C^*-algebras generated by mappings with an inverse semigroup of monomials

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C^*-algebra generated by mapping

The starting point is a selfmapping $\varphi : X \longrightarrow X$ on a countable set X. The mapping generates an oriented graph (X, φ) with the elements of X as vertices and pairs $(x, \varphi(x))$ as edges. We assume the cardinality of the preimage set of each point is finite and a number m exists such that $m = \sup_{x \in X} \operatorname{card}\{\varphi^{-1}(x)\} < \infty$.

The mapping φ induces the composition operator $T_\varphi : l^2(X) \longrightarrow l^2(X)$:

$$T_\varphi f = f \circ \varphi.$$

Definition

C^*-algebra $C^*_\varphi(X)$ generated by the mapping φ is the C^*-algebra generated by the operator T_φ.

In what follows $T_\varphi = T$.

The conjugate operator T^* can be calculated by the formula

$$(T^*f)(y) = \begin{cases} \sum\limits_{x\in\varphi^{-1}(y)} f(x), & \text{if } \varphi^{-1}(y) \neq \emptyset; \\ 0, & \text{if } \varphi^{-1}(y) = \emptyset. \end{cases}$$

Using positive operators TT^* and T^*T we obtain an orthogonal decompositions:

$$l^2(X) = \oplus_{k=0}^{m} l^2(X_k), \quad l^2(X_k) = \{f \in l^2(X) : T^*Tf = kf\};$$

$$l^2(X) = \oplus_{k=0}^{m} l_k^2, \quad l_k^2 = \{f \in l^2(X) : TT^*f = kf\},$$

and

$$TT^* = \oplus_{k=1}^{m} kQ_k, \quad T^*T = \oplus_{k=1}^{m} kP_k,$$

where P_k is the projection onto $l^2(X_k)$ and Q_k is the projection onto l_k^2.
Projections P_k and Q_k are equivalent and mutually non permutable in general.

Family of partial isometries $\mathcal{U}$

U_k is the respective partial isometry with the initial space $l^2(X_k)$ and the final space l_k^2 and

$$T = U_1 + \sqrt{2}U_2 + \cdots + \sqrt{m}U_m$$

(certain of summand-operators can be zero).
We denote by $\mathcal{U}$ the set of respective partial isometries.

Theorem

$C_\varphi^*(X)$ is generated by the set $\mathcal{U}$ of partial isometries satisfying the equalities:

$$U_1^* U_1 + U_2^* U_2 + \cdots + U_m^* U_m = P_\varphi;$$

$$U_1 U_1^* + U_2 U_2^* + \cdots + U_m U_m^* = Q_\varphi;$$

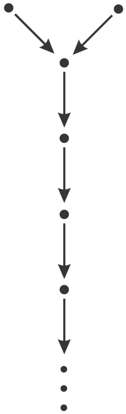
where P_φ and Q_φ are projections defined by the mapping φ.

Some examples of C^*-algebras generated by mappings

pic. 1

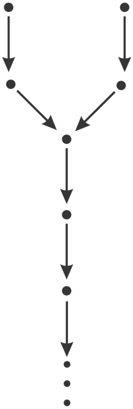
$C(S^1)$

pic. 2



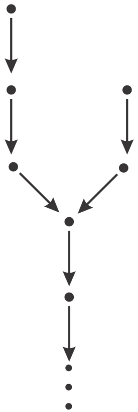
$\mathcal{T}$

pic. 3



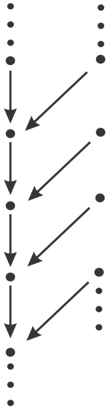
$\mathcal{T} \bigoplus M_2(\mathbb{C})$

pic. 4



$\mathcal{T}$

pic. 5



$C(S^1, M_2(\mathbb{C}))$

pic. 6

$\mathcal{T}$

In the report we suppose the mapping φ to be fixed satisfying the following conditions:

- there is no cyclic element in X,
- the graph (X, φ) is connected,
- the number of preimages is uniformly bounded i.e. a number m exists such that $m = \sup\limits_{x \in X} \operatorname{card}\{\varphi^{-1}(x)\} < \infty$.

Definition

Elements of the set $\mathcal{U} \bigcup \mathcal{U}^*$ we call elementary monomials.
Each finite product of elementary monomials we call a monomial.

The set of all monomials $\mathrm{Mon}(X)$ forms a semigroup with respect to multiplication operation or, in other words, $\mathrm{Mon}(X)$ is an involutive semigroup generated by the set $\mathcal{U}$.
In general $\mathrm{Mon}(X)$ is not inverse and the semigroup of idempotents is not commutative.

For each $a \in \mathcal{U} \bigcup \mathcal{U}^*$ we define

$$\operatorname{ind} a = 1,\ a \in \mathcal{U}, \quad \operatorname{ind} a = -1,\ a \in \mathcal{U}^*.$$

We consider $\operatorname{ind} V$ for $V \neq 0 \in \operatorname{Mon}(X)$ as the sum of the indices of the factors. We assume the index of zero operator to be 0.

Lemma

The index of monomial is well-defined, and for $V_1, V_2 \in \operatorname{Mon}(X)$

$$\operatorname{ind} V_1 V_2 = \operatorname{ind} V_1 + \operatorname{ind} V_2,\ V_1 V_2 \neq 0.$$

Let $C_{\varphi,n}$ be an operator subspace generated by monomials of index n. So $C_{\varphi,0}$ is a subalgebra generated by monomials of zero index.

We can equip $C^*_\varphi(X)$ with a circle action $\alpha : S^1 \longrightarrow \operatorname{Aut} C^*_\varphi(X)$, namely

$$\alpha_z(V) = z^{\operatorname{ind} V}\, V,\ V \in \operatorname{Mon}(X),\ z \in S^1.$$

Then $C_{\varphi,n} = \{A \in C^*_\varphi(X) : \alpha_z(A) = z^n A \quad \text{for} \quad z \in S^1\}$ is the n-th spectral subspace for the action α, and $C_{\varphi,0}$ is a fixed point subalgebra of $C^*_\varphi(X)$.

Theorem

Algebra $C^*_\varphi(X)$ is $\mathbb{Z}$-graded and the grading is generated by the covariant system $(C^*_\varphi(X), S^1, \alpha)$.
The fixed point subalgebra $C_{\varphi,0}$ is an AF-algebra, and hence $C^*_\varphi(X)$ is nuclear.

Theorem

The algebra $C^*_\varphi(X)$ is irreducible if and only if the equality

$$(We_x, e_x) = (We_y, e_y) \text{ for all } W \in \mathrm{Mon}(X) \quad \text{implies}$$

$$e_x = e_y.$$

Family $\{e_x\}_{x\in X}$ $(e_x(y) = \delta_{x,y})$ forms a natural basis in $l^2(X)$.

When $\mathrm{Mon}(X)$ is inverse

Mapping φ induces a partial order and an equivalence on X. We write

$$x \prec y \text{ if } \exists m \text{ s.t } \varphi^m(y) = x \text{ and } x \sim y \text{ if } \exists m \text{ s.t. } \varphi^m(x) = \varphi^m(y).$$

Theorem

The semigroup $\mathrm{Mon}(X)$ is inverse if and only if for any pair $x \sim y$ there is a bijection $\chi : X \longrightarrow X$ such that $\chi \circ \varphi = \varphi \circ \chi$ and $\chi(x) = y$.

Using the criterion of irreducibility we can determine another equivalence on X,

$$x \overset{\omega}{\sim} y \text{ if } (We_x, e_x) = (We_y, e_y) \text{ for all } W \in \mathrm{Mon}(X).$$

Structure of $C^*_\varphi(X)$ if $\mathrm{Mon}(X)$ is inverse

Let $\mathcal{U}$ be a tame set i.e. generates an inverse semigroup.

Theorem

The fixed point subalgebra $C_{\varphi,0}$ is commutative if and only if the semigroup $\mathrm{Mon}(X)$ is inverse. If this is the case, the subsemigroup $Mon(X)_0$ of monomials of zero index is a commutative subsemigroup of idempotents.

Let $\mathcal{E}(x) = \{y \in X : x \prec y\}$, and $\mathfrak{E}(x) = \{y \in X : x \sim y\}$.

Definition

We call X φ-tame if the corresponding family of partial isometries $\mathcal{U}$ is a tame set.

Structure of X if $\mathrm{Mon}(X)$ is inverse

Proposition

Let X be a φ-tame set. Let at list one $x \in X$ be such that $\mathfrak{E}(x)$ is finite. Then $\mathfrak{E}(x)$ is finite for all $x \in X$.

Proposition

Let X be a φ-tame set. Let x be such that there exists an $y \in \mathcal{E}(x)$ such that $x \overset{\omega}{\sim} y$. Then this is true for all $x \in X$.

In this case we will say that X satisfies the ω-property.
Let $\omega[x] = \{y \in \mathcal{E}(x) : x \overset{\omega}{\sim} y\}$. For all $y \in \omega[x]$ we have $\varphi^{k_n}(y) = x$, for some k_n. The minimal of such $\{k_n\}$ we call the period of the set $\omega[x]$.

Proposition

Let X be a φ-tame set satisfying the ω-property. Then for all $x \in X$ the sets $\omega[x]$ have the same period.

Invariant subspaces

Theorem (1)

Let $\mathcal{U}$ be a tame set and the mapping φ be not surjective. Let $\mathfrak{E}(x)$ be countable for all $x \in X$. Then

- $l^2(X) \simeq \bigoplus_{n=1}^{+\infty} \big(\bigoplus_{i \in I_n} H_i\big)$, *where I_n is countable;*
- *subspaces H_i are only invariant subspaces for the algebra $C^*_\varphi(X)$, they are finite dimensional and mutually isomorphic for n fixed;*
- *every $A \in C^*_\varphi(X)$ can be represented in the form $A = \big(\bigoplus_{\mathbb{Z}} A_1\big) \oplus A_0$, where $A_0 = A_{|_{(\bigoplus_{i \in I_n} H_i)^{\perp}}}$, $A_1 = A_{|_{H_1}}$;*
- *$C^*_\varphi(X)$ is isomorphic to a subalgebra of $\bigoplus_{k=1}^{\infty} M_{n_k}(\mathbb{C})$.*

Theorem (2)

Let $\mathcal{U}$ be a tame set and the mapping φ be not surjective. Let $\mathfrak{E}(x)$ be finite for all $x \in X$. Then

- $l^2(X) \simeq l^2(\mathbb{Z}_+) \oplus \Big(\bigoplus\limits_{\text{fin}}\big(\bigoplus\limits_{i \in I_n} H_i\big)\Big)$, *where I_n is finite;*
- *subspaces H_i (as well as $l^2(\mathbb{Z}_+)$) are only invariant subspaces for the algebra $C^*_\varphi(X)$, all H_i are finite dimensional and mutually isomorphic for n fixed;*
- *every $A \in C^*_\varphi(X)$ can be represented in the form $A = \big(\bigoplus\limits_{\text{fin}} A_1\big) \oplus A_0$, where $A_0 = A|_{(\bigoplus\limits_{i \in I_n} H_i)^\perp}$, $A_1 = A|_{H_1}$;*
- *$C^*_\varphi(X)$ is isomorphic to the direct sum $\mathcal{T} \oplus \big(\bigoplus\limits_k M_{n_k}\big)$, where $\mathcal{T}$ is the Toeplitz algebra.*

Theorem (3)

Let $\mathcal{U}$ be a tame set and the mapping φ be surjective. Let $\mathfrak{E}(x)$ be countable for all $x \in X$. Then

- $l^2(X) \simeq \bigoplus_{n=-\infty}^{+\infty} \big(\bigoplus_{i \in I_n} H_i \big)$, *where* I_n *is countable;*
- $H_i \simeq l^2(\mathbb{Z}_+)$ *are only invariant subspaces for* $C^*_\varphi(X)$*;*
- *every* $A \in C^*_\varphi(X)$ *can be represented in the form* $A = \big(\bigoplus_{\mathbb{Z}} A_1 \big) \oplus A_0$, *where* $A_0 = A|_{(\bigoplus_{i \in I_n} H_i)^\perp}$, $A_1 = A|_{H_1}$*;*
- *the restriction of* $C^*_\varphi(X)$ *to* H_i *is isomorphic to a* C^**-algebra generated by a weighted shift (w.s.). If* $H_{(n)}$ $(H_{(m)})$ *is one of the summands in* $\bigoplus_{i \in I_n} H_i$ *(corr.* $\bigoplus_{i \in I_m} H_i$*),* $n > m$*, then the respective operators of w.s. have the coefficients* $(\alpha_1, \alpha_2, \dots)$ *and* $(\beta_1, \cdots \beta_{n-m}, \alpha_1, \alpha_2, \dots)$*.*

Theorem (4)

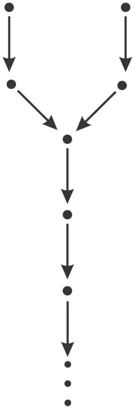
Let $\mathcal{U}$ be a tame set and the mapping φ be surjective. Let $\mathfrak{E}(x)$ be finite for all $x \in X$. Then

- $l^2(X) \simeq l^2(\mathbb{Z}) \oplus \left(\bigoplus_{n=0}^{\infty} \left(\bigoplus_{i \in I_n} H_i \right) \right)$, *where I_n is finite;*
- *subspaces H_i (as well as $l^2(\mathbb{Z})$) are only invariant subspaces of the algebra $C^*_\varphi(X)$, each H_i is isomorphic to $l^2(\mathbb{Z}_+)$;*
- *every $A \in C^*_\varphi(X)$ can be represented in the form $A = (\bigoplus_{\mathbb{Z}} A_1) \oplus A_0$, where $A_0 = A|_{(\bigoplus_{i \in I_n} H_i)^\perp}$, $A_1 = A|_{H_1}$;*
- *the restriction of $C^*_\varphi(X)$ to H_i ($l^2(\mathbb{Z})$) is isomorphic to the C^*-algebra generated by a w.s. (corr. bilateral w.s.). If $H_{(n)}$ ($H_{(m)}$) is one of the summands in $\bigoplus_{i \in I_n} H_i$ ($\bigoplus_{i \in I_m} H_i$), $n > m$, then the respective w.s. have the coefficients $(\alpha_1, \alpha_2, \ldots)$ and $(\beta_1, \cdots \beta_{n-m}, \alpha_1, \alpha_2, \ldots)$.*

Theorem (5)

Let $\mathcal{U}$ be a tame set and let X has the ω-property. Then in Theorem 3 the last statement is replaced by the following one: the restriction of $C^*_\varphi(X)$ to each invariant subspace is isomorphic to the C^*-algebra generated by a periodic weighted shift. The algebra $C^*_\varphi(X)$ is isomorphic to $M_n(\mathcal{T})$ where n is the period of the set $\omega[x]$.

Example to Theorem 2

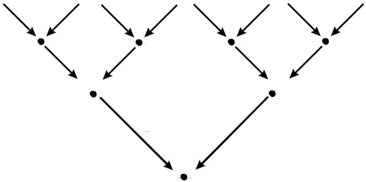


$$l^2(X) = l^2(\mathbb{Z}_+) \oplus H_2,$$

$$C^*_\varphi(X) \simeq \mathcal{T} \oplus M_2(\mathbb{C}),$$

$$C_{\varphi,0} \simeq \mathbb{C}^2 \oplus C_0(\mathbb{Z}_+) + \mathbb{C}I.$$

Example to Theorem 3



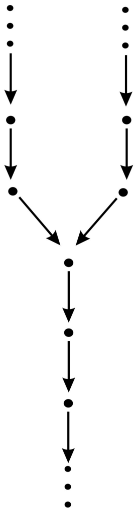
$$l^2(X) = \bigoplus_{\mathbb{Z}} l^2(\mathbb{Z}_+),$$

$$C^*_\varphi(X)|_{l^2(\mathbb{Z}_+)} = \mathcal{T},$$

$$C^*_\varphi(X) \simeq \mathcal{T},$$

$$C_{\varphi,0} \simeq C_0(\mathbb{Z}_+) + \mathbb{C}I.$$

Example to Theorem 4



$$l^2(X) \simeq l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z}_+),$$

$C^*_\varphi(X)|_{l^2(\mathbb{Z})}$ is the Toeplitz type algebra,

$$C^*_\varphi(X)|_{l^2(\mathbb{Z}_+)} = \mathcal{T},$$

$C^*_\varphi(X)$ is the Toeplitz type algebra,

$$C_{\varphi,0} \simeq C_0(\mathbb{Z}) + \mathbb{C}I.$$

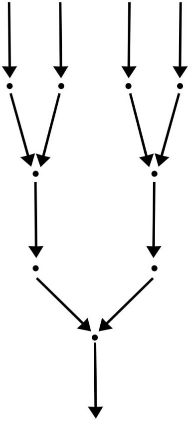
$$0 \longrightarrow K_1 \longrightarrow C^*_\varphi(X)_{|l^2(\mathbb{Z})} \longrightarrow C(S^1) \longrightarrow 0.$$

$$0 \longrightarrow K_1 \oplus K_2 \longrightarrow C^*_\varphi(X) \longrightarrow C(S^1) \longrightarrow 0.$$

$$0 \longrightarrow K_1 \longrightarrow C^*_\varphi(X) \longrightarrow \mathcal{T} \longrightarrow 0.$$

$$0 \longrightarrow K_2 \longrightarrow C^*_\varphi(X) \longrightarrow C(S^1) + K_1 \longrightarrow 0.$$

Example to Theorem 5



$$l^2(X) \simeq \bigoplus_{\mathbb{Z}} l^2(\mathbb{Z}_+),$$

$$C^*_\varphi(X)|_{l^2(\mathbb{Z}_+)} \simeq M_2(\mathcal{T}),$$

$$C^*_\varphi(X) \simeq M_2(\mathcal{T}),$$

$$C_{\varphi,0} \simeq \mathbb{C}^2 \otimes \big(C_0(\mathbb{Z}_+) + \mathbb{C}I\big).$$

Thank you!