A limiting characteristic polynomial of GUE

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Outline

1 Introduction

2 Motivation and related works

3 Initial Settings and Known Results

4 Main Result

5 Open Questions

Introduction

- Let H_n be a random $n \times n$ Hermitian matrix such that its probability distribution is

$$P(dH_n) \propto e^{-\mathrm{Tr}(V(H_n))} dH_n,$$

where $V : \mathbb{R} \to \mathbb{R}$ is the potential.

- Hence, its eigenvalue density function is

$$\frac{1}{Z(n)} e^{-\sum_{i=1}^{n} V(\lambda_i)} \prod_{i>j} |\lambda_i - \lambda_j|^2.$$

- We consider ratios of characteristic polynomials:

$$\frac{\det(\alpha_1 - H_n) \cdots \det(\alpha_k - H_n)}{\det(\beta_1 - H_n) \cdots \det(\beta_k - H_n)}. \tag{1}$$

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- Methods used: **classical analysis, representation theory, supersymmetry techniques.**

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- **Andreev, Simons**[1]: study expectation of type (1) in quantum chaotic systems with broken T-invariance in the case of GUE.
- **Borodin, Strahov** [2] expectation of the ratios of type (1) in the case of GUE, GOE and GSE. Explicit determinantal expressions have been obtained.
- **Fyodorov, Strahov** [3] ratios and products have been studied for the even polynomial potential V. Exact and asymptotic determinantal expressions have been obtained (Riemann-Hilbert approach).

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Related works

Chhaibi, Najnudel and Nikeghbali: [4]

- U_n is $n \times n$ CUE, i.e. its eigenphases density function is given by

$$\frac{1}{Z_n^{\text{CUE}}} \prod_{k<j} \left| e^{i\theta_j} - e^{i\theta_k} \right|^2 .$$

- $\xi_n^{CUE}(s) := \frac{\det\left(\text{Id}-U_n^{-1} e^{2i\pi s/n}\right)}{\det(\text{Id}-U_n^{-1})}$.

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Related Works

- $\xi_\infty(s) := \lim_{Y\to\infty} \prod_{|y_i|\le Y} \left(1 - \frac{s}{y_i}\right)$,
  - converges for all $s \in \mathbb{C}$,
  - is a random entire function.

  $\xi_\infty(s)$ is called the **limiting characteristic polynomial**.
- Points y_i form a determinantal point process with sine kernel.
- The determinantal sine-kernel point process is a point process with the r-point correlation function ρ_r given as follows:

$$\rho_r(x_1, \dots, x_r) = \det\left(\frac{\sin(\pi(x_i - x_j))}{\pi(x_i - x_j)}\right)_{1\le i,j\le r}.$$

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Theorem (Chhaibi, Najnudel and Nikeghbali)

$\xi_n^{CUE}(s)$ converge in law to $e^{i\pi s}\xi_\infty(s)$, endowed with the topology of uniform convergence on compact sets.

- In order to prove the above mentioned convergence in distribution, a stronger result, a. s. convergence, was also shown. This has been achieved by using the recursive representation of Haar measure and virtual isometries (Bourgade, Hughes, Najnudel Nikeghbali, Yor).

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Semicircular law

From now on we restrict ourselves to the case of GUE ($V(x) = \frac{x^2}{2}$).

Theorem (The semicircular law)

For any continuous and compactly supported function $f : \mathbb{R} \to \mathbb{C}$,

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{\lambda_i}{\sqrt{n}}\right)\right] = \int \rho_{sc}(x) f(x)\, dx,$$

with

$$\rho_{sc}(x) := \frac{1}{2\pi}\sqrt{(4 - x^2)_+}.$$

Determinantal Point Processes and GUE

- The eigenvalue density for GUE is

$$\frac{1}{Z^{GUE}(n)} e^{-\sum_{i=1}^{n} \frac{\lambda_i^2}{2}} \prod_{i>j} |\lambda_i - \lambda_j|^2 \propto \det\left(K_n(\lambda_i, \lambda_j)\right)_{1\leq i,j\leq n},$$

where $K_n(x,y) = \sum_{k=0}^{n-1} H_k(x)e^{-\frac{x^2}{4}} H_k(y)e^{-\frac{y^2}{4}}$.

- Hence, the eigenvalues of GUE form a determinantal point process.
- Gaudin, Mehta: Moreover, after normalization they converge to sine-determinantal PP.

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Determinantal Point Processes and GUE

Theorem (Gaudin-Mehta)

For any $E \in (-2, 2)$ and any piecewise continuous and compactly supported function η, the random variable

$$\sum_{i=1}^{\infty} \eta\Big(n\rho_{sc}(E)\Big(\frac{\lambda_i}{\sqrt{n}} - E\Big)\Big),$$

tends as $n \to \infty$ in distribution to

$$\sum_i \eta(y_i).$$

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Main Result

Theorem (Main result)

We define the random analytic function

$$\xi_n^{GUE}(s) := \frac{\det\left(-\frac{s\pi}{\sqrt{n}} + H_n\right)}{\det(H_n)},$$

Then in the topology of uniform convergence on compact sets in the variable s, in distribution,

$$\xi_n^{GUE}(s) \to \xi_\infty(s).$$

Idea of the Proof

- For any fixed $s \in \mathbb{C}$ and $K > 0$

$$\mathbb{P}\left(\left|\log\left|\xi_n^{GUE}(s)\right|\right| \geq x\right) = O_K(e^{-Kx}),$$

uniformly in n.

- We need to obtain fine estimates for

$$\mathbb{E}(\sum_i f(\lambda_i)) = \int_{\mathbb{R}} f(x)K_n(x,x)dx,$$

$$\mathrm{Var}\left(\sum_i f(\lambda_i)\right) = \frac{1}{2}\iint_{\mathbb{R}^2}(f(x)-f(y))^2 K_n^2(x,y)dxdy.$$

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Consequence

Remark

The similar result has been obtained for the ratios of the type

$$\frac{\det\Big(-\frac{s}{\rho_{sc}(E)\sqrt{n}}-E\sqrt{n}+H_n\Big)}{\det(-E\sqrt{n}+H_n)},$$

for fixed $E\in(-2,2)$.

Consequence

$$\prod_{i=1}^{k}\frac{\det\Big(-\frac{s_i}{\rho_{sc}(E)\sqrt{n}}-E\sqrt{n}+H_n\Big)}{\det\Big(-\frac{u_i}{\rho_{sc}(E)\sqrt{n}}-E\sqrt{n}+H_n\Big)}\to\prod_{i=1}^{k}\frac{\xi^{\infty}(s_i)}{\xi^{\infty}(u_i)}.$$

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- The ratios for the general potential V.
  For example: when V is a polynomial with even degree with a positive leading coefficient. Eigenvalues are determinantal PP. Instead of Hermite polynomials we have orthonormal polynomial w.r.t the weight $e^{-V(x)}$ (Deift 2000).
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Open Questions

- Ratios at the edge:

$$\frac{\det\left(2\sqrt{n} + \frac{s}{n^{1/6}} - H_n\right)}{\det\left(2\sqrt{n} - H_n\right)}$$

The zeros of the limiting characteristic polynomial will form Airy determinantal PP.

End

Thank you for the attention!!