$C^*$-algebras generated by mappings with an inverse semigroup of monomials

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09.09.2016
The starting point is a selfmapping $\varphi : X \to X$ on a countable set $X$. The mapping generates an oriented graph $(X, \varphi)$ with the elements of $X$ as vertices and pairs $(x, \varphi(x))$ as edges. We assume the cardinality of the preimage set of each point is finite and a number $m$ exists such that $m = \sup_{x \in X} \text{card} \{\varphi^{-1}(x)\} < \infty$.

The mapping $\varphi$ induces the composition operator $T_\varphi : l^2(X) \to l^2(X)$:

$$T_\varphi f = f \circ \varphi.$$

**Definition**

$C^*$-algebra $C^*_\varphi(X)$ generated by the mapping $\varphi$ is the $C^*$-algebra generated by the operator $T_\varphi$.

In what follows $T_\varphi = T$. 

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The conjugate operator $T^*$ can be calculated by the formula

$$(T^* f)(y) = \begin{cases} \sum_{x \in \varphi^{-1}(y)} f(x), & \text{if } \varphi^{-1}(y) \neq \emptyset; \\ 0, & \text{if } \varphi^{-1}(y) = \emptyset. \end{cases}$$

Using positive operators $TT^*$ and $T^* T$ we obtain an orthogonal decompositions:

$$l^2(X) = \bigoplus_{k=0}^{m} l^2(X_k), \quad l^2(X_k) = \{ f \in l^2(X) : T^* Tf = kf \};$$

$$l^2(X) = \bigoplus_{k=0}^{m} l^2_k, \quad l^2_k = \{ f \in l^2(X) : TT^* f = kf \},$$

and

$$TT^* = \bigoplus_{k=1}^{m} kQ_k, \quad T^* T = \bigoplus_{k=1}^{m} kP_k,$$

where $P_k$ is the projection onto $l^2(X_k)$ and $Q_k$ is the projection onto $l^2_k$.

Projections $P_k$ and $Q_k$ are equivalent and mutually non permutable in general.
Family of partial isometries $\mathcal{U}$

$U_k$ is the respective partial isometry with the initial space $l^2(X_k)$ and the final space $l^2_k$ and

$$T = U_1 + \sqrt{2}U_2 + \cdots + \sqrt{m}U_m$$

(certain of summand-operators can be zero).

We denote by $\mathcal{U}$ the set of respective partial isometries.

**Theorem**

$C^*_\varphi(X)$ is generated by the set $\mathcal{U}$ of partial isometries satisfying the equalities:

$$U_1^*U_1 + U_2^*U_2 + \cdots + U_m^*U_m = P_\varphi;$$

$$U_1U_1^* + U_2U_2^* + \cdots + U_mU_m^* = Q_\varphi;$$

where $P_\varphi$ and $Q_\varphi$ are projections defined by the mapping $\varphi$. 
Some examples of $C^*$-algebras generated by mappings

pic. 1

\[ C(S^1) \]

pic. 2

\[ \mathcal{T} \]

pic. 3

\[ \mathcal{T} \oplus M_2(\mathbb{C}) \]
pic. 4

\[ \mathcal{T} \]

pic. 5

\[ C(S^1, M_2(\mathbb{C})) \]

pic. 6

\[ \mathcal{T} \]
In the report we suppose the mapping \( \varphi \) to be fixed satisfying the following conditions:

- there is no cyclic element in \( X \),
- the graph \((X, \varphi)\) is connected,
- the number of preimages is uniformly bounded i.e. a number \( m \) exists such that
  \[
  m = \sup_{x \in X} \text{card}\{ \varphi^{-1}(x) \} < \infty.
  \]

**Definition**

*Elements of the set \( U \cup U^* \) we call elementary monomials. Each finite product of elementary monomials we call a monomial.*

The set of all monomials \( \text{Mon}(X) \) forms a semigroup with respect to multiplication operation or, in other words, \( \text{Mon}(X) \) is an involutive semigroup generated by the set \( U \).

In general \( \text{Mon}(X) \) is not inverse and the semigroup of idempotents is not commutative.
For each $a \in U \cup U^*$ we define

$$\text{ind } a = 1, \ a \in U, \ \text{ind } a = -1, \ a \in U^*.$$

We consider $\text{ind } V$ for $V \neq 0 \in \text{Mon}(X)$ as the sum of the indices of the factors. We assume the index of zero operator to be 0.

**Lemma**

*The index of monomial is well-defined, and for $V_1, V_2 \in \text{Mon}(X)$*

$$\text{ind } V_1 V_2 = \text{ind } V_1 + \text{ind } V_2, \ V_1 V_2 \neq 0.$$

Let $C_{\varphi,n}$ be an operator subspace generated by monomials of index $n$. So $C_{\varphi,0}$ is a subalgebra generated by monomials of zero index.

We can equip $C_{\varphi}^*(X)$ with a circle action $\alpha : S^1 \rightarrow \text{Aut } C_{\varphi}^*(X)$, namely

$$\alpha_z(V) = z^{\text{ind } V} V, \ V \in \text{Mon}(X), \ z \in S^1.$$
Grading and irreducibility

Then $C_{\varphi, n} = \{ A \in C_{\varphi}^*(X) : \alpha_z(A) = z^n A \text{ for } z \in S^1 \}$ is the $n$-th spectral subspace for the action $\alpha$, and $C_{\varphi, 0}$ is a fixed point subalgebra of $C_{\varphi}^*(X)$.

**Theorem**

Algebra $C_{\varphi}^*(X)$ is $\mathbb{Z}$-graded and the grading is generated by the covariant system $(C_{\varphi}^*(X), S^1, \alpha)$.

The fixed point subalgebra $C_{\varphi, 0}$ is an AF-algebra, and hence $C_{\varphi}^*(X)$ is nuclear.

**Theorem**

The algebra $C_{\varphi}^*(X)$ is irreducible if and only if the equality

$$(We_x, e_x) = (We_y, e_y) \text{ for all } W \in \text{Mon}(X)$$

implies

$$e_x = e_y.$$ 

Family $\{e_x\}_{x \in X}$ ($e_x(y) = \delta_{x,y}$) forms a natural basis in $l^2(X)$.
When $\text{Mon}(X)$ is inverse

Mapping $\varphi$ induces a partial order and an equivalence on $X$. We write

$$x \prec y \text{ if } \exists m \text{ s.t. } \varphi^m(y) = x \text{ and } x \sim y \text{ if } \exists m \text{ s.t. } \varphi^m(x) = \varphi^m(y).$$

**Theorem**

*The semigroup $\text{Mon}(X)$ is inverse if and only if for any pair $x \sim y$ there is a bijection $\chi : X \to X$ such that $\chi \circ \varphi = \varphi \circ \chi$ and $\chi(x) = y$.***

Using the criterion of irreducibility we can determine another equivalence on $X$,

$$x \overset{\omega}{\sim} y \text{ if } (W e_x, e_x) = (W e_y, e_y) \text{ for all } W \in \text{Mon}(X).$$
Let $\mathcal{U}$ be a tame set i.e. generates an inverse semigroup.

**Theorem**

The fixed point subalgebra $C_{\varphi, 0}$ is commutative if and only if the semigroup $\text{Mon}(X)$ is inverse. If this is the case, the subsemigroup $\text{Mon}(X)_0$ of monomials of zero index is a commutative subsemigroup of idempotents.

Let $\mathcal{E}(x) = \{y \in X : x \prec y\}$, and $\mathcal{C}(x) = \{y \in X : x \sim y\}$.

**Definition**

We call $X$ $\varphi$-tame if the corresponding family of partial isometries $\mathcal{U}$ is a tame set.
Structure of $X$ if $\text{Mon}(X)$ is inverse

**Proposition**

Let $X$ be a $\varphi$-tame set. Let at list one $x \in X$ be such that $\mathcal{E}(x)$ is finite. Then $\mathcal{E}(x)$ is finite for all $x \in X$.

**Proposition**

Let $X$ be a $\varphi$-tame set. Let $x$ be such that there exists an $y \in \mathcal{E}(x)$ such that $x \sim y$. Then this is true for all $x \in X$.

In this case we will say that $X$ satisfies the $\omega$-property.

Let $\omega[x] = \{y \in \mathcal{E}(x) : x \sim y\}$. For all $y \in \omega[x]$ we have $\varphi^{k_n}(y) = x$, for some $k_n$. The minimal of such $\{k_n\}$ we call the period of the set $\omega[x]$.

**Proposition**

Let $X$ be a $\varphi$-tame set satisfying the $\omega$-property. Then for all $x \in X$ the sets $\omega[x]$ have the same period.
Theorem (1)

Let $\mathcal{U}$ be a tame set and the mapping $\varphi$ be not surjective. Let $\mathcal{E}(x)$ be countable for all $x \in X$. Then

- $l^2(X) \cong \bigoplus_{n=1}^{+\infty} \left( \bigoplus_{i \in l_n} H_i \right)$, where $l_n$ is countable;
- subspaces $H_i$ are only invariant subspaces for the algebra $C^*_\varphi(X)$, they are finite dimensional and mutually isomorphic for $n$ fixed;
- every $A \in C^*_\varphi(X)$ can be represented in the form $A = \left( \bigoplus_{\mathbb{Z}} A_1 \right) \oplus A_0$, where $A_0 = A|_{(\bigoplus_{i \in l_n} H_i)^\perp}$, $A_1 = A|_{H_1}$;
- $C^*_\varphi(X)$ is isomorphic to a subalgebra of $\bigoplus_{k=1}^{\infty} M_{n_k}(\mathbb{C})$. 
Theorem (2)

Let $\mathcal{U}$ be a tame set and the mapping $\varphi$ be not surjective. Let $\mathcal{E}(x)$ be finite for all $x \in X$. Then

$\triangleright$ $l^2(X) \simeq l^2(\mathbb{Z}_+) \oplus \left( \bigoplus_{\text{fin}} \bigoplus_{i \in I_n} H_i \right)$, where $I_n$ is finite;

$\triangleright$ subspaces $H_i$ (as well as $l^2(\mathbb{Z}_+)$) are only invariant subspaces for the algebra $C^*_\varphi(X)$, all $H_i$ are finite dimensional and mutually isomorphic for $n$ fixed;

$\triangleright$ every $A \in C^*_\varphi(X)$ can be represented in the form $A = \left( \bigoplus_{\text{fin}} A_1 \right) \oplus A_0$, where $A_0 = A|_{\left( \bigoplus_{i \in I_n} H_i \right)^\perp}$, $A_1 = A|_{H_1}$;

$\triangleright$ $C^*_\varphi(X)$ is isomorphic to the direct sum $\mathcal{T} \oplus \left( \bigoplus_k M_{n_k} \right)$, where $\mathcal{T}$ is the Toeplitz algebra.
Invariant subspaces

Theorem (3)

Let \( \mathcal{U} \) be a tame set and the mapping \( \varphi \) be surjective. Let \( \mathcal{E}(x) \) be countable for all \( x \in X \). Then

- \( l^2(X) \simeq \bigoplus_{n=-\infty}^{+\infty} \left( \bigoplus_{i \in I_n} H_i \right) \), where \( I_n \) is countable;
- \( H_i \simeq l^2(\mathbb{Z}+) \) are only invariant subspaces for \( C^*_\varphi(X) \);
- every \( A \in C^*_\varphi(X) \) can be represented in the form \( A = \left( \bigoplus_{\mathbb{Z}} A_1 \right) \oplus A_0 \), where \( A_0 = A\bigg|_{\left( \bigoplus_{i \in I_n} H_i \right)^\perp} \), \( A_1 = A\bigg|_{H_1} \);
- the restriction of \( C^*_\varphi(X) \) to \( H_i \) is isomorphic to a \( C^* \)-algebra generated by a weighted shift (w.s.). If \( H_{(n)} \) (\( H_{(m)} \)) is one of the summands in \( \bigoplus_{i \in I_n} H_i \) (corr. \( \bigoplus_{i \in I_m} H_i \)), \( n > m \), then the respective operators of w.s. have the coefficients \( (\alpha_1, \alpha_2, \ldots) \) and \( (\beta_1, \ldots, \beta_{n-m}, \alpha_1, \alpha_2, \ldots) \).
Invariant subspaces

**Theorem (4)**

Let $\mathcal{U}$ be a tame set and the mapping $\varphi$ be surjective. Let $\mathfrak{C}(x)$ be finite for all $x \in X$. Then

- $l^2(X) \simeq l^2(\mathbb{Z}) \oplus \left( \bigoplus_{n=0}^{\infty} \left( \bigoplus_{i \in I_n} H_i \right) \right)$, where $I_n$ is finite;

- subspaces $H_i$ (as well as $l^2(\mathbb{Z})$) are only invariant subspaces of the algebra $C^*_\varphi(X)$, each $H_i$ is isomorphic to $l^2(\mathbb{Z}_+)$;

- every $A \in C^*_\varphi(X)$ can be represented in the form $A = (\bigoplus A_1) \oplus A_0$, where $A_0 = A_{\mid (\bigoplus_{i \in I_n} H_i)^\perp}$, $A_1 = A_{\mid H_1}$;

- the restriction of $C^*_\varphi(X)$ to $H_i$ ($l^2(\mathbb{Z})$) is isomorphic to the $C^*$-algebra generated by a w.s. (corr. bilateral w.s.). If $H_{(n)}$ ($H_{(m)}$) is one of the summands in $\bigoplus_{i \in I_n} H_i$ ($\bigoplus_{i \in I_m} H_i$), $n > m$, then the respective w.s. have the coefficients $(\alpha_1, \alpha_2, \ldots)$ and $(\beta_1, \cdots \beta_{n-m}, \alpha_1, \alpha_2, \ldots)$. 

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Theorem (5)

Let $\mathcal{U}$ be a tame set and let $X$ has the $\omega$-property. Then in Theorem 3 the last statement is replaced by the following one: the restriction of $C^*_\varphi(X)$ to each invariant subspace is isomorphic to the $C^*$-algebra generated by a periodic weighted shift. The algebra $C^*_\varphi(X)$ is isomorphic to $M_n(T)$ where $n$ is the period of the set $\omega[x]$. 
Example to Theorem 2

\[ l^2(X) = l^2(\mathbb{Z}_+) \oplus H_2, \]

\[ C^*_\varphi(X) \cong \mathcal{T} \oplus M_2(\mathbb{C}), \]

\[ C_{\varphi,0} \cong \mathbb{C}^2 \oplus C_0(\mathbb{Z}_+) + \mathbb{C}I. \]
Example to Theorem 3

\[ l^2(X) = \bigoplus_{\mathbb{Z}} l^2(\mathbb{Z}_+) , \]

\[ C_{\varphi}^*(X)|_{l^2(\mathbb{Z}_+)} = \mathcal{T} , \]

\[ C_{\varphi}(X) \simeq \mathcal{T} , \]

\[ C_{\varphi,0} \simeq C_0(\mathbb{Z}_+) + \mathbb{C}I . \]
Example to Theorem 4

\[ l^2(X) \cong l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z}_+), \]
\[ C^*_\varphi(X)|_{l^2(\mathbb{Z})} \text{ is the Toeplitz type algebra,} \]
\[ C^*_\varphi(X)|_{l^2(\mathbb{Z}_+)} = T, \]
\[ C^*_\varphi(X) \text{ is the Toeplitz type algebra,} \]
\[ C^*_{\varphi,0} \cong C_0(\mathbb{Z}) + \mathbb{C}I. \]

\[ 0 \rightarrow K_1 \rightarrow C^*_\varphi(X)|_{l^2(\mathbb{Z})} \rightarrow C(S^1) \rightarrow 0. \]
\[ 0 \rightarrow K_1 \oplus K_2 \rightarrow C^*_\varphi(X) \rightarrow C(S^1) \rightarrow 0. \]
\[ 0 \rightarrow K_1 \rightarrow C^*_\varphi(X) \rightarrow T \rightarrow 0. \]
\[ 0 \rightarrow K_2 \rightarrow C^*_\varphi(X) \rightarrow C(S^1) + K_1 \rightarrow 0. \]
Example to Theorem 5

\[ l^2(X) \cong \bigoplus_{\mathbb{Z}} l^2(\mathbb{Z}_+) , \]

\[ C_\varphi^*(X)|_{l^2(\mathbb{Z}_+)} \cong M_2(\mathcal{I}) , \]

\[ C_\varphi^*(X) \cong M_2(\mathcal{I}) , \]

\[ C_{\varphi,0} \cong \mathbb{C}^2 \otimes (C_0(\mathbb{Z}_+) + \mathbb{C}1) . \]
Thank you!