Infinite volume continuum random cluster model

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1 Finite volume CRCM

Definition

- $\Lambda \subset \mathbb{R}^d$
- parameters : $R > 0$, $q > 0$ and $z > 0$
- π_Λ^z the distribution of a Poisson point process on Λ with intensity $z > 0$.

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The finite volume CRCM on Λ is defined by

$$P_\Lambda^{z,R}(d\gamma) = \frac{1}{Z_\Lambda} q^{N_{cc}(\gamma)} \pi_\Lambda^z(d\gamma),$$

where $N_{cc}(\gamma)$ is the number of connected components in

$$\bigcup_{x \in \gamma} B(x, R).$$

Motivations

- Statistical Physics : Gray representation of the Widom-Rowlinson model (Chayes, Chayes and Kotecký 95)
- Stochastic geometry : for modelling germ-grain structures (Helisova, Möller 2008)

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The setting of our work :

- Random radii : R is random and follow the distribution Q on $\mathbb{R}^+$
- Infinite volume version (DLR equations)
- Phase transition phenomena

2 Results

DLR equations

Definition

Let $\Lambda \subset \mathbb{R}^d$. The local modification of N_{cc} in Λ is defined by

$$N_{cc}^{\Lambda}(\gamma) = \lim_{\Delta \mapsto \mathbb{R}^d} N_{cc}(\gamma_\Lambda) - N_{cc}(\gamma_{\Lambda \setminus \Delta})$$

DLR equations

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Definition (DLR equations)

A probability measure P on the set of infinite configurations is a CRCM if for any bounded $\Lambda \subset \mathbb{R}^d$

$$P(d\gamma_\Lambda | \gamma_{\Lambda_c}) = \frac{1}{Z_\Lambda(\gamma_{\Lambda^c})} q^{N_{cc}^{\Lambda}(\gamma)} \pi_{\Lambda}^{z,Q}(d\gamma_\Lambda).$$

Existence results

Theorem (Der.-Houdebert)

- *If there exits* $R_0 > 0$ *such that* $Q([0, R_0]) = 1$, *then for any* $z > 0$ *and any* $q > 0$ *it exists at least one CRCM.*
- *If* $\int_0^{+\infty} R^d Q(dR) < +\infty$, *then for any* $z > 0$ *and any* $q \geq 1$ *it exists at least one CRCM.*

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Ingredients for the proof :

- entropy tightness tool $\mapsto$ accumulation point
- Burton and Keane argument $\mapsto$ uniqueness of the infinite connected component
- fine study of Gibbs kernels $\mapsto$ DLR equations

Phase transition results

Corollary

In the setting $\int_0^{+\infty} R^d Q(dR) < +\infty$, *the Widom-Rowlinson model exhibits a phase transition. There exists* $0 < z_0 < z_1 < \infty$, *such that*

- *for* $z < z_0$, *the WR model is unique.*
- *for* $z > z_1$, *the WR model is not unique.*

3 An extreme regime

Which extreme regime ?

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Question : Is there exist different CRCM from $\pi^{z,Q}$?

Result

Theorem (Der.-Houdebert)

For any integer $q \geq 2$, there exists $z_0 > 0$ such that for any $z < z_0$, there exists a CRCM different from $\pi^{z,Q}$.

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Conjecture

For any $q > 1$, there exists $z_1 > 0$ such that for any $z > z_1$, there exists an unique CRCM which is simply $\pi^{z,Q}$.

We proved the conjecture for $d = 1$. The general case is in progress...

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Specific entropy : $I(Q) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} I(P_{\Lambda_n} | \pi_{\Lambda_n}^{\otimes q})$

Lemma

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Lemma

For z small enough

$$I(P) < (q-1)z.$$