BLOW-UP OF COMPLEX SOLUTIONS
OF THE 3-d NAVIER-STOKES EQUATIONS
AND BEHAVIOR OF RELATED REAL SOLUTIONS
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1. INTRODUCTION

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\[
\frac{\partial \mathbf{u}}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} \mathbf{u} = \Delta \mathbf{u} - \nabla p, \quad \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3.
\]

\[\nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}(\cdot, 0) = \mathbf{u}_0.\]

\[\mathbf{u} : \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^3\] is the velocity field, \(p\) is the pressure and we assume for the viscosity \(\nu = 1\) (always possible by rescaling).
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\( \mathbf{u} : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3 \) is the velocity field, \( p \) is the pressure and we assume for the viscosity \( \nu = 1 \) (always possible by rescaling). In spite of considerable progress, it is still unknown whether there are initial conditions for which the solution becomes singular in a finite time (global regularity problem).
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The singular solutions, if they exist, would have physical relevance:

\[ E(t) + \int_0^t S(\tau) \, d\tau = E(0), \]

where \( E(t) \) is the total energy and \( S(t) \) the total enstrophy,

\[ E(t) = \frac{1}{2} \int_{\mathbb{R}^3} |u(x, t)|^2 \, dx, \]
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where a blowup is proved for a modified NS equations, which preserve the the energy identity

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\[ v(k, t) = \frac{i}{(2\pi)^3} \int_{\mathbb{R}^3} u(x, t) e^{i(k \cdot x)} \, dx. \]
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The NS equations go, by a Duhamel formula, into

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v(k, t) = e^{-t k^2} v_0(k) + \\
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where \( v_0(k) = v(k, 0) \) is the initial data and \( P_k \) the orthogonal projector expressing incompressibility:

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2. Li-Sinai solutions. Theory.

Behavior of energy and enstrophy at $t \uparrow \tau$. 

\[
E(t) = (2\pi)^{3/2} \int_{\mathbb{R}^3} |v(k,t)|^2 \, dk, \quad \sim C(\alpha) E(\tau - t)^{\beta_\alpha},
\]

\[
S(t) = (2\pi)^{3/2} \int_{\mathbb{R}^3} k^2 |v(k,t)|^2 \, dk, \quad \sim C(\alpha) S(\tau - t)^{\beta_\alpha + 2},
\]

where $\beta_I = 1$, $\beta_{II} = \frac{1}{2}$ and $C(\alpha)$ are constants.
Behavior of energy and enstrophy at $t \uparrow \tau$.

The total energy $E(t)$ and the total enstrophy $S(t)$ blow up as $t \uparrow \tau$, with different rates for the two types $\alpha = I, II$: 

$$E(t) = (2\pi)^3 \int |v(k,t)|^2 \, dk \sim C(\alpha) E(\tau - t)^{\beta I},$$

$$S(t) = (2\pi)^3 \int k^2 |v(k,t)|^2 \, dk \sim C(\alpha) S(\tau - t)^{\beta I + 2},$$

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$$E(t) = \frac{(2\pi)^3}{2} \int_{\mathbb{R}^3} |\mathbf{v}(\mathbf{k}, t)|^2 d\mathbf{k}, \sim \frac{C_E^{(\alpha)}}{(\tau - t)^{\beta_\alpha}},$$

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where $\beta_I = 1$, $\beta_{II} = \frac{1}{2}$ and $C_E^{(\alpha)}$, $C_S^{(\alpha)}$ are constants.
2. Li-Sinai solutions. Theory

The rigorous results give the following predictions:

i) The solution has its main support within a thin cone along the $k_3$-axis, and is represented as a sum of modulated gaussian terms concentrated around the points $(0, 0, p \alpha)$ and multiplied by $\exp\{-\kappa p (\tau - t)\}$;

ii) For large $k_3$, the velocity field is approximately orthogonal to the $k_3$-axis and its direction is approximately radial;

iii) The solutions converge point-wise in $k$-space as $t \uparrow \tau$, while $E(t)$ and $S(t)$ diverge as inverse powers of $\tau - t$. 
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The computer simulations for the complex Li-Sinai solutions reveal important properties which are not, so far, predicted by the theory. Computer simulations for the Li-Sinai solutions were first performed by Arnol’d and Khokhlov in 2009. However, due to computational limitations, they could only get a qualitative description of the blow-up.
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i) the blow-up takes place in a very short time $\approx 10^{-5}$ time units (t.u), so that the time step has to be small;

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3. Li-Sinai solutions: simulations.

According to a preliminary screening on a sample of “good‘ initial data it appears that the “best” initial data are

$$v_0^\pm(k) = \pm K \bar{v}_0(k),$$

where $k(0) = (0, 0, a)$, $g(3)$ is the standard gaussian on $\mathbb{R}^3$, $I_D$ is the indicator function of the set $D = \{k: |k| \leq 17\}$, and in all cases $a \geq 20$. The positive constant $K$ controls the initial energy.

We get solutions of type I for the initial data $v^+_0$, and of type II (alternating signs) for $v^-_0$. 
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Most simulations were done for solutions of type II. Their behavior is more similar to that of the related real solutions. The mesh $R$ in $k$-space is taken with step 1:

$$R = [-127, 127] \times [-127, 127] \times [-19, L] \subset \mathbb{Z}^3,$$

where the critical parameter $L$ takes the values 2028, 2528, 3028.
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where the critical parameter \( L \) takes the values 2028, 2528, 3028. Control simulations with finer meshes confirm stability.
3. Li-Sinai solutions: simulations.

For the description of the behavior we use the energy and enstrophy marginals in $\mathbf{k}$ along the main axis

$$E_3(k_3, t) = \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}} dk_1 dk_2 |\mathbf{v}(\mathbf{k}, t)|^2,$$

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$$E_3(k_3, t) = \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}} dk_1 dk_2 |v(k, t)|^2,$$

$$S_3(k_3, t) = \int_{\mathbb{R} \times \mathbb{R}} dk_1 dk_2 |k|^2 |v(k, t)|^2$$

and the analogous marginals $E_j(k_j, t), S_j(k_j, t), j = 1, 2.$
3. Li-Sinai solutions: simulations.

For the description of the behavior we use the energy and enstrophy marginals in $\mathbf{k}$ along the main axis

$$E_3(k_3, t) = \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}} dk_1 dk_2 |\mathbf{v}(k, t)|^2,$$

$$S_3(k_3, t) = \int_{\mathbb{R} \times \mathbb{R}} dk_1 dk_2 |\mathbf{k}|^2 |\mathbf{v}(k, t)|^2$$

and the analogous marginals $E_j(k_j, t), S_j(k_j, t), j = 1, 2.$

The marginals in $\mathbf{x}$-space are denoted $\tilde{E}_j(x_j, t), \tilde{S}_j(x_j, t), j = 1, 2, 3,$ with

$$\tilde{E}_3(k_3, t) = \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}} dx_1 dx_2 |\mathbf{u}(x, t)|^2,$$

$$\tilde{S}_3(k_3, t) = \int_{\mathbb{R} \times \mathbb{R}} dx_1 dx_2 |\nabla \mathbf{u}(x, t)|^2,$$

e tc.
We now illustrate some of the main features of the solutions near the blow-up provided by the computer simulations.

- The component of \( v(k, t) \) orthogonal to the \( k_3 \)-axis is roughly radial already at the beginning of the blow-up and for relatively small \( k_3 \).
- For \( k_1 \) and \( k_2 \) fixed the type II solutions describe, as a function of \( k_3 \), a damped oscillation with approximate period \( 2a \) and vanish on the planes \( k_3 \approx (j + 1/2)a \).
- The total enstrophy \( S(t) \) starts growing much earlier than the total energy \( E(t) \).
- The solution of type I blow up much earlier than the solutions of type II with the same initial energy and same \( a \).
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3. Li-Sinai solutions: simulations. The fixed point $H(0)$.

Figure 1: Type II, $a = 20$, $E_0 = 5 \times 10^4$. The arrows indicate the direction of $v(k, t)$ on a regular point lattice on a section of the plane $k_3 = 100$ with sides of length 100, $t = 1521 \times 10^{-7}$. Simulation range $k_3 \in [-19, 2528]$. Magnitude refers to $|v(k, t)|$. In the grey external region $|v(k, t)| < 10^{-6}$.
3. Li-Sinai solutions: simulations. Oscillations type I.

Figure 2: Type I, $a = 20$, $E_0 = 5 \times 10^4$. Enstrophy marginal density $S_3(k_3, t)$ at the beginning of the blow-up. $t = 900 \times 10^{-7}$. Simulation range $k_3 \in [-19, 2528]$. 
3. Li-Sinai solutions: simulations. Oscillations type II.

**Figure 3:** Type II, $a = 20$, $E_0 = 5 \times 10^4$. Enstrophy marginal density $S_3(k_3, t)$ at the beginning of the blow-up. $t = 1125 \times 10^{-7}$. The zeroes are approximately periodic with period $a$. Simulation range $k_3 \in [-19, 2528]$. 
3. Li-Sinai solutions: simulations. Oscillations type II.

Figure 4: Type II, $a = 20$, $E_0 = 5 \times 10^4$. $v_1(k, t)$ vs $k_3$ for $k_1, k_2$ fixed, at the times $t \times 10^7 = 1342, 1500, 1544, 1574, 1600$. The amplitudes increase as time grows, and tend to a limit. Simulation range $k_3 \in [-19, 3028]$. 
3. Li-Sinai solutions: simulations. Type I: compared growth.

Figure 5: Type I, \( a = 20, E_0 = 5 \times 10^4 \). Compared growth of the total enstrophy \( S(t) \) and the total energy \( E(t) \). Simulation range \( k_3 \in [-19, 2528] \).
3. Li-Sinai solutions: simulations. Type II: compared growth

![Graph showing logarithmic growth of enstrophy and energy over time.]

**Figure 6:** *Type II, $a = 20, E_0 = 5 \times 10^4$. Compared growth of the total enstrophy $S(t)$ and the total energy $E(t)$. Simulation range $k_3 \in [-19, 2528]$.***
3. Li-Sinai solutions: simulations. Enstrophy distribution

Figure 7: Type I, \(a = 20, E_0 = 5 \times 10^4\). Plot of the marginal enstrophy density \(S_3(k_3, t)\) on the whole simulation range \([-19 \leq k_3 \leq 2528]\), at \(t \cdot 10^7 = 1060, 1075, 1080\).
3. Li-Sinai solutions: simulations. Enstrophy distribution

![Graph showing the enstrophy distribution $S_3(k_3, t)$ for Type II, $a = 20$, $E_0 = 5 \times 10^4$.]

Figure 8: Type II, $a = 20$, $E_0 = 5 \times 10^4$. Plot of the marginal enstrophy density $S_3(k_3, t)$ on the whole simulation range $-19 \leq k_3 \leq 2528$, at $t \cdot 10^7 = 1521, 1544, 1560$. 

![Graph showing marginal enstrophy density](image)

Figure 9: Type II, $a = 20$, $E_0 = 5 \times 10^4$. Plot of the marginal enstrophy density $S_1(k_1, t)$ at $t \cdot 10^7 = 1521, 1544, 1560$. Simulation range $k_3 \in [-19, 2528]$. 
3. Li-Sinai solutions: simulations. Critical time

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In fact, while $S(t)$ shows a significant growth before the significant support gets out of the simulation region, the growth of $E(t)$ is hard to follow with the present computer resources.
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The best way of estimating the critical time $\tau$ is based on the fact that, as predicted by the theory, the high $k_3$-modes fall off exponentially fast in $k_3$ with a rate proportional to $(\tau - t)$. 
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Already at times relatively far from the critical time, the decay rate of the marginal energy density $E_3(k_3, t)$ in the region $k_3 > 400$ turns out to be exponential decreasing in time with great accuracy.
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Figure 10: Type II, $a = 20$, $E_0 = 5 \times 10^4$. Plot of $\log(E_3(k_3, t))$, where $E_3$ is the marginal energy density along the $k_3$-axis for $k_3 \geq 400$ at two different times. The dots represent the local maxima of the oscillations of $E_3(k_3, t)$. Simulation range $k_3 \in [-19, 2028]$. 

Figure 11: Type I, \( a = 20, \ E_0 = 5 \times 10^4 \). Exponential decay rate for the marginal density \( E_3(k_3, t) \), taken for \( k_3 \geq 400 \), vs magnified time \( t \times 10^7 \), with linear regression (dashed line). Simulation range \( k_3 \in [-19, 2528] \).

Figure 12: Type II, $a = 20, E_0 = 5 \times 10^4$. Exponential decay rate for the marginal density $E_3(k_3, t)$, taken for $k_3 \geq 400$, vs magnified time $t \times 10^7$, with linear regression (dashed line). Simulation range $k_3 \in [-19, 2528]$. 

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Observe that the parameter $a$ controls the initial enstrophy $S(0)$ independently of the initial energy $E_0$. 

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The data show that when we increase \( a \) the excitation of the high \( k_3 \)-modes is accelerated and the critical time decreases (at least in the range we considered).

Figure 13: Type II, $E_0 = 5 \times 10^4$. Behavior of the exponential decay rates of $E_3(k_3, t)$ vs. magnified time $t \times 10^7$ for $a = 20, 25, 30$. Simulation range $k_3 \in [-19, 3028]$
3. Li-Sinai solutions: simulations.

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Figure 14: Type I, \( a = 20, \ E_0 = 5 \times 10^4 \). Log-plot of the total enstrophy \( S(t) \) vs \( \log \frac{1}{\tau^* - t} \), at times near the blow-up, with linear regression (dashed line, the prediction for the slope is 3.0). Simulation range \( k_3 \in [-19, 2528] \).
For the behavior in \( \mathbf{x} \)-space the data show convergence everywhere as \( t \uparrow \tau \), except for a singularity at the origin for type I solutions.
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$$\mathbf{x}^{(0)}_{\pm} = (0, 0, \pm x_3^{(0)}), \quad x_3^{(0)} \approx \frac{\pi}{a}$$

for the solutions of type $\text{II}$. 
3. Li-Sinai solutions: simulations. x-space.

Figure 15: Type I, \( a = 20, E_0 = 5 \times 10^4 \). Plot of the marginal energy density \( \tilde{E}_3(x_3, t) \) at \( t \cdot 10^7 = 1021 \) (dotted line) and \( t \times 10^7 = 1044 \) (continuous line). Simulation range \( k_3 \in [-19, 2528] \).
3. Li-Sinai solutions: simulations. $x$-space.

Figure 16: Type II, $a = 20$, $E_0 = 5 \times 10^4$. Plot of the marginal energy density $\tilde{E}_3(x_3, t)$ at $t \cdot 10^7 = 1521$ (continuous line) and $t \times 10^7 = 1544$ (dotted line). Simulation range $k_3 \in [-19, 2528]$

Figure 17: Type II, $a = 20$, $E_0 = 5 \times 10^4$. Plot of the marginal energy density $\tilde{E}_1(x_1, t)$ at $t \cdot 10^7 = 1521$ (continuous line), and $t \cdot 10^7 = 1544$ (dotted line). Simulation range $k_3 \in [-19, 2528]$. 
4. REAL SOLUTIONS.

Assuming antisymmetric initial data $A\mathbf{v}_0(k)$

$$\mathbf{v}_0(k) = \mathbf{v}_0^{\pm}(k) - \mathbf{v}_0^{\pm}(-k),$$
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where $\mathbf{v}_0^\pm$ are as above, we get a real solution. (The choice $\pm$ amounts to a change of sign.) $\mathbf{v}_0(k)$ has support in two separate regions around the points $\pm(0,0,a)$. 
4. Real solutions.

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- The support in $k$-space restricted to a thin (double) cone around the main axis,
- The solution shows modulated oscillations along the $k_3$-axis, as the complex solutions of type $II$. 
The simulation show, if the initial energy is large enough, that:

\[ S(t) \]

grows, reaches a maximum at a time \( t^* \) (which for a fixed \( E_0 \) depends on \( E_0 \)), then falls;
The simulation show, if the initial energy is large enough, that:
- The total enstrophy $S(t)$ grows, reaches a maximum at a time $t_*$ (which for a fixed depends on $E_0$), then falls;
Figure 18: *Plot of the total enstrophy $S(t)$ vs. magnified time $t \times 3.2 \times 10^7$. Initial energy $\bar{E}_0 = 2.5 \times 10^4$, $a = 20$.**
- The large $k_3$ modes fall off exponentially fast, with a rate decreasing in absolute value up to $t \approx t_*$, then stays constant.
4. Real solutions.

Figure 19: Logarithmic plot of the marginal density $\tilde{E}_3(k_3, t)$ at the (magnified) times $t \times 3.2 \times 10^7 = 100, 200, 300, 400, 500$. Initial energy $\tilde{E}_0 = 2.5 \times 10^4$, $a = 20$. 
- At the time $t_*$, the energy and the enstrophy concentrate in two (pseudo)-spikes, close to the singularities of the complex solution of type II with the same $a$. 
4. Real solutions.

Figure 20: Plot of the marginal density $\tilde{S}_3(x_3, t)$ at $t = 1.27 \times 10^{-5}$. Initial energy $\bar{E}_0 = 2.5 \times 10^4$, $a = 20$. 