A limiting characteristic polynomial of GUE

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Outline

1. Introduction
2. Motivation and related works
3. Initial Settings and Known Results
4. Main Result
5. Open Questions
Let $H_n$ be a random $n \times n$ Hermitian matrix such that its probability distribution is

$$P(dH_n) \propto e^{-\text{Tr}(V(H_n))} dH_n,$$

where $V : \mathbb{R} \rightarrow \mathbb{R}$ is the potential.

Hence, its eigenvalue density function is

$$\frac{1}{Z(n)} e^{-\sum_{i=1}^{n} V(\lambda_i)} \prod_{i > j} |\lambda_i - \lambda_j|^2.$$

We consider ratios of characteristic polynomials:

$$\frac{\det(\alpha_1 - H_n) \cdots \det(\alpha_k - H_n)}{\det(\beta_1 - H_n) \cdots \det(\beta_k - H_n)}. \quad (1)$$
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Motivation and related works

- **Andreev, Simons**\(^1\): study expectation of type (1) in quantum chaotic systems with broken \(T\)-invariance in the case of GUE.

- **Borodin, Strahov**\(^2\) expectation of the ratios of type (1) in the case of GUE, GOE and GSE. Explicit determinantal expressions have been obtained.

- **Fyodorov, Strahov**\(^3\) ratios and products have been studied for the even polynomial potential \(V\). Exact and asymptotic determinantal expressions have been obtained (Riemann-Hilbert approach).

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\(^1\) *Phys. Rev. Lett.*, 75, 1995


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Chhaibi, Najnudel and Nikeghbali:  

- $U_n$ is $n \times n$ CUE, i.e. its eigenphases density function is given by

\[
\frac{1}{Z_n^{\text{CUE}}} \prod_{k<j} |e^{i\theta_j} - e^{i\theta_k}|^2.
\]

- $\xi_{\text{CUE}}(s) := \frac{\det(\text{Id} - U_n^{-1} e^{2i\pi s/n})}{\det(\text{Id} - U_n^{-1})}$.  

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\( \xi_\infty(s) := \lim_{Y \to \infty} \prod_{|y_i| \leq Y} \left( 1 - \frac{s}{y_i} \right), \)

- converges for all \( s \in \mathbb{C}, \)
- is a random entire function.

\( \xi_\infty(s) \) is called the **limiting characteristic polynomial**.

- Points \( y_i \) form a determinantal point process with sine kernel.
- The determinantal sine-kernel point process is a point process with the \( r \)-point correlation function \( \rho_r \) given as follows:

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\rho_r(x_1, \ldots, x_r) = \det \left( \frac{\sin(\pi(x_i - x_j))}{\pi(x_i - x_j)} \right)_{1 \leq i, j \leq r}.
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**Theorem** (Chhaibi, Najnudel and Nikeghbali)

\( \xi_n^{CUE}(s) \) converge in law to \( e^{i\pi s} \xi_\infty(s) \), endowed with the topology of uniform convergence on compact sets.

In order to prove the above mentioned convergence in distribution, a stronger result, a. s. convergence, was also shown. This has been achieved by using the recursive representation of Haar measure and virtual isometries (Bourgade, Hughes, Najnudel Nikeghbali, Yor).
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From now on we restrict ourselves to the case of GUE \((V(x) = \frac{x^2}{2})\).

**Theorem** (The semicircular law)

For any continuous and compactly supported function \(f : \mathbb{R} \rightarrow \mathbb{C}\),

\[
\lim_{n \to \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} f \left( \frac{\lambda_i}{\sqrt{n}} \right) \right] = \int \rho_{sc}(x) f(x) \, dx,
\]

with

\[
\rho_{sc}(x) := \frac{1}{2\pi} \sqrt{(4 - x^2)_+}.
\]
The eigenvalue density for GUE is

\[
\frac{1}{Z_{\text{GUE}}(n)} e^{-\sum_{i=1}^{n} \frac{\lambda_i^2}{2}} \prod_{i>j} |\lambda_i - \lambda_j|^2 \propto \det(K_n(\lambda_i, \lambda_j))_{1 \leq i,j \leq n},
\]

where \( K_n(x, y) = \sum_{k=0}^{n-1} H_k(x) e^{-\frac{x^2}{4}} H_k(y) e^{-\frac{y^2}{4}} \).

Hence, the eigenvalues of GUE form a determinantal point process.

Gaudin, Mehta: Moreover, after normalization they converge to sine-determinantal PP.
Determinantal Point Processes and GUE

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Gaudin, Mehta: Moreover, after normalization they converge to sine-determinantal PP.
**Theorem** (Gaudin-Mehta)

For any $E \in (-2, 2)$ and any piecewise continuous and compactly supported function $\eta$, the random variable

$$\sum_{i=1}^{\infty} \eta\left(n \rho_{sc}(E) \left(\frac{\lambda_i}{\sqrt{n}} - E\right)\right),$$

tends as $n \to \infty$ in distribution to

$$\sum_{i} \eta(y_i).$$
Theorem (Main result)

*We define the random analytic function*

\[ \xi_{n}^{GUE}(s) := \frac{\det \left( -\frac{s\pi}{\sqrt{n}} + H_{n} \right)}{\det(H_{n})}, \]

*Then in the topology of uniform convergence on compact sets in the variable s, in distribution,*

\[ \xi_{n}^{GUE}(s) \to \xi_{\infty}(s). \]
For any fixed $s \in \mathbb{C}$ and $K > 0$

$$
\mathbb{P}\left( |\log |\xi_n^{GUE}(s)|| \geq x \right) = O_K(e^{-Kx}),
$$

uniformly in $n$.

We need to obtain fine estimates for

$$
\mathbb{E}\left( \sum_i f(\lambda_i) \right) = \int_{\mathbb{R}} f(x) K_n(x, x) dx,
$$

$$
\text{Var}\left( \sum_i f(\lambda_i) \right) = \frac{1}{2} \iint_{\mathbb{R}^2} (f(x) - f(y))^2 K_n^2(x, y) dxdy.
$$

It has been achieved by using Plancherel-Rotach asymptotics:
For any fixed $s \in \mathbb{C}$ and $K > 0$

$$\mathbb{P}(\|\log |\xi^{GUE}_n(s)|\| \geq x) = O_K(e^{-Kx}),$$

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$$\mathbb{E}(\sum_{i} f(\lambda_i)) = \int_{\mathbb{R}} f(x)K_n(x,x)dx,$$

$$\text{Var} \left( \sum_{i} f(\lambda_i) \right) = \frac{1}{2} \iint_{\mathbb{R}^2} (f(x) - f(y))^2 K^2_n(x,y)dxdy.$$
Remark

The similar result has been obtained for the ratios of the type

$$
\frac{\det \left( - \frac{s_i}{\rho_{sc}(E) \sqrt{n}} - E \sqrt{n} + H_n \right)}{\det \left( - E \sqrt{n} + H_n \right)},
$$

for fixed $E \in (-2, 2)$.

Consequence

$$
\prod_{i=1}^{k} \frac{\det \left( - \frac{s_i}{\rho_{sc}(E) \sqrt{n}} - E \sqrt{n} + H_n \right)}{\det \left( - \frac{u_i}{\rho_{sc}(E) \sqrt{n}} - E \sqrt{n} + H_n \right)} \to \prod_{i=1}^{k} \frac{\xi^\infty(s_i)}{\xi^\infty(u_i)}.
$$
The ratios for the general potential $V$. For example: when $V$ is a polynomial with even degree with a positive leading coefficient. Eigenvalues are determinantal PP. Instead of Hermite polynomials we have orthonormal polynomial w.r.t the weight $e^{-V(x)}$ (Deift 2000).

Ratios for Wigner matrices: the eigenvalues are not determinantal PP, however, they converge to sine determinantal PP.
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Ratios for Wigner matrices: the eigenvalues are not determinantal PP, however, they converge to sine determinantal PP.
Ratios at the edge:

\[
\frac{\det \left( 2\sqrt{n} + \frac{s}{n^{1/6}} - H_n \right)}{\det \left( 2\sqrt{n} - H_n \right)}
\]

The zeros of the limiting characteristic polynomial will form Airy determinantal PP.
Thank you for the attention!!