Spectral properties of Schrödinger operators with singular interactions on hypersurfaces

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- Some open questions

Operators to deal with

The simplest example of the singular Schrödinger operators we are going to consider here can formally written as

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x-\Gamma)\,, \quad \alpha > 0\,,$$

in $L^2(\mathbb{R}^n)$, where Γ is a zero-measure subset of $\mathbb{R}^n$, for instance, a manifold, a metric graph, etc.

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- the coupling strength may vary along the interaction support
- δ may be replaced by other, more singular interactions
- on the other hand, we restrict ourselves to the situations with $\operatorname{codim}\Gamma = 1$. Note that there are various results for $\operatorname{codim}\Gamma = 2$, cf. [E-Kondej'02,'15; E-Frank'07], while the remaining nontrivial case $\operatorname{codim}\Gamma = 3$ has not been studied so far

The δ-interaction

A natural tool to define the corresponding singular Schrödinger operator is to employ the appropriate quadratic form, namely

$$q_{\delta,\alpha}[\psi] := \|\nabla\psi\|^2_{L^2(\mathbb{R}^d)} - \alpha\|f|_\Gamma\|^2_{L^2(\Gamma)}$$

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If Γ is a *smooth manifold* with $\operatorname{codim}\Gamma = 1$ one can easily check that the form defines a unique self-adjoint operator $H_{\alpha,\Gamma}$, which can alternatively characterized by boundary conditions: it acts as $-\Delta$ on functions from $H^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \Gamma)$, which are continuous and exhibit a normal-derivative jump,

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Alternatively, one sometimes uses the symbol $-\Delta_{\delta,\alpha}$ for this operator.

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The union $\cup_{k=1}^n \partial\Omega_k =: \Gamma$ is the *boundary* of $\mathcal{P}$. For $k \neq l$ we set $\Gamma_{kl} := \partial\Omega_k \cap \partial\Omega_l$ and we say that Ω_k and Ω_l, $k \neq l$, are neighboring domains if $\sigma_k(\Gamma_{kl}) > 0$, where σ_k is the Lebesgue measure on $\partial\Omega_k$.

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Using standard coloring maps, we define the *chromatic number* $\chi_{\mathcal{P}}$ of $\mathcal{P}$ as the smallest number of colors allowed by the partition 'map'. In particular, we know that $\chi_{\mathcal{P}} \leq 4$ if $d = 2$.

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Then we have the following result [Behrndt-E-Lotoreichik'14]:

Proposition

Let $\mathcal{P} = \{\Omega_k\}_{k=1}^n$ be a Lipschitz partition of $\mathbb{R}^d$ with the boundary Γ, and let $\alpha : \Gamma \to \mathbb{R}$ belong to $L^\infty(\Gamma)$. Then the quadratic form $q_{\delta,\alpha}$ defined above is closed and semibounded from below.

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Note that the interaction support may be a *proper subset* of Γ, since α may vanish on a part of Γ, hence it may be, e.g., a finite non-closed curve, a manifold with a boundary, etc.

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On the other hand, the essential spectrum may change if the support Γ is non-compact. As an example, take a line in the plane and suppose that α is *constant and positive*; by separation of variables we find easily that $\sigma_{\mathrm{ess}}(-\Delta_{\delta,\alpha}) = [-\frac{1}{4}\alpha^2, \infty)$.

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The question about the *discrete spectrum* is more involved. Suppose first that interaction support is *finite*, $|\Gamma| < \infty$. It is clear that $\sigma_{\mathrm{disc}}(-\Delta_{\delta,\alpha})$ is empty if the interaction is repulsive, $\alpha \le 0$.

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$$\lambda(\alpha) = \big(C_\Gamma + o(1)\big) \exp\left(-\frac{4\pi}{\alpha|\Gamma|}\right) \quad \text{as} \quad \alpha|\Gamma| \to 0+$$

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and the same obviously holds in dimensions $d > 3$.

A δ-interaction supported by infinite curves

A geometrically induced discrete spectrum may exist even if Γ is infinite and $\inf \sigma_{\mathrm{ess}}(-\Delta_{\delta,\alpha}) < 0$. Consider, for instance, a *non-straight, piecewise C^1-smooth curve* $\Gamma : \mathbb{R} \to \mathbb{R}^2$ parameterized by its arc length, $|\Gamma(s) - \Gamma(s')| \leq |s - s'|$, assuming in addition that

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- $|\Gamma(s) - \Gamma(s')| \ge c|s - s'|$ holds for some $c \in (0,1)$
- Γ is *asymptotically straight*: there are $d > 0$, $\mu > \frac{1}{2}$ and $\omega \in (0,1)$ such that

$$1 - \frac{|\Gamma(s) - \Gamma(s')|}{|s - s'|} \le d\left[1 + |s + s'|^{2\mu}\right]^{-1/2}$$

in the sector $S_\omega := \left\{(s,s') : \omega < \frac{s}{s'} < \omega^{-1}\right\}$

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Theorem (E-Ichinose'01)

Under these assumptions, $\sigma_{\rm ess}(-\Delta_{\delta,\alpha}) = [-\frac{1}{4}\alpha^2, \infty)$ and $-\Delta_{\delta,\alpha}$ has at least one eigenvalue below the threshold $-\frac{1}{4}\alpha^2$.

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- On the other hand, we have an example of a *conical surface* of an opening angle $\theta \in (0, \frac{1}{2}\pi)$ in $\mathbb{R}^3$, where for any constant $\alpha > 0$ we have $\sigma_{\text{ess}}(-\Delta_{\delta,\alpha}) = \mathbb{R}_+$ and an *infinite numbers of negative eigenvalues* accumulating at zero, cf. [Behrndt-E-Lotoreichik'14]

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- Moreover, the above result remain valid for any *local deformation* of the conical surface. We also know the accumulation rate for conical layers: by [Lotoreichik–Ourmières-Bonafos'16] it is

$$\mathcal{N}_{-\frac{1}{4}\alpha^2 - E}(-\Delta_{\delta,\alpha}) \sim \frac{\cot\theta}{4\pi}\,|\ln E|\,, \quad E \to 0+\,.$$

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- On the other hand, the result is again dimension-dependent: for a conical surface in $\mathbb{R}^d$, $d > 3$, we have $\sigma_{\rm disc}(-\Delta_{\delta,\alpha}) = \emptyset$, cf. [Lotoreichik–Ourmières-Bonafos'16].

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- Implications for *more complicated Lipschitz partitions*: let $\tilde\Gamma \supset \Gamma$ holds in the set sense, then $H_{\alpha,\tilde\Gamma} \le H_{\alpha,\Gamma}$. If the essential spectrum thresholds are the same – which is often easy to establish – then $\sigma_{\rm disc}(H_{\alpha,\tilde\Gamma}) \neq \emptyset$ whenever the same is true for $\sigma_{\rm disc}(H_{\alpha,\Gamma})$

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- Many other results, for instance, concerning the *strong coupling asymptotics*: for a C^4 smooth *curve in $\mathbb{R}^2$ without ends* the j-th eigenvalue of $-\Delta_{\delta,\alpha}$ behaves as

$$\lambda_j(\alpha) = -\frac{\alpha^2}{4} + \mu_j + \mathcal{O}(\alpha^{-1} \ln \alpha)$$

in the limit $\alpha \to \infty$, where μ_j is the j-th ev of $S_\Gamma = -\frac{\mathrm{d}}{\mathrm{d}s^2} - \frac{1}{4}\kappa(s)^2$ on $L^2((0,|\Gamma|))$, where κ is the *signed curvature* of Γ.

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- For infinite curves in $\mathbb{R}^2$ we have also a *weak bending asymptotics*: for a family Γ_θ parametrized by the bending angle θ one proves $\lambda(H_{\alpha,\Gamma_\theta}) = -\frac{1}{4}\alpha^2 + a\theta^4 + o(\theta^4)$ with an explicit $a < 0$ as $\theta \to 0+$ under some technical assumptions [E-Kondej'16]. In particular, for broken line we have $a = -\frac{\alpha^2}{36\pi^2}$.

Geometrically induced bound states, continued

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- Also various other results are known ...

More singular operators: the δ'-interaction

Having in mind the one-dimensional point interaction, we can define for a smooth planar curve the operator $-\Delta_{\delta',\beta}$ using boundary conditions: it acts as Laplacian outside the interaction support,

$$(H_{\beta,\Gamma}\psi)(x) = -(\Delta\psi)(x)\,, x \in \mathbb{R}^2 \setminus \Gamma\,,$$

with the domain consisting of functions $\psi \in H^2(\mathbb{R}^2 \setminus \Gamma)$ that satisfy the b.c. $\partial_{n_\Gamma}\psi(x) = \partial_{-n_\Gamma}\psi(x) =: \psi'(x)|_\Gamma$, $-\beta\psi'(x)|_\Gamma = \psi(x)|_{\partial_+\Gamma} - \psi(x)|_{\partial_-\Gamma}\}$, where n_Γ is the normal to Γ and $\psi(x)|_{\partial_\pm\Gamma}$ are the appropriate traces.

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The corresponding quadratic form is easily seen to be

$$h_{\beta,\Gamma}[\psi] = \|\nabla\psi\|^2 - \beta^{-1}\int_\Gamma |\psi(s,0_+) - \psi(s,0_-)|^2\,\mathrm{d}s$$

defined on functions $\psi \in H^1(\mathbb{R}^2 \setminus \Gamma)$ as $\psi(s,u)$, where s, u are the natural curvilinear coordinates in the vicinity of Γ.

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Note that the strong-coupling in this case means $\beta \to 0+$.

The δ'-interaction

Let $\mathcal{P} = \{\Omega_k\}_{k=1}^n$ be a Lipschitz partition of $\mathbb{R}^d$ with the boundary Γ, and let $\beta : \Gamma \to \mathbb{R}$ be such that $\beta^{-1} \in L^\infty(\Gamma)$. Then we define the form

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$$q_{\delta',\beta}[f,g] := \sum_{k=1}^{n} \big(\nabla f_k, \nabla g_k\big)_{L^2(\Omega_k}$$

$$- \sum_{k=1}^{n-1} \sum_{l=k+1}^{n} \big(\beta_{kl}^{-1}(f_k|_{\Gamma_{kl}} - f_l|_{\Gamma_{kl}}), g_k|_{\Gamma_{kl}} - g_l|_{\Gamma_{kl}}\big)_{L^2(\Gamma_{kl})}$$

with the domain $\bigoplus_{k=1}^n H^1(\Omega_k)$; we denote here $\Gamma_{kl} = \partial\Omega_k \cap \partial\Omega_l$ for $k, l = 1, 2, \dots, n$, $k \neq l$, and β_{kl} means the restrictions of β to Γ_{kl}.

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As in the δ case, we have the following result [Behrndt-E-Lotoreichik'14]:

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The s-a operator associated with $q_{\delta',\beta}$ will be denoted as $-\Delta_{\delta',\beta}$ or $H_{\beta.\Gamma}$

Spectrum of $-\Delta_{\delta',\beta}$

Similarly to the δ case, we have $\sigma_{\mathrm{ess}}(-\Delta_{\delta',\beta}) = \mathbb{R}_+$ if Γ is compact.

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On the other hand, relations between the discrete spectrum and the form of Γ are, in general, different from the δ situation. It is now the *topology* of the interaction support which plays role.

The δ' interaction in the plane

Consider a finite curve Γ in $\mathbb{R}^2$. If it is a *loop*, then it is easy to see that $\sigma_{\mathrm{disc}}(-\Delta_{\delta',\beta}) \neq \emptyset$ for any constant $\beta > 0$

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Theorem (Jex-Lotoreichik'16)

We have $\sigma(-\Delta_{\delta',\beta}) \subset \mathbb{R}_+$ if $\beta > 2\pi r\sqrt{1+(r\varphi'(r))^2}$ for all $r \in (0,R)$.

An operator inequality

Spectral analysis of $-\Delta_{\delta',\beta}$ is more difficult because we lack a direct counterpart to some of the tools used before, in particular, to the (generalized) Birman-Schwinger principle.

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One the other hand, there is a useful relation between the two cases:

Theorem (Behrndt-E-Lotoreichik'14)

Let $\mathcal{P} = \{\Omega_k\}_{k=1}^n$ be a Lipschitz partition of $\mathbb{R}^d$ with boundary Γ and chromatic number $\chi_\mathcal{P}$. Let $\alpha, \beta \colon \Gamma \to \mathbb{R}$ be such that $\alpha, \beta^{-1} \in L^\infty(\Gamma)$ and assume that

$$0 < \beta \leq \frac{4}{\alpha} \sin^2\left(\frac{\pi}{\chi_\mathcal{P}}\right).$$

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$$0 < \beta \leq \frac{4}{\alpha} \sin^2\left(\frac{\pi}{\chi_{\mathcal{P}}}\right).$$

Then there exists a unitary operator $U \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that the self-adjoint operators $-\Delta_{\delta,\alpha}$ and $-\Delta_{\delta',\beta}$ satisfy the inequality

$$U^{-1}(-\Delta_{\delta',\beta})U \leq -\Delta_{\delta,\alpha}.$$

Sketch of the proof

By assumption, to the given $\mathcal{P}$ there is an optimal *coloring map*

$$\phi\colon \{1,2,\dots,n\} \to \{0,1,\dots,\chi_{\mathcal{P}}-1\}$$

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Then we define n complex numbers $\mathcal{Z} := \{z_k\}_{k=1}^n$ on the unit circle,

$$z_k := \exp\left(i\frac{2\pi\phi(k)}{\chi_{\mathcal{P}}}\right), \quad k = 1,2,\dots,n\,;$$

it is easy to see that for $k \neq l$ such that $\sigma_k(\Gamma_{kl}) > 0$ they satisfy

$$|z_k - z_l|^2 \geq 2 - 2\cos\left(\tfrac{2\pi}{\chi_{\mathcal{P}}}\right),$$

in other words $4\sin^2\left(\frac{2\pi}{\chi_{\mathcal{P}}}\right) \leq |z_k - z_l|^2$.

Sketch of the proof

Putting now $\alpha_{\mathcal{Z}}(x) := |z_k - z_l|^2 \beta_{kl}^{-1}(x)$ for $x \in \Gamma_{kl}$ with $k \neq l$, we find

$$0 < \alpha \leq \frac{4}{\beta} \sin^2\left(\frac{2\pi}{\chi_{\mathcal{P}}}\right) \leq \alpha_{\mathcal{Z}}\,.$$

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Now we define the unitary operator $U_{\mathcal{Z}}\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$(U_{\mathcal{Z}} f)(x) := z_k f_k(x)\,, \quad x \in \Omega_k\,, \quad k = 1, \dots, n\,.$$

Using then the above inequality in combination with the explicit expressions of the involved quadratic forms, it is not difficult to derive the sought result.

Consequences of the inequality

The above result allows to draw conclusions from an operator comparison.

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Denote by $\{\lambda_k(-\Delta_{\delta,\alpha})\}_{k=1}^{\infty}$ and $\{\lambda_k(-\Delta_{\delta',\beta})\}_{k=1}^{\infty}$ the eigenvalues of the operators $-\Delta_{\delta,\alpha}$ and $-\Delta_{\delta',\beta}$, respectively, below the bottom of their essential spectra, enumerated in non-decreasing order and repeated with multiplicities, and let $N(-\Delta_{\delta,\alpha})$ and $N(-\Delta_{\delta',\beta})$ be their total numbers.

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Corollary

Under the assumption of the theorem, we have

(i) $\lambda_k(-\Delta_{\delta',\beta}) \leq \lambda_k(-\Delta_{\delta,\alpha})$

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Corollary

Under the assumption of the theorem, we have

(i) $\lambda_k(-\Delta_{\delta',\beta}) \le \lambda_k(-\Delta_{\delta,\alpha})$ *for all* $k \in \mathbb{N}$;

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(ii) $\min \sigma_{\mathrm{ess}}(-\Delta_{\delta',\beta}) \le \min \sigma_{\mathrm{ess}}(-\Delta_{\delta,\alpha})$;

(iii) *If* $\min \sigma_{\mathrm{ess}}(-\Delta_{\delta,\alpha}) = \min \sigma_{\mathrm{ess}}(-\Delta_{\delta',\beta})$, *then* $N(-\Delta_{\delta,\alpha}) \le N(-\Delta_{\delta',\beta})$.

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The estimates are the better the smaller the chromatic number is.

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Corollary

Under the stated assumptions, let $\chi_{\mathcal{P}} = 2$ *and* $0 < \beta \leq \frac{4}{\alpha}$, *then there is a unitary operator such that*

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Moreover, the examples with Γ being a line in the plane show that the inequality $0 < \beta \le \frac{4}{\alpha}$ cannot be improved.

Example: Let Γ be a bent, asymptotically straight curve considered above, now supporting the δ'-interaction with a constant $\beta > 0$. Choose $\alpha = \frac{4}{\beta}$, then $-\Delta_{\delta',\beta}$ and $-\Delta_{\delta,\alpha}$ have the same essential spectrum. Since we know that $\sigma_{\text{disc}}(-\Delta_{\delta,\alpha}) \ne \emptyset$, *the same is true for* $-\Delta_{\delta',\beta}$.

Strong coupling on a δ' loop

Some δ arguments, though, can be adapted easily to the δ' situation.

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Theorem (E-Jex'13)

Let Γ be a C^4-smooth closed curve without self-intersections. Then $\sigma_{\mathrm{ess}}(H_{\beta,\Gamma}) = [0,\infty)$ and to any $n \in \mathbb{N}$ there is a $\beta_n > 0$ such that $\#\sigma_{\mathrm{disc}}(H_{\beta,\Gamma}) \ge n$ holds for $\beta \in (0,\beta_n)$. Denoting by $\lambda_j(\beta)$ the j-th eigenvalue of $H_{\beta,\Gamma}$, counted with multiplicity, we have the expansion

$$\lambda_j(\beta) = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}\big(\beta|\ln\beta|\big)\,, \quad j = 1,\dots,n\,,$$

valid as $\beta \to 0_+$, where μ_j is the j-th eigenvalue of the comparison operator S_Γ, the same as before.

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$$\sharp\sigma_{\rm disc}(H_{\beta,\Gamma}) = \frac{2L}{\pi\beta} + \mathcal{O}(|\ln\beta|) \quad \text{as} \;\; \beta\to 0_+\,.$$

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A similar result holds for infinite curves, cf. [Jex'14], and for strong δ' interaction supported by surfaces *without boundary*, cf. [E-Jex'14]

More general interactions

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For simplicity we restrict ourselves to the simplest partition of the space, namely we assume that $\Gamma \subset \mathbb{R}^d$, $d \geq 2$, is the boundary of a (bounded or unbounded) Lipschitz domain $\Omega = \Omega_{\rm i}$ and $\Omega_{\rm e} := \mathbb{R}^d \setminus (\Omega_{\rm i} \cup \Gamma)$; for $f \in L^2(\mathbb{R}^d)$ we write $f_j = f|_{\Omega_j}$, $j = {\rm i}, {\rm e}$, and $f = f_{\rm i} \oplus f_{\rm e}$.

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The trace of $f \in H^1(\Omega_j)$ on Γ is denoted by $f|_\Gamma \in H^{1/2}(\Gamma)$. For each $f \in H^1(\Omega_j)$ we define the derivative of f with respect to the outer unit normal on $\Gamma = \partial\Omega_j$ using Green's first identity; if Γ is sufficiently smooth and f is differentiable up to the boundary then $\partial_{\nu_j} f|_\Gamma$ is the usual derivative. The outer unit normals for Ω_i and Ω_e coincide up to a minus sign, in particular, for $f \in H^2(\mathbb{R}^d)$ we have $\partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Gamma + \partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Gamma = 0$.

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The interactions supported on Γ will be thus described by Laplacian on $\mathbb{R}^d \setminus \Gamma$ subject to the interface conditions

$$\partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Gamma + \partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Gamma = \frac{\alpha}{2}\left(f_\mathrm{i}|_\Gamma + f_\mathrm{e}|_\Gamma\right) + \frac{\gamma}{2}\left(\partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Gamma - \partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Gamma\right),$$

$$f_\mathrm{i}|_\Gamma - f_\mathrm{e}|_\Gamma = -\frac{\overline{\gamma}}{2}\left(f_\mathrm{i}|_\Gamma + f_\mathrm{e}|_\Gamma\right) + \frac{\beta}{2}\left(\partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Gamma - \partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Gamma\right).$$

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Concerning the *coefficient functions*, we assume that $\alpha : \Gamma \to \mathbb{R}$ and $\gamma : \Gamma \to \mathbb{C}$ are bounded, measurable functions. Moreover, let $\Gamma_\beta \subset \Gamma$ be a relatively open subset and let $\beta : \Gamma \to \mathbb{R}$ be a function such that β^{-1} is measurable and bounded on Γ_β and $\beta = 0$ identically on $\Gamma_0 := \Gamma \setminus \Gamma_\beta$.

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For some of them, however, the above conditions are formal and we have to seek an alternative way to define the operators in question.

The quadratic form definition

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$$\Theta_{\mathcal{A}} = \begin{pmatrix} \frac{|1+\frac{\gamma}{2}|^2}{\beta}\mathbb{I}_{\Gamma_\beta} + \frac{\alpha}{4} & \frac{(\frac{\overline{\gamma}}{2}-1)(1+\frac{\gamma}{2})}{\beta}\mathbb{I}_{\Gamma_\beta} + \frac{\alpha}{4} \\ \frac{(\frac{\gamma}{2}-1)(1+\frac{\overline{\gamma}}{2})}{\beta}\mathbb{I}_{\Gamma_\beta} + \frac{\alpha}{4} & \frac{|1-\frac{\gamma}{2}|^2}{\beta}\mathbb{I}_{\Gamma_\beta} + \frac{\alpha}{4} \end{pmatrix}$$

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with the convention that $\frac{1}{\beta}\mathbb{I}_{\Gamma_\beta}$ equals zero on Γ_0.

Then we define a quadratic form $h_\mathcal{A}$ in $L^2(\mathbb{R}^d)$ in the following way,

$$q_\mathcal{A}(f,g) = \int_{\Omega_\mathrm{i}} \nabla f_\mathrm{i} \cdot \overline{\nabla g_\mathrm{i}}\,\mathrm{d}x + \int_{\Omega_\mathrm{e}} \nabla f_\mathrm{e} \cdot \overline{\nabla g_\mathrm{e}}\,\mathrm{d}x - \int_\Gamma \left\langle \Theta_\mathcal{A}\begin{pmatrix} f_\mathrm{i} \\ f_\mathrm{e}\end{pmatrix}, \begin{pmatrix} g_\mathrm{i} \\ g_\mathrm{e}\end{pmatrix} \right\rangle \mathrm{d}\sigma\,,$$

$$\mathcal{D}(q_\mathcal{A}) = \left\{ f_\mathrm{i} \oplus f_\mathrm{e} \in H^1(\Omega_\mathrm{i}) \oplus H^1(\Omega_\mathrm{e}) : (1+\tfrac{\bar\gamma}{2})f_\mathrm{i} = (1-\tfrac{\bar\gamma}{2})f_\mathrm{e} \text{ on } \Gamma_0 \right\},$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{C}^2$ and σ is the surface measure on Γ. Note that $q_\mathcal{A}$ is well-defined since the entries of $\Theta_\mathcal{A}$ are bounded functions.

The quadratic form definition

Under the stated assumption we have [E-Rohleder'16]:

Proposition

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Remark: The definition includes not only the δ- ($\beta = \gamma = 0$) and δ'-interaction ($\alpha = \gamma = 0$), but also other cases of interest. For instance, given real constants c_{i}, c_{e} with $c_{\mathrm{i}} + c_{\mathrm{e}} \neq 0$ and choosing

$$\alpha = \frac{4c_{\mathrm{i}}c_{\mathrm{e}}}{c_{\mathrm{i}} + c_{\mathrm{e}}}, \quad \beta = \frac{4}{c_{\mathrm{i}} + c_{\mathrm{e}}}, \quad \gamma = \frac{2(c_{\mathrm{i}} - c_{\mathrm{e}})}{c_{\mathrm{i}} + c_{\mathrm{e}}},$$

we get separated regions with Robin conditions, $\partial_{\nu_j} f_j = c_j f_j\,,\; j = \mathrm{i}, \mathrm{e}$.

Spectral properties of $-\Delta_{\mathcal{A}}$

A lot can be said about spectrum of $-\Delta_{\mathcal{A}}$, let us mention a few results.

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Theorem (E-Rohleder'16)

Let Ω_{i} be bounded, i.e. Γ is compact. Then the resolvent difference

$$(-\Delta_{\mathcal{A}} - \lambda)^{-1} - (-\Delta_{\mathrm{free}} - \lambda)^{-1}\,, \quad \lambda \in \rho(-\Delta_{\mathcal{A}}) \cap \rho(-\Delta_{\mathrm{free}})$$

is compact. In particular, $\sigma_{\mathrm{ess}}(-\Delta_{\mathcal{A}}) = \mathbb{R}_+$ and the discrete spectrum $\sigma(-\Delta_{\mathcal{A}}) \cap (-\infty, 0)$ is finite.

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Concerning the existence of $\sigma_{\rm disc}(-\Delta_{\mathcal{A}})$, in the presence of δ' we have the following sufficient condition:

Theorem (E-Rohleder'16)

In addition the hypotheses of the previous theorem, let $\Gamma=\Gamma_\beta$, i.e., $\beta(s)\neq 0$ for all $s\in\Gamma$. If $\int_\Gamma\left(\frac{|1+\frac{\gamma}{2}|^2}{\beta}+\frac{\alpha}{4}\right)\mathrm{d}\sigma>0$ holds, $N(-\Delta_{\mathcal{A}})>0$.

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Let Γ be compact in dimension $d = 2$. Assume that $\beta = 0$ identically on Γ, and moreover, $\alpha(s) \geq \alpha_{\min} > 0$ for all $s \in \Sigma$ and let $\gamma \in \mathbb{C}$ be constant. Then $N(-\Delta_{\mathcal{A}}) > 0$.

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If $d \geq 3$ the situation is different:

Proposition

Let Γ be compact, $d \geq 3$, and $\beta = 0$ identically on Γ. Moreover, let $0 \leq \alpha(s) \leq \alpha_{\text{max}}$ for all $s \in \Sigma$ and let $\gamma \in \mathbb{C}$ be constant. Define

$$\widetilde{\alpha} = \frac{\alpha_{\text{max}}}{\min\{|1+\gamma/2|^2, |1-\gamma/2|^2\}} \geq 0$$

and let $-\Delta_{\delta,\widetilde{\alpha}}$ be the Schrödinger operator in $L^2(\mathbb{R}^d)$ with δ-interaction of strength $\widetilde{\alpha}$ on Γ. If $N(-\Delta_{\delta,\widetilde{\alpha}}) = 0$ the same is true for $N(-\Delta_{\mathcal{A}})$.

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Let Γ be a surface in $\mathbb{R}^3$ homeomorphic to the plane which is C^2 smooth outside a compact and asymptotically planar in the sense that K, M vanish asymptotically. Suppose further that the functions α, β, γ are constant outside a compact and $\alpha(s), \beta(s)$ are nonnegative for all $s \in \Gamma$, then under additional mild assumptions we have $\sigma_{\mathrm{ess}}(-\Delta_{\mathcal{A}}) \subset [m_{\mathcal{A}}, \infty)$, where

$$m_{\mathcal{A}} = \begin{cases} -\frac{4\alpha^2}{(4+|\gamma|^2)^2}, & \text{if } \beta = 0 \\ -\frac{\left(4+\det\mathcal{A}+\sqrt{-16\alpha\beta+(4+\det\mathcal{A})^2}\right)^2}{16\beta^2} & \text{if } \beta \neq 0. \end{cases}$$

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In some case one can prove equality, $\sigma_{\mathrm{ess}}(-\Delta_{\mathcal{A}}) = [m_{\mathcal{A}}, \infty)$, for instance if Γ is a plane outside a compact.

Operator inequalities

To prove the existence of a non-void discrete spectrum one can combine known results in particular case with *operator inequalities*. In various particular situations one can prove the existence of a unitary operator, denoted generically as U, which make it possible:

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$$U^*(-\Delta_{\mathcal{A}})U \leq -\Delta_{\delta,\alpha} \, .$$

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The first three can be used to estimate the spectra from the known results about the δ-interaction, the last one includes also the *intermediate class* which occurs if $\mathrm{Re}\,\gamma \neq 0$.

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$$H = -\Delta + V(x) - \alpha\delta(x - \Gamma), \quad \alpha > 0,$$

in $L^2(\mathbb{R}^2)$ understood in the sense discussed above, where the δ-potential is supported by an infinite, piecewise smooth curve Γ dividing the plane into two regions.

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We will be interested in the situation where the potential is *constant, positive, and supported in one of those regions*. Our aim is to show that 'binding-by-bending' effect of [E-Ichinose'01] acquires in this case a *distinguished asymmetry* reminiscent that known for waveguides with a combined Dirichlet-Neumann boundary known from [Dittrich-Kříž'02].

An auxiliary problem

If Γ is a straight line the problem is solved by separation of variable. Let us inspect the transverse part, i.e. the operator

$$h = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha\delta(x) + V(x)\,,$$

where $V(x) = V_0$ for $x > 0$ and $V(x) = 0$ otherwise, associated with the form $\phi \mapsto \|\phi'\|^2 - \alpha|\phi(0)|^2 + \langle V\phi, \phi\rangle$ defined on $H^1(\mathbb{R})$.

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(iv) *If* $V_0 = \alpha^2$ *there is a bounded weak solution* $\psi \notin L^2(\mathbb{R}^1)$ *to* $h\psi = 0$.

We are particularly interested in the *critical case*, $V_0 = \alpha^2$.

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Theorem (E-Vugalter'16)

Under the assumptions (a)–(e) we have $\sigma_{\mathrm{ess}}(H) = [\mu, \infty)$, where $\mu = -\frac{1}{4}\alpha^{-2}(\alpha^2 - V_0)^2$ for $V_0 < \alpha^2$ and $\mu = 0$ otherwise.

An exterior positive potential

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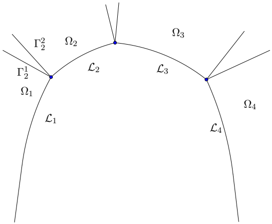
Proof sketch: We employ Neumann bracketing as indicated and in $\Omega_1, \Omega_2, \ldots$ we use the natural locally orthogonal coordinated to show that the corresponding operator are $\geq h$ 1



The subcritical case

Let the potential be positive in the exterior region and *subcritical*. The discrete spectrum then depends on V_0 and the geometry. Consider the *example* of a *broken line* $\Gamma_{\pi-2\varphi}$ of the opening angle 2φ.

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Let $2\varphi < \pi$ *be fixed. Then there exists a* $V_c \in (0, \alpha^2)$ *such that for all* $0 \le V_0 \le V_c$ *the operator* H *has at least one isolated eigenvalue below the threshold* μ *of its essential spectrum.*

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Let $V_0 < \alpha^2$ be fixed. Then to any given $n \in \mathbb{N}$ there is a $\varphi_n \in (0, \frac{\pi}{2})$ such that for all $0 < \varphi \leq \varphi_n$ we have $\sharp\sigma_{\mathrm{disc}}(H) \geq n$.

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Proof sketch: Assuming the existence of ψ_0 such that $H\psi_0 = \lambda\psi_0$ with $\lambda < 0$ we get a contradiction by angular rescaling of ψ_0.

More results

While the number of eigenvalues can be large for a sharply bent Γ it remains nevertheless finite:

Theorem (E-Vugalter'16)

In addition to (a)–(e) assume that $\mathrm{dist}(\Gamma(s) - \Gamma_{\mathrm{asympt}}(s)) = o(s^{-1})$ *as* $|s| \to \infty$, *then* $\sharp\sigma_{\mathrm{disc}}(H) < \infty$.

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and, of course, various questions remain open ...

Open questions

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The talk sources

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as well as the other papers mentioned in the course of the presentation.

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Thank you for your attention!

Shnorhakalutyun dzez hamar ushadrutyan!