deterministic rules (protocols, control) for any car $k \geq 1$ to perform deterministic movement. These rules should garanty SAFETY (no collisions), EFFICIENCY (sufficiently extensive flow, that is to maximize the current and the density of cars, and should be as SIMPLE as possible). More exactly, denoting

$$I = \inf_{k \geqslant 1} \inf_{t \geqslant 0} r_k(t), \quad S = \sup_{k \geqslant 1} \sup_{t \geqslant 0} r_k(t),$$

we try to get the bounds - lower positive bound on I and upper bound on S - as close as possible.

Stability depends not only on the parameters α, ω, d but also on the initial conditions and on the movement of the leader (on its velocity and acceleration). With increase of N the situation becomes more and more complicated, and its study has no much sense. That is why we study, in the space of two parameters α, ω (for fixed d), stability conditions, which are uniform in N and in large class of reasonable initial conditions and reasonable movement of the leader.

It appears that under these conditions there are 3 sectors in the quarter-plane $R_+^2 = \{(\alpha, \omega)\}$:
1) $\alpha > 2\omega$, where we can prove stability,
2) $\alpha < \sqrt{2}\omega$, where we can prove instability, and the sector
3) $\sqrt{2}\omega \leq \alpha \leq 2\omega$, where we can prove stability only for more restricted classes of initial conditions and of the leader motion.


a
a=2w
Stability
a=sqrt(2)w
Stability*
Instability
w

Here we consider the region $\alpha > 2\omega$, and this is always assumed in this section.

Movement close to stationary

For any given $d > 0$ there are special initial conditions (which can be called equilibrium configuration) when the force acting on any particle is zero:

$$z_k(0) = -ka, \;\; \dot{z}_k(0) = v, \;\; k = 0, 1, 2, \ldots, \tag{29}$$

with

$$a = a(\alpha, \omega, d, v) = d + \frac{\alpha}{\omega^2} v \tag{30}$$

If the leader moves as $z_0(t) = vt, t \geq 0$, then for any $k > 0$ also

$$z_k(t) = z_k(0) + vt$$

Such movement we call stationary. Now consider a perturbation of this situation

Theorem

For given d and v assume

$$|z_k(0) + ka| \leqslant \theta a, \quad |\dot{z}_k(0) - v| \leqslant \beta v, \tag{31}$$

$$\sup_{t \geqslant 0} |z_0(t) - vt| \leqslant \delta a \tag{32}$$

for $0 \leq \delta < \frac{1}{2}$ *and* $\theta, \beta \geqslant 0$ *such that*

$$\epsilon = 2 \max\{\delta, \frac{\theta + \frac{2v\beta}{\alpha a}}{\sqrt{1 - \left(\frac{2\omega}{\alpha}\right)^2}}\} < 1 \tag{33}$$

Then

$$(1 - \epsilon)a \leqslant I \leqslant S \leqslant (1 + \epsilon)a \tag{34}$$

Theorem

1) Assume (29) and (32). Then for all $k = 1, 2, \ldots$ *we have:* $\sup_{t \geqslant 0} |\dot{z}_k(t) - v| \leqslant \sup_{t \geqslant 0} |\dot{z}_0(t) - v|$. *If the righthand side is sufficiently small then* $v_k(t) > 0$ *for any* k, t.

2) Assume $z_0(t) = vt$, *(31) and that for some parameters* $\theta, \beta > 0$

$$\zeta = 2\frac{\theta + 2\beta/\alpha}{\sqrt{1 - \left(\frac{2\omega}{\alpha}\right)^2}} < 1 \tag{35}$$

Then for all k

$\lim_{t \to \infty}(z_k(t) - (vt - ka)) = 0$ *where* a *is in (30) and*

$$(1 - \zeta)a \leqslant I \leqslant S \leqslant (1 + \zeta)a.$$

Non-stationary initial conditions

In Theorems (8) and (9) we considered initial coordinates of the particles close to the lattice points $-ka$. Here we consider more general initial conditions with restrictions only on the distances between particles. Denote

$$d^* = \frac{a_{\max} + \alpha v_{\max}}{\omega^2} \qquad (36)$$

Theorem

Let the initial conditions be

$$(1-\theta)d \leqslant r_k(0) \leqslant (1+\theta)d, \quad |\dot{z}_{k-1}(0)-\dot{z}_k(0)| \leqslant \beta, \quad k = 1, 2, \ldots . \tag{37}$$

for some $\beta \geqslant 0, 0 \leqslant \theta < 1$. *Assume moreover that*

$$\eta = \max(\frac{d^*}{d}, \frac{\theta + \frac{2v\beta}{\alpha a}}{\sqrt{1-(\frac{2\omega}{\alpha})^2}}) < 1 \tag{38}$$

Then we have the following stability bounds

$$(1-\eta)d \leqslant I \leqslant S \leqslant (1+\eta)d \tag{39}$$

Theorem 8 and 10

1) garanty that there are no collisions between particles (by the lower bound),

2) provide lower bound for the density of the flow (by the upper bound), that is the mean distance between particles remains bounded.

3) do not garanty that the velocities remain positive. Such conditions were obtained above in Theorem 9.

Restricted stability

Here we consider the region $\sqrt{2}\omega \leq \alpha \leq 2\omega$, where we can prove stability only for asymptotically homogeneous initial conditions.

Theorem

Let $\sqrt{2}\omega \leqslant \alpha \leq 2\omega$ *and let* $z_0(t)$ *be such that*

$$\omega \int_0^{\infty} |z_0(t) - vt| dt = \sigma a < \infty$$

for some $\sigma \geqslant 0$. *Assume also that the initial conditions are "summable", that is*

$$\sum_{k=1}^{\infty} |z_k(0) + ka| \leqslant \theta a, \quad \sum_{k=1}^{\infty} |\dot{z}_k(0) - v| \leqslant \beta v$$

for some $\theta, \beta \geqslant 0$.

Then

$$I \geqslant (1-2\eta)a, \quad S \leqslant (1+2\eta)a,$$

where

$$\eta = 2\left(\theta + \frac{\beta v}{a\omega} + \sigma\right)$$

It follows from this theorem that the upper bound $S < \infty$ holds for all parameters, but the lower (safety) bound $I > 0$ holds if

$$\theta + \frac{\beta v}{a\omega} + \sigma < \frac{1}{4}.$$

Theorem

Assume again that $\sqrt{2}\omega \leqslant \alpha \leq 2\omega$, *the initial conditions satisfy*

$$\sum_{k=1}^{\infty} |r_k(0) - d| \leqslant \theta d, \quad \sum_{k=1}^{\infty} \frac{|\dot{z}_{k-1}(0) - \dot{z}_k(0)|}{\alpha} \leqslant \beta d, \quad k = 1, 2, \ldots,$$

for some $\beta \geqslant 0, 0 \leqslant \theta \leqslant 1$, *and the leader moves as*

$$\frac{1}{\omega} \int_0^{+\infty} |\ddot{z}(t) + \alpha \dot{z}_0(t)| dt = \sigma d < \infty$$

for some $\sigma \geqslant 0$.

Then

$$I > d(1-\eta), \quad S < d(1+\eta),$$

where

$$\eta = 2\left(\theta + x\beta + \sigma\right), \quad x = \frac{\alpha}{\omega}.$$

It follows that the safety condition $I > 0$ holds if

$$\theta + x\beta + \sigma < \frac{1}{2}.$$

Here we will prove instability for the region $\alpha < \sqrt{2}\omega$. The first reason for the instability is the absence of dissipation, that is if $\alpha = 0$. The following result shows this even for the most favorable initial conditions.

Theorem

Assume that $\alpha = 0$ *and*

$$z_k(0) = -ka, \ \frac{dz_k}{dt}(0) = v, \ k \geq 0 \tag{40}$$

$$z_0(t) = tv + \sin \omega_0 t, \ \ v > 0, \ \ \omega_0 \neq 0$$

Then for any $k \geqslant 2$ *we have due to resonance*

$$\inf_{t \geqslant 0} r_k(t) = -\infty$$

Now we show that, even under the smallest perturbation of the initial conditions (40) and even simpler leader trajectory, we get much more general instability condition.

Theorem

Assume $\alpha < 2\omega$, $z_0(t) = vt$ *and initial conditions such that:*

$$z_k(0) = -ka,\ \dot{z}_k(0) = \begin{cases} v, & k > 1 \\ v + \epsilon, & k = 1 \end{cases}, \qquad (41)$$

where ϵ *is some real number. Then for any* $\mu > \frac{1}{\tau}$, $\tau = \sqrt{\omega^2 - \frac{\alpha^2}{4}}$ *the following asymptotic formula takes place:*

$$z_{k+1}(t) - (vt - (k+1)a) \sim \frac{c}{\sqrt{k}} e^{kf(\mu)} \sin(\Omega(\mu)k + \phi_0(\mu)),$$

$$\text{when } t = \mu k, \;\; k \to \infty$$

where

$$f(\mu) = -\frac{\alpha\mu}{2} + 1 - \ln\left(\frac{2\tau}{\mu\omega^2}\right),$$

$$\phi_0(\mu) = \arctan(\nu), \nu = \sqrt{\mu^2\tau^2 - 1},$$

$$\Omega(\mu) = \nu - \phi_0(\mu) = \nu - \arctan(\nu), c = \epsilon\sqrt{\frac{2\tau}{\pi\nu\mu}}$$

Corollary

Assume $\alpha < \sqrt{2}\omega$ and $z_0(t) = vt$. Assume the initial conditions (41). Then

$$I = -\infty, \quad S = \infty.$$

While proving the theorem we will see that corollary (15) holds even for more general initial conditions:

$$\max_k |z_k(0) + ka| \leqslant \epsilon_1, \quad \max_k |\dot{z}_k(0) - v| \leqslant \epsilon_2.$$

with some nonnegative $\epsilon_1, \epsilon_2 \geqslant 0$.

Thank you for the attention!