Sigma-model solitons from the Schrödinger representation

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**Sigma-model solitons on noncommutative spaces**

L. Dabrowski, G.L. Franz Luef  

and earlier work with L. Dabrowski and T. Krajewski  
and more recent work by J. Rosenberg and Mathai V.

Abstract

Sigma-model solitons over the Moyal plane and noncommutative tori  
(also known as Irrational Rotation Algebra),  
as source spaces, with a target space made of two points

A natural action functional leads to self-duality equations for projections in  
the source algebra

Solutions, having non-trivial topological content (instantons),  
are constructed via suitable Morita duality bimodules  
(obtained using the Schrödinger representation).

Inputs from time-frequency analysis and Gabor analysis
Non-linear $\sigma$-models:

field theories of maps $X$ between the source space $(\Sigma, g)$, and the target space $(M, G)$. The action functional

$$S[X] = \frac{1}{2\pi} \int_{\Sigma} \sqrt{g} \, g^{\mu\nu} G_{ij}(X) \partial_\mu X^i \partial_\nu X^j,$$

The stationary points:

harmonic maps from $\Sigma$ to $M$; describe minimal surfaces embedded in $M$.

$\Sigma$ two dimensional, conformal invariance:

the action $S$ is invariant for any rescaling of the metric $g \rightarrow e^\sigma g$.

Thus the action only depends on the conformal class of the metric and may be rewritten using a complex structure on $\Sigma$

$$S[X] = \frac{i}{\pi} \int_{\Sigma} G_{ij}(X) \partial X^i \wedge \bar{\partial} X^j,$$
Here $\partial$ and $\bar{\partial}$, a complex structure and $d = \partial + \bar{\partial}$.

In two dimensions

**complex and conformal**

are the same thing.

In two-dimensions, the conformal class of a general constant metric is parametrized by a complex number $\tau \in \mathbb{C}$, $\Im \tau > 0$.

Up to a conformal factor, the metric is

$$g = (g_{\mu \nu}) = \begin{pmatrix} 1 & \Re \tau \\ \Re \tau & |\tau|^2 \end{pmatrix}.$$
An algebraic generalization: by dualization and reformulation in terms of the ∗-algebras \( \mathcal{A} = C^\infty(\Sigma, \mathbb{C}) \) and \( \mathcal{B} = C^\infty(M, \mathbb{C}) \).

Embeddings \( X \) of \( \Sigma \) into \( M \) correspond to ∗-algebra morphisms \( \pi_X : \mathcal{B} \to \mathcal{A} \), with correspondence

\[
f \mapsto \pi_X(f) = f \circ X.
\]

∗-algebra morphisms make sense for general algebras \( \mathcal{A} \) and \( \mathcal{B} \).

The configuration space is all ∗-algebra morphisms from \( \mathcal{B} \) to \( \mathcal{A} \)

The definition of the action functional involves generalizations of the conformal and Riemannian geometries.
Connes: conformal is understood in the framework of positive Hochschild cohomology.

The tri-linear map $\phi : \mathcal{A}^3 \rightarrow \mathbb{R}$,

$$\phi(f_0, f_1, f_2) = \frac{i}{\pi} \int_\Sigma f_0 \partial f_1 \wedge \bar{\partial} f_2$$

is an extremal of positive Hochschild cocycles belonging to the Hochschild cohomology class of the cyclic cocycle $\psi$ defined by

$$\psi(f_0, f_1, f_2) = \frac{i}{2\pi} \int_\Sigma f_0 d f_1 \wedge d f_2.$$ 

On the one hand $\psi$, the fundamental class, allows one to integrate 2-forms in dimension 2, so it is a metric independent object.

On the other hand, $\phi$ defines a suitable positive scalar product

$$\langle a_0 da_1, b_0 db_1 \rangle = \phi(b_0^* a_0, a_1, b_1^*)$$

on 1-forms and depends on the conformal class of the metric.
Expressions like $\phi$ and $\psi$ make sense for a general algebra $\mathcal{A}$.

Compose the cocycle $\phi$ with a morphism $\pi : \mathcal{B} \to \mathcal{A}$ to obtain a positive cocycle on $\mathcal{B}$

$$\phi_\pi = \phi \circ (\pi \otimes \pi \otimes \pi)$$

Evaluate the cocycle $\phi_\pi$ on a suitably element of $\mathcal{B}^{\otimes 3}$ which provides the noncommutative analogue of the metric on the target;

Easiest choice for this metric: a positive element $G = \sum_i b^i_0 \delta b^i_1 \delta b^i_2$ of the space of universal 2-forms $\Omega^2(\mathcal{B})$.

Thus, a well defined and positive quantity

$$S[\pi] = \phi_\pi(G)$$  \hspace{1cm} (1)

a noncommutative analogue of the action functional of the non linear $\sigma$-model.
Here:

π is the dynamical variable (the embedding)

whereas φ (the conformal structure on the source) and G (the metric on the target) are background structures that have been fixed.

The critical points of the σ-model for the action functional (1): generalizations of harmonic maps; “minimally embedded surfaces” in the (noncommutative) space associated with B.

The role of the other cocycle ψ is to give a topological ‘charge’.

More on this late on.
Two points as a target space \( M = \{1, 2\} \)

Any continuous map from a connected surface \( \Sigma \) to a discrete space is constant, a commutative theory would be trivial.

Not the case for a “noncommutative” source space: in general, not trivial such maps, ‘dually’, as \( \ast \)-algebra morphisms from the algebra of functions over \( M = \{1, 2\} \), that is \( \mathbb{C}^2 \), to the algebra \( A \) of the noncommutative source.

As a vector space \( \mathbb{C}^2 \) is generated by the projection function \( e \) defined by \( e(1) = 1 \) and \( e(2) = 0 \);

\[ \Rightarrow \] any \( \ast \)-algebra morphism \( \pi : \mathbb{C}^2 \to A \) is the same as a projection \( p = \pi(e) \in A \).

The configuration space of a two point target space sigma-model is the collection of all projections \( P(A) \) in the algebra \( A \).

Choosing the metric \( G = \delta e \delta e \) on the space \( M = \{1, 2\} \), and a Hochschild cocycle \( \phi \) for the conformal structure, the action functional is simply

\[ S[p] = \phi(1, p, p), \]

Positivity in Hochschild cohom implies it is bounded by a topological term
Noncommutative torus and Moyal plane as source space $\mathcal{A}$:

the action functional is

$$S[p] = \frac{1}{4\pi} \text{tr}(\partial p \bar{\partial} p).$$

with the natural complex structure on $\mathcal{A}$ given by

$$\partial = \partial_1 - i \partial_2, \quad \bar{\partial} = \partial_1 + i \partial_2,$$

and derivations $\partial_1$ and $\partial_2$ infinitesimal generators of a $\mathbb{T}^2$-action

and $\text{tr}$ an invariant trace.

All of above can be extended to more general metrics.

In two dimensions: Up to a conformal factor the general constant metric is parametrized by a complex number $\tau \in \mathbb{C}$, $\Im \tau > 0$.

The corresponding ‘complex torus’ $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z} + \tau \mathbb{Z}$ would act infinitesimally on $\mathcal{A}$ with two complex derivations

$$\partial = \partial_1 + \bar{\tau} \partial_2, \quad \bar{\partial} = \partial_1 + \tau \partial_2.$$
As usual, the critical points of the action functional are obtained by equating to zero its first variation, that is the linear term in an infinitesimal variation

\[ \delta S[p] = S[p + \delta p] - S[p], \quad \text{for} \quad \delta p \in T_p(P(A)). \]

One gets

\[ p \Delta(p) (1 - p) = 0 \quad \text{and} \quad (1 - p) \Delta(p) p = 0, \]

or, equivalently the non-linear equations of the second order

\[ p \Delta(p) - \Delta(p) p = 0. \quad (2) \]

with the Laplacian of the metric \( \Delta = \frac{1}{2}(\partial \overline{\partial} + \overline{\partial} \partial) \)
The cyclic 2-cocycle giving the fundamental class is
\[ \psi(a_0, a_1, a_2) = \frac{1}{2\pi i} \text{tr} \left( a_0 (\partial_1 a_1 \partial_2 a_2 - \partial_2 a_1 \partial_1 a_2) \right), \]

For any projection \( p \in P(A) \), the quantity
\[ c_1(p) := \psi(p, p, p) \]
is an integer: the index of a Fredholm operator.

For any \( p \in P(A) \) it holds that
\[ S[p] \geq |c_1(p)|. \]

The equality for projection \( p \) satisfying self-duality or anti-self duality eqns
\[ p(\partial_1 \pm i \partial_2)(p) = 0 \quad (3) \]
These equation imply the EOM (2).
Projection from Morita equivalence (Rieffel)

A Morita equivalence between (pre $C^*$-algebras) $\mathcal{A}$ and $\mathcal{B}$:

a $\mathcal{A}–\mathcal{B}$-bimodule $\mathcal{E}$

with a left-linear $\mathcal{A}$-valued $h\mathcal{p} \langle \cdot, \cdot \rangle$ and a right-linear $\mathcal{B}$-valued $h\mathcal{p} \langle \cdot, \cdot \rangle$.

There is an associativity condition:

$$\mathcal{p}\langle \xi, \eta \rangle \zeta = \xi \langle \eta, \zeta \rangle$$

It follows an identification $\mathcal{B} \simeq \mathcal{K}_A(\mathcal{E})$ (compact endomorphisms).

In particular, there exist elements $\{\eta_1, ..., \eta_n\}$ in $\mathcal{E}$ such that

$$\sum_j \langle \eta_j, \eta_j \rangle = 1_B.$$

Then, the associativity condition gives that the matrix $p = (p_{jk})$

$$p_{jk} = \mathcal{p}\langle \eta_j, \eta_k \rangle$$

is a projection in the matrix algebra $M_n(\mathcal{A})$. 
Both algebras $\mathcal{A}$ and $\mathcal{B}$ are in the joint smooth domain of two commuting derivations $\partial_1$ and $\partial_2$; and have faithful invariant tracial states, which are compatible in the sense that:

$$\text{tr} \langle \xi, \eta \rangle = \text{tr} \langle \eta, \xi \rangle.$$  

Derivations are lifted to $\mathcal{E}$ as (left and right) **covariant derivatives**:

$$\nabla_j : \mathcal{E} \to \mathcal{E}, \quad j = 1, 2,$$

$$\nabla_j(a\xi) = (\partial_j a)\xi + a(\nabla_j \xi) \quad \text{and} \quad \nabla_j(\xi b) = (\nabla_j \xi)b + \xi(\partial_j b)$$

compatible with both the $\mathcal{A}$-valued hermitian structure $\langle \cdot, \cdot \rangle$; the $\mathcal{B}$-valued hermitian structure $\langle \cdot, \cdot \rangle$.

\[\partial_j(\langle \xi, \eta \rangle) = \langle \nabla_j \xi, \eta \rangle + \langle \xi, \nabla_j \eta \rangle\]

and

\[\partial_j(\langle \xi, \eta \rangle) = \langle \nabla_j \xi, \eta \rangle + \langle \xi, \nabla_j \eta \rangle\]
Lifting self-duality equations: solitons

The holomorphic/anti-holomorphic, connection on $\mathcal{E}$,

$$\nabla = \nabla_1 - i \nabla_2, \quad \bar{\nabla} = \nabla_1 + i \nabla_2,$$

lift to $\mathcal{E}$ the complex derivations $\partial = \partial_1 - i \partial_2$ or $\bar{\partial} = \partial_1 + i \partial_2$.

The ‘rank’ one case:

Seek solutions of the s-d eqs (3) of the form

$$p_\psi := \langle \psi, \psi \rangle \in \mathcal{A} \quad \text{with} \quad \psi \in \mathcal{E} \quad \text{such that} \quad \langle \psi, \psi \rangle = 1_\mathcal{B}.$$  

The projection $p_\psi$ is a solution of the s-d eqs:

$$p_\psi \partial (p_\psi) = 0,$$

if and only if the vector $\psi$ is a generalized eigenvector of $\bar{\nabla}$ i.e. there exists $\lambda \in \mathcal{B}$ such that

$$\bar{\nabla} \psi = \psi \lambda.$$
How to compute the topological charge:

The curvature of the covariant derivatives is defined as

\[ F_{12} := \nabla_1 \nabla_2 - \nabla_2 \nabla_1 \]

Let \( \psi \in \mathcal{E} \) be such that \( \langle \psi, \psi \rangle \rho = 1_B \) and \( p_\psi := \rho \langle \psi, \psi \rangle \in \mathcal{A} \) the corresponding projection. Then, its topological charge is:

\[ c_1(p_\psi) = -\frac{1}{2\pi i} \text{tr} \langle \psi, F_{12} \psi \rangle \rho. \]

Constant curvature: \( F_{12} = -2\pi i q \text{id}_\mathcal{E} \)

the projection \( p_\psi = \rho \langle \psi, \psi \rangle \) has then topological charge

\[ c_1(p) = q \text{tr}(1_B) \in \mathbb{Z} \]

note that neither \( q \) nor \( \text{tr}(1_B) \) need be an integer
Moyal plane from Schrödinger representation

The projective representation of $\mathbb{R}^2$ on $L^2(\mathbb{R})$ defined for $\xi \in L^2(\mathbb{R})$ by

$$(\pi(z)\xi)(t) = e^{2\pi it\omega}\xi(t-x), \quad \text{for } z = (x, \omega).$$

(4)

$\Rightarrow \pi(z)\pi(z') = e^{-2\pi ix\omega'}\pi(z + z').$

The map $c : \mathbb{R} \times \mathbb{R} \to T, c(z, z') = e^{-2\pi i(x\omega')} \text{ is a 2-cocycle.}$

Its matrix-coefficients are defined for $\xi, \eta \in L^2(\mathbb{R})$ by

$$V_{\eta}\xi(z) := \langle \xi, \pi(z)\eta \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \xi(t)\overline{\eta}(t-x)e^{-2\pi it\omega}dt$$

In signal analysis $V_{\eta}\xi$ is known as the short time Fourier transform.
Moyal’s identity: \[ \langle V_\eta \xi, V_\psi \varphi \rangle_{L^2(\mathbb{R}^2)} = \langle \xi, \varphi \rangle_{L^2(\mathbb{R})} \overline{\langle \eta, \psi \rangle_{L^2(\mathbb{R})}} \]

An additional important consequence of this identity: is a reconstruction formula for \( \xi \in L^2(\mathbb{R}) \) in terms of \( \{ \pi(z) \eta : z \in \mathbb{R}^2 \} \).

Let \( \eta \) and \( \psi \) be in \( L^2(\mathbb{R}) \) such that \( \langle \psi, \eta \rangle \neq 0 \). Then for any \( \xi \in L^2(\mathbb{R}) \),

\[
\xi = \langle \psi, \eta \rangle^{-1} \iint_{\mathbb{R}^2} \langle \xi, \pi(z) \eta \rangle \pi(z) \psi \, dz = \langle \psi, \eta \rangle^{-1} \iint_{\mathbb{R}^2} V_\eta \xi(z) \pi(z) \psi \, dz.
\]

The twisted group algebra \( L^1(\mathbb{R}^2, c) \) of \( \mathbb{R}^2 \) associated to the cocycle \( c \).

For \( k \) and \( l \) in \( L^1(\mathbb{R}^2) \), the twisted convolution \( (k \natural l) \):

\[
(k \natural l)(z) = \iint k(z')l(z - z')c(z', z - z') \, dz'
\]

and twisted involution of \( k \in L^1(\mathbb{R}^2) \):

\[
k^*(z) = c(z, z) \overline{k(-z)} = e^{-2\pi ix\omega} \overline{k(-z)}
\]
The integrated representation

\[ K = \pi(k) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} k(z)\pi(z)dz \]

for \( k \in L^1(\mathbb{R}^2) \), is a non-degenerate bounded representation of the twisted convolution algebra \( L^1(\mathbb{R}, c) \) on \( L^2(\mathbb{R}^2) \).

The adjoint of \( K = \pi(k) \) is given by \( K^* = \pi(k^*) \) and the composition of \( K = \pi(k) \) and \( L = \pi(l) \) corresponds to \( (k \natural l) \):

\[ KL = \int_{\mathbb{R}} \int_{\mathbb{R}^2} (k \natural l)(z)\pi(z)dz.\]

Denote by \( \mathcal{A} \) the class of all operators \( K = \pi(k) \) for \( k \in S(\mathbb{R}^2) \); they are all trace-class. Its norm closure is all compact operators.

\( \mathcal{A} \) is a model of the Moyal plane: the Fourier transforms of the symbols defining elements of \( \mathcal{A} \) yield the Moyal product:

\[ k \star l = \mathcal{F}^{-1}(\mathcal{F}(k)\natural\mathcal{F}(l)) \quad \text{for} \quad k, l \in S(\mathbb{R}^2).\]
Rieffel:
(a result equivalent to the uniqueness of the Schroëdinger representation of
the CCR (for the two-dimensional quantum phase space of a free particle):

The space $\mathcal{E} = \mathcal{S}(\mathbb{R})$ is an equivalence bimodule between $\mathcal{A}$ and $\mathbb{C}$

with respect to the actions:

$$K \cdot \xi = \iiint k(z) \pi(z) \xi \, dz,$$

$$\xi \cdot \lambda = \xi \bar{\lambda}$$

and $\mathcal{A}$ and $\mathbb{C}$-valued hermitian products:

$$\langle \xi, \eta \rangle = \iiint \langle \xi, \pi(z) \eta \rangle_{L^2(\mathbb{R})} \pi(z) \, dz = \iiint V_\eta \xi(z) \pi(z) \, dz = \pi(V_\eta \xi)$$

$$\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle_{L^2(\mathbb{R})}.$$
A two dimensional spectral geometry

Commuting derivations (an infinitesimal action of $\mathbb{T}^2$) $\partial_1, \partial_2$:

$$\partial_1 K = 2\pi i \int \int_{\mathbb{R}^2} xk(x, \omega)\pi(x, \omega) \, dx \, d\omega,$$

$$\partial_2 K = 2\pi i \int \int_{\mathbb{R}^2} \omega k(x, \omega)\pi(x, \omega) \, dx \, d\omega.$$

They lift to covariant derivatives on the equivalence bimodule $\mathcal{E}$:

$$(\nabla_1 \xi)(t) = 2\pi i t \xi(t) \quad \text{and} \quad (\nabla_2 \xi)(t) = \xi'(t)$$

They are compatible with both left and right hermitian structures.

The connection has constant curvature:

$$F_{1,2} := [\nabla_1, \nabla_2] = -2\pi i \text{id}_{\mathcal{E}}$$
For $\psi \in S(\mathbb{R})$ normalized as $\langle \psi, \psi \rangle \bullet = \| \psi \|_2 = 1$, 
\Rightarrow a non-trivial projection $p_\psi = \bullet \langle \psi, \psi \rangle$ in $A$.

The projection $p_\psi$ is a solution of the self-duality equations,
$$p_\psi (\partial p_\psi) = 0$$
if and only if, for some $\lambda \in \mathbb{C}$, the element $\psi$ satisfies
$$\overrightarrow{\nabla} \psi = \psi \lambda.$$

Eigenfunction equations for $\overrightarrow{\nabla}$; solutions are generalized Gaussians:
$$\psi_\lambda (t) = c e^{-\pi t^2 - i \lambda t}.$$

Explicitly,
$$p_\psi = \bullet \langle \psi, \psi \rangle = \int \int_{\mathbb{R}^2} V_\psi \psi(z) \pi(z) \, dz$$
$$V_\psi \psi(x, \omega) = e^{-\frac{x^2}{2} (x^2 + \omega^2)} e^{-\pi i x \omega - \frac{1}{2} (\bar{\lambda} + \lambda) x + \frac{1}{2} (\bar{\lambda} - \lambda) \omega}.$$

For its topological charge:
$$c_1(p_\psi) = \text{tr}(p_\psi) = V_\psi \psi(0) = 1.$$
The constant curvature is none other than the Heisenberg commutation relations (in the Schrödinger representation).

The anti-holomorphic connection $\nabla = \nabla_1 + i\nabla_2$ is the annihilation operator; the holomorphic $\nabla = \nabla_1 - i\nabla_2$ is the creation operator.

The self-duality equation for the projections is the equation for the minimizers of the Heisenberg uncertainty relation, which explains why they are Gaussian $\psi_\lambda$. 
The irrational rotation algebra (aka the noncommutative torus).

For $\theta \in \mathbb{R}$, the $C^*$-algebra $A_\theta$ of the noncommutative torus is the norm closure of the span of $\{\pi(\theta k, l) : k, l \in \mathbb{Z}\}$: the restriction of the Schrödinger rep (4) of $\mathbb{R}^2$ on $L^2(\mathbb{R})$ to $\theta \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R}^2$.

Denoting $\pi(0, 1) = M_1$ and $\pi(\theta, 0) = T_\theta$ they satisfy:

$$M_1 T_\theta = e^{2\pi i \theta} T_\theta M_1.$$

The smooth torus: subalgebra $A_\theta$ of $A_\theta$ consisting of operators

$$\pi(a) = \sum_{k,l \in \mathbb{Z}} a_{kl} \pi(\theta k, l), \quad \text{for} \quad a = (a_{kl}) \in \mathcal{S}(\mathbb{Z}^2).$$

With $a$ and $b$ in $\mathcal{S}(\mathbb{R})$ we have for their product

$$\pi(a) \pi(b) = \pi(a \natural b)$$

where $a \natural b$ is the twisted convolution

$$(a \natural b)(k, l) = \sum_{m,n \in \mathbb{Z}} a_{mn} b_{k-m,n-l} e^{-2\pi i \theta n(k-m)}.$$
while $\pi(a)^* = \pi(a^*)$, where $a^*$ is the twisted involution of $a$:

$$(a_{kl})^* = e^{-2\pi i\theta kl} a_{-k,-l}.$$ 

Operators commuting with $\pi(\theta k, l)$ are associated with the lattice $\mathbb{Z} \times \theta^{-1}\mathbb{Z}$. They make up the algebra $A_{1/\theta}$ of elements

$$b = \sum_{k,l \in \mathbb{Z}} b_{kl} \pi(k, \theta^{-1}l), \quad \text{for} \quad b = (b_{kl}) \in \mathcal{S}(\mathbb{Z}^2)$$

Take $A = A_\theta$ and $B = (A_{1/\theta})^{\text{op}} \simeq A_{-1/\theta}$ (this acts from the right)
The space $\mathcal{E} = S(\mathbb{R})$ is an equivalence bimodule between the noncommutative tori $A$ and $B$ with respect to the actions:

$$a \cdot \xi = \sum_{k,l \in \mathbb{Z}} a_{kl} \pi(\theta k, l) \xi,$$

and

$$\xi \cdot b = \sum_{k,l \in \mathbb{Z}} b_{kl} \pi(k, \theta^{-1} l)^* \xi,$$

and with hermitian products

$$\langle \xi, \eta \rangle = \theta \sum_{k,l \in \mathbb{Z}} V_{\eta} \xi(\theta k, l) \pi(\theta k, l),$$

and

$$\langle \xi, \eta \rangle^* = \sum_{k,l \in \mathbb{Z}} V_{\xi} \eta(k, l \theta^{-1}) \pi(k, \theta^{-1} l).$$
A two dimensional spectral geometry

The infinitesimal action of an ordinary torus $\mathbb{T}^2$ on both algebras $A_\theta$ and $A_{-1/\theta}$, are derivations. On $A_\theta$ they are

$$\partial_1(a) = 2\pi i \sum_{k,l} ka_{k,l} \pi(\theta k, l)$$

and

$$\partial_2(a) = 2\pi i \sum_{k,l} la_{k,l} \pi(\theta k, l),$$

and the dual ones on $A_{-1/\theta}$ are then

$$\partial_1(b) = -2\pi i \theta^{-1} \sum_{k,l} b_{k,l} \pi(k, \theta^{-1} l)^*$$

and

$$\partial_2(b) = -2\pi i \theta^{-1} \sum_{k,l} l b_{k,l} \pi(k, \theta^{-1} l)^*.$$

Lift to covariant derivatives $\nabla_1, \nabla_2$ on the bimodules $\mathcal{E} = \mathcal{S}(\mathbb{R})$:

$$(\nabla_1 \xi)(t) = 2\pi i \theta^{-1} t \xi(t) \quad \text{and} \quad (\nabla_2 \xi)(t) = \frac{d\xi(t)}{dt} =: \xi'(t).$$

The curvature is constant:

$$F_{1,2} := [\nabla_1, \nabla_2] = -2\pi i \theta^{-1} \text{id}_\mathcal{E}.$$
Frames

As a module over $A_\theta$, the space $E = S(\mathbb{R})$ is of finite rank and projective and it admits a standard module Parseval frame $\{\eta_1, ..., \eta_n\}$ for $S(\mathbb{R})$, that is each $\xi \in S(\mathbb{R})$ has an expansion,

$$\xi = \langle \xi, \eta_1 \rangle \eta_1 + \cdots + \langle \xi, \eta_n \rangle \eta_n.$$

For $0 < \theta < 1$, the module $S(\mathbb{R})$, is given by a projection in $A_{\theta}$ itself: one can use a one-element Parseval frame $\eta$

From a standard module frame $\eta$ one gets a Parseval frame $\tilde{\eta}$ by taking the element $\tilde{\eta} := \eta(\langle \eta, \eta \rangle \cdot)^{-1/2}$

Then $\langle \tilde{\eta}, \tilde{\eta} \rangle \cdot = 1$ and $\langle \tilde{\eta}, \tilde{\eta} \rangle$ is a projection in $A_{\theta}$. 
Frames and projections:

• The Hermite function

\[ \eta = \psi_k(t) = c_k e^{\pi t^2} \frac{d^k}{dt^k} e^{-2\pi t^2} \]

gives a projection \( p_k = \langle \tilde{\eta}, \tilde{\eta} \rangle \in A_\theta \), if \( 0 < \theta < (k + 1)^{-1} \).

• Let \( \eta \in S(\mathbb{R}) \) be a totally positive function of finite type greater than 2. Then, \( p_{\tilde{\eta}} = \langle \tilde{\eta}, \tilde{\eta} \rangle \) is a projection in \( A_\theta \) for \( 0 < \theta < 1 \).

All of these projections have topological charge equal to 1. From

\[ c_1(p) = q \text{ tr}(1_B) \]

with now \( q = \theta^{-1} \) (the curvature) and \( \text{tr}(1_B) = \text{tr}(A_{-1/\theta}) = \theta \).
Duality and Gabor frames

For a Parseval frame, the duality principle (Wexler-Raz identity), reads as an expansion of each \( \xi \) in \( S(\mathbb{R}) \) both over \( A \) and \( B \),

\[
\xi = \langle \xi, \tilde{\eta} \rangle \tilde{\eta} = \tilde{\eta} \langle \tilde{\eta}, \xi \rangle, \]

with \( \langle \xi, \tilde{\eta} \rangle \in A \) and \( \langle \tilde{\eta}, \xi \rangle \in B \) which are uniquely determined.
This helps for the soliton equation.

As before, the s-d eqs for the projection $p_\psi$ obeys $p_\psi \partial (p_\psi) = 0$ translate to a generalized eigenvector equation

$$\overline{\nabla} \psi = \psi \lambda,$$

with now $\lambda = \langle \psi, \overline{\nabla} \psi \rangle \in A_{-1/\theta}$.

Using the duality principle we have that

$$\text{with } \psi := \eta (\langle \eta, \eta \rangle^{-1/2})$$

the projection $p_\psi = \langle \psi, \psi \rangle \in A_\theta$

satisfies the s-d eqs:

- For $0 < \theta < (k + 1)^{-1}$, if $\eta$ is the k-th Hermite functions $\psi_k$.
- For $0 < \theta < 1$, if $\eta$ is a tot pos fun in $S(\mathbb{R})$ of finite type greater than 2.
In particular, the Gaussian function
\[ \psi_\lambda(t) = c e^{-\frac{\pi}{\theta} t^2 - i \lambda t}, \quad \text{for } \lambda \in \mathbb{C}, \]
obey the equation \( \nabla \psi_\lambda = \psi_\lambda \lambda \).

The right hermitian product \( \langle \psi_\lambda, \psi_\lambda \rangle \) is invertible in \( \mathcal{A}_{-1/\theta} \) for all \( 0 < \theta < 1 \), so that the projections \( p_\lambda = \langle \widetilde{\psi}_\lambda, \widetilde{\psi}_\lambda \rangle \), with \( \widetilde{\psi}_\lambda := \psi_\lambda (\langle \psi_\lambda, \psi_\lambda \rangle \bullet)^{-1/2} \)
are solutions of the self-duality equations

The moduli space of such Gaussian solutions, is parametrised by possible \( \lambda \)'s modulo gauge transformations

(implemented by invertible elements in \( \mathcal{A}_{-1/\theta} \))

is a copy of the complex torus.
More examples and applications
Sigma-model solitons over the Moyal plane and noncommutative tori, as source spaces, with a target space made of two points

A natural action functional leads to self-duality equations for projections in the source algebra

Solutions, having non-trivial topological content, are constructed via suitable Morita duality bimodules,

Inputs from time-frequency analysis and Gabor analysis

More interesting cases

Uses in time-frequency analysis and Gabor analysis coming up
Thank you