Diffusive behaviour of ergodic sums over rotations

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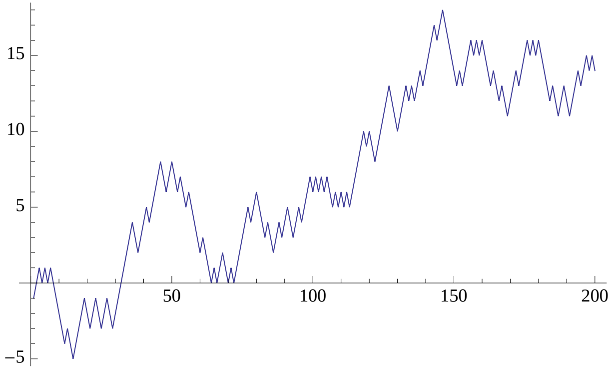

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For an irrational rotation $T(x) := x + \alpha \pmod 1$ let $[a_1, a_2, a_3, \dots]$ be the continued fraction expansion of α and

$$\frac{p_k}{q_k} := [a_1, a_2, \dots, a_k] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_k}}}}$$

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For $q_k \leq n < q_{k+1}$ one can use the Ostrowski representation: $n = \sum_{i=0}^{k} c_i q_i$ with $0 \leq c_i \leq a_{i+1}$ to get the upper bound,

$$\|S_n(f, \alpha)\|_\infty \leq V(f) \sum_{i=1}^{k+1} a_i$$

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  Example: for $\alpha = e-2$ we have $a_{3k-1} = 2k$ and $a_n = 1$ otherwise (so $\tau = 0$), and one finds

$$\|S_n(f,\alpha)\|_\infty = O(\log^2 n / \log^2(\log n))$$

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Some basic spectral theory

- $\rho(k) := \mu(f \cdot f \circ T^k) = \int_0^1 e^{2\pi i k\lambda}\sigma_f(d\lambda)$, where the measure σ_f on $(0,1]$ is the *spectral type* of f, and

$$DS_n = \sum_{k=-n+1}^{n-1}(n-|k|)\rho(k) = \int_0^1 \Phi_n(\lambda)\sigma_f(d\lambda),$$

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- $\langle DS_n\rangle := \frac{1}{n}\sum_{k=0}^{n-1} DS_k$ satisfies (finite or infinite)

$$\lim_{n\to\infty}\langle DS_n\rangle = \int_0^1 (2\sin^2(\pi\lambda))^{-1}\sigma_f(d\lambda)$$

The α-rotation has eigenvalues $\lambda_r = e^{2\pi i\, r\, \alpha}$ with eigenvectors $e_r(x) = e^{2\pi i\, r\, x}$, hence

$$\sigma_f(d\lambda) = \sum_{r\in\mathbb{Z}} |f_r|^2 \delta(\lambda - \{r\alpha\})\, d\lambda \quad , \quad f_r = (f, e_r)$$

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- $DS_n \to 0$ along the subsequence $n = q_k$, $k \to \infty$.
- $\lim_{n\to\infty} \langle DS_n \rangle = \infty$.

Upper bounds for DS_n

Let $\mathcal{C}$ be the class of functions $f \in L_2(X, \mu)$ with $\mu(f) = 0$ and Fourier coefficients

$$f_r = \frac{c_r}{r} \quad \text{with} \quad K(f) = \sup_{r \neq 0} |c_r| < \infty$$

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Theorem For $f \in \mathcal{C}$ and $q_k \leq n < q_{k+1}$ we have

$$DS_n \leq C \sum_{i=0}^{k} a_{i+1}^2$$

In particular for α of bounded type we have

$$\max_{q_k \leq n < q_{k+1}} DS_n = O(k) = O(\log n)$$

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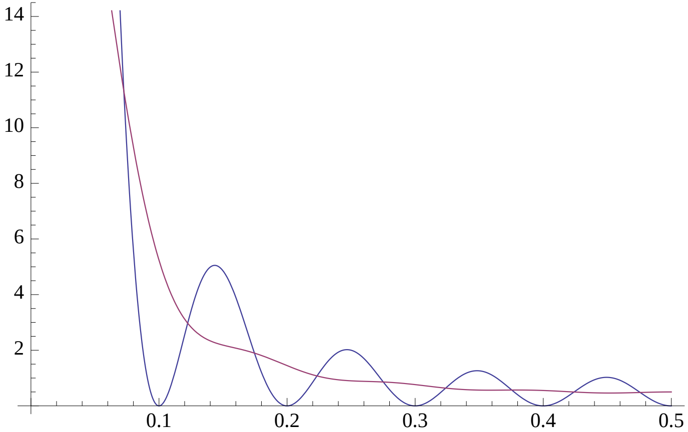

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If $a_i = O(1)$ and $\beta = 1/2$ we get a logarithmic lower bound.

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Indeed, it turns out that (using the lower bounds obtained above) we can use a slight extension of a theorem due to Berkes and Philipp (Acta Math. Acad. Sci. Hungar. 34 (1979), no. 1-2, 141-155) giving an approximation of lacunary series of the form $\sum_{k=1}^{N} f_k(n_k x)$ by a Wiener process (ASIP).

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If, moreover, $a_{n_k+1} \geq k^{\gamma}$, with $\gamma > 1$, then, keeping its distribution, the sequence $(S_{\ell_N})_{N\geq 1}$ can be redefined on a larger probability space together with a Wiener process $W(t)$ s.t.

$$S_{\ell_N} = W(\tau_N) + o(N^{\frac{1}{2}-\lambda}) \quad a.s.$$

where $\lambda > 0$ is an absolute constant and τ_N is an increasing sequence of r.v.'s such that $\tau_N/\sqrt{DS_{\ell_N}} \to 1$ a.s.

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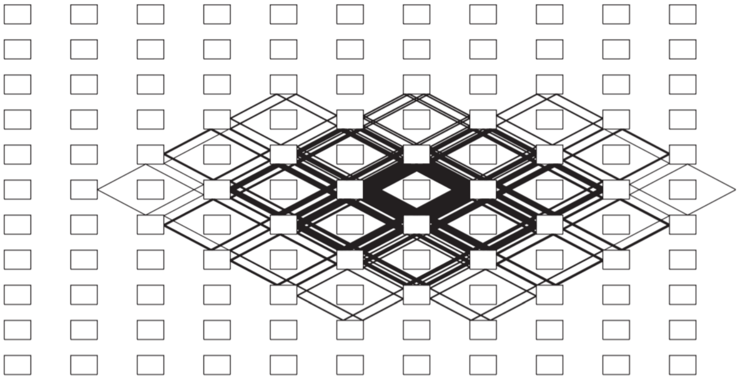

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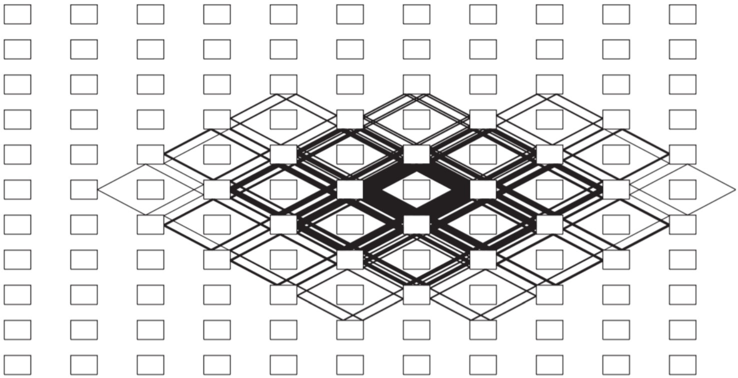

An orbit of the billiard flow in direction (1, 1) for the rectangular Lorenz gas.

Under the geometrical condition of "small obstacles":

$$pa + qb \leq 1$$

one can decompose (the conservative part of) the billiard map in the (rational) direction (p, q), on the polygonal surface $\mathbb{R}^2 \setminus \cup_{(m,n)\in\mathbb{Z}^2} R_{(m,n)}(a, b)$, into $2pq$ isomorphic ergodic components, where it can be expressed as a skew product over a rotation, namely a map of the form:

$$(x, y) \to (x + \alpha, y + \varphi(x))$$

where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and the displacement function $\varphi : \mathbb{R}/\mathbb{Z} \to \mathbb{Z}^2$ is a suitable step function.

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Then apply the above results...