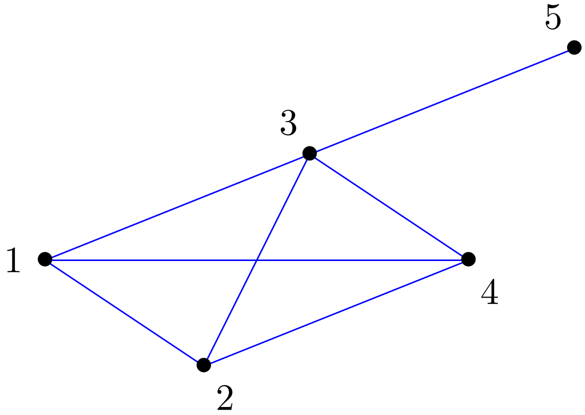
The random interchange process on the hypercube

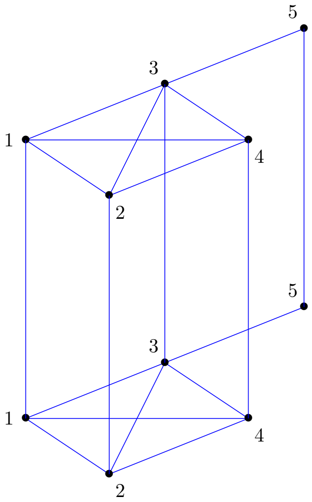
Roman Kotecký

A joint work with P. Miłoś (Warsaw) and D. Ueltschi (Warwick)

Warwick/Prague


5
3
1
4
2

Random transpositions, resulting permutations and their cycles



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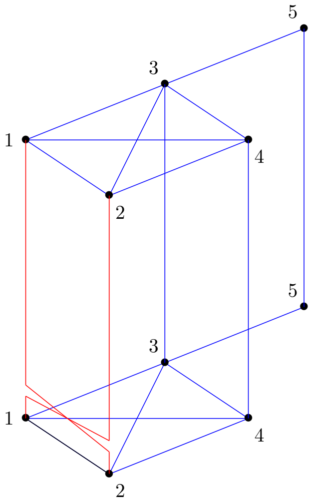

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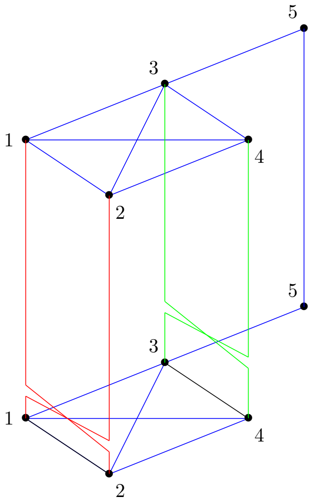

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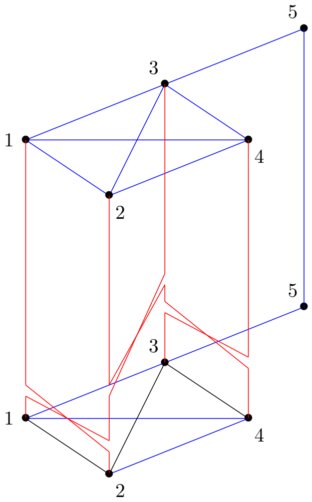

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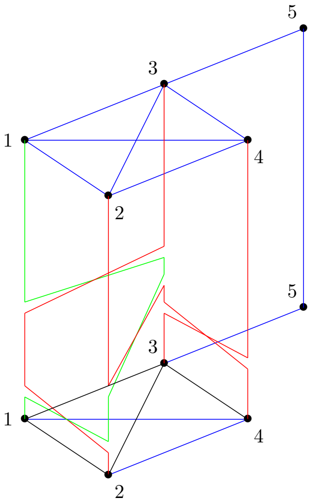

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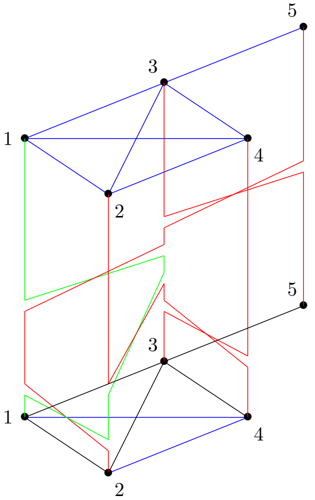

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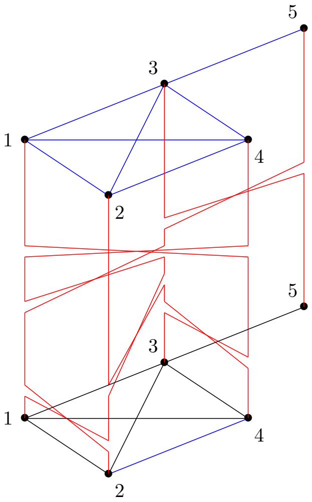

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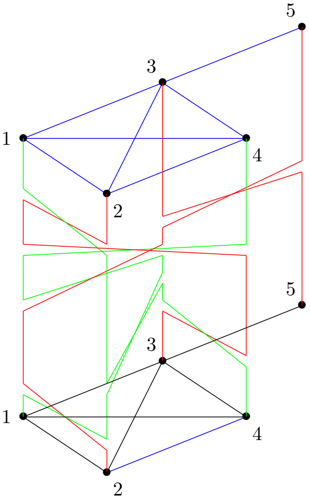

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Random transpositions on a complete graph

Card shuffling

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Problem makes sense for an arbitrary graph: for large times t, do cycles of length $\sim N$ exist?

Connection with physics

Bálint Tóth, 93 : Heisenberg ferromagnet via random interchange process on $\mathbb{Z}^d$:

the spontaneous magnetization

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Phase transition is linked with the ocurrence of long cycles

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No giant component $\implies$ only small cycles

$t = cN$
$p = \frac{cN}{Nd/2} = \frac{2c}{d} \implies t_c \sim \frac{1}{2}N$

Theorem

Let $c < 1/2$ and $\epsilon > 0$. Then there exists d_0 such that

$$\mathbb{P}(|V_t(\kappa d)| = 0) > 1 - \epsilon\kappa^{-3/2}$$

for all $t \leq cN$, all $\kappa \geq \frac{2\ln 2}{(1-2c)^2}$, and all $d > d_0$.

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$$\frac{1}{\Delta_d T} \sum_{t=T+1}^{\lfloor (1+\Delta_d)T \rfloor} \mathbb{E}\Big(\frac{|V_t(N^a)|}{N} \Big) \geq \eta(c) - a.$$

For $c > 1$, *we can take* $\eta(c) = \frac{1}{2}(1 - \frac{1}{c})$.

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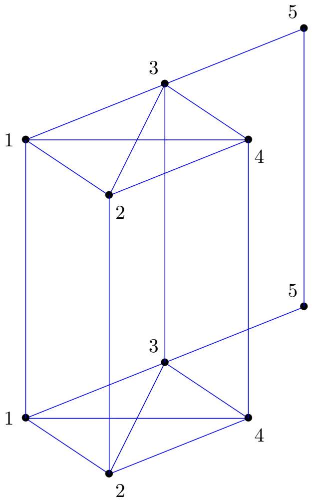
The main obstacle:
Unlike percolation clusters, the cycles do not grow monotonously —
an interchange over an edge with endpoints in a cycle splits it into two shorter cycles.

Dynamics of merging and splitting cycles



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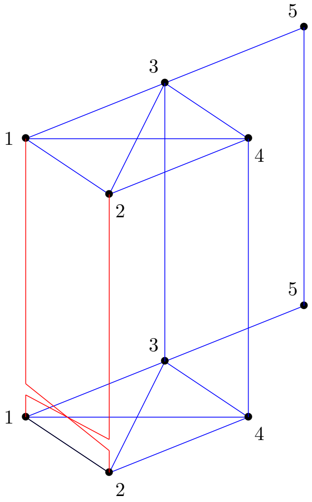
$N_t = \tilde{N}_t = 5$

Dynamics of merging and splitting cycles

Merging clusters (and corresponding cycles): $\widetilde{M}_t$



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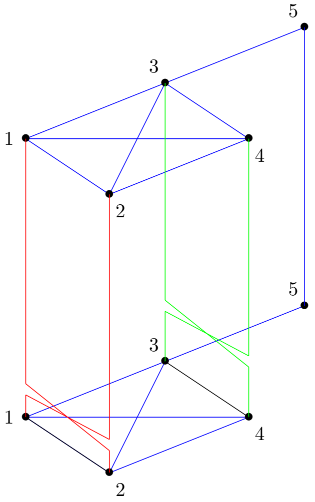

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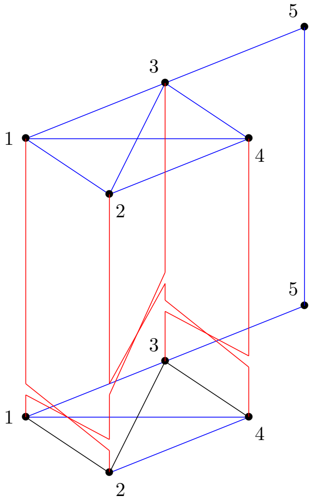

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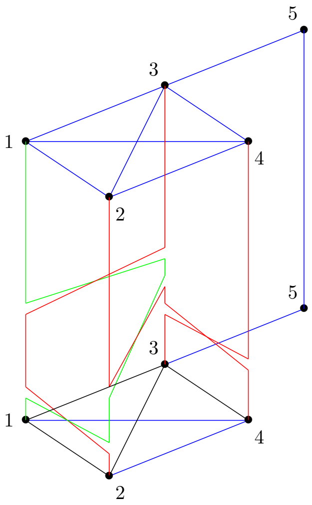
Dynamics of merging and splitting cycles

Splitting a cycle (within a cluster) S_t



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$$N_t = 3, \tilde{N}_t = 2, \Delta N_t = 1, \Delta \tilde{N}_t = 0$$

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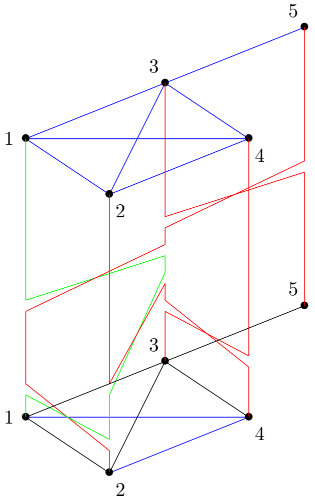

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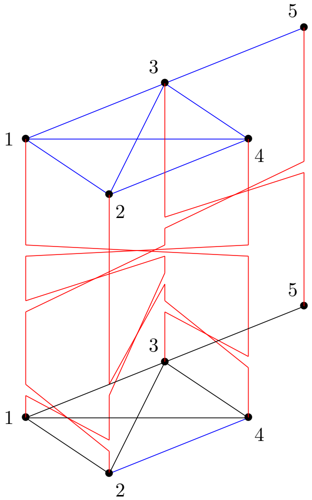
Dynamics of merging and splitting cycles
Merging cycles (within a cluster) M_t



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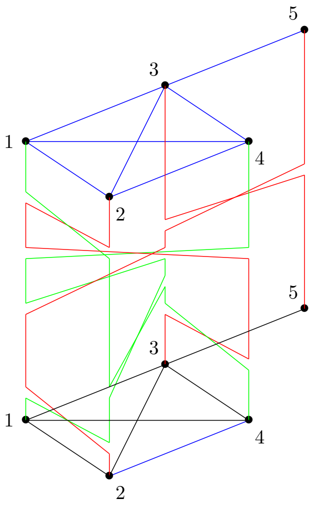

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6 steps of the proof:

1) Slightly paradoxically: if $\mathbb{P}(S_t) \geq \lambda$ then $\mathbb{E}\left(\frac{|V_t(N^a)|}{N}\right) \geq \frac{\lambda - a}{1-a}$.

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2) $\sum_{t=T+1}^{T+L}\mathbb{P}(S_t) \geq \frac{1}{2}\sum_{t=T+1}^{T+L}\mathbb{P}(I_t) - \frac{1}{2}\mathbb{E}(N_T - \widetilde{N}_T)$

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$$N_t^{>2d} \leq \frac{|V_t(2d)|}{2d} \implies \mathbb{E}(N_t^{>2d}) \leq \frac{\epsilon(d)\kappa^{-3/2}}{2d}$$

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By considering transpositions within $\widetilde{V}_t$:

$\mathbb{P}(I_t) = \frac{c'd}{Nd}\mathbb{E}(|\widetilde{V}_t \cap W_t|) \geq \frac{c'}{N}(|\widetilde{V}_t| - |W_t^{\mathsf{c}}|)$