Infinite volume continuum random cluster model

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Finite volume CRCM
Definition

- $\Lambda \subset \mathbb{R}^d$
- parameters: $R > 0$, $q > 0$ and $z > 0$
- $\pi_{\Lambda}$ the distribution of a Poisson point process on $\Lambda$ with intensity $z > 0$. 
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Definition

The finite volume CRCM on $\Lambda$ is defined by

$$P_{\Lambda}^{\tilde{z}, R}(d\gamma) = \frac{1}{Z_\Lambda} q^{N_{cc}(\gamma)} \pi^\Lambda(d\gamma),$$

where $N_{cc}(\gamma)$ is the number of connected components in

$$\bigcup_{x \in \gamma} B(x, R).$$
Motivations

- Statistical Physics: Gray representation of the Widom-Rowlinson model (Chayes, Chayes and Kotecký 95)
- Stochastic geometry: for modelling germ-grain structures (Helisova, Möller 2008)
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The setting of our work:

- Random radii: $R$ is random and follow the distribution $Q$ on $\mathbb{R}^+$
- Infinite volume version (DLR equations)
- Phase transition phenomena
2 Results
Definition

Let \( \Lambda \subset \mathbb{R}^d \). The local modification of \( N_{cc} \) in \( \Lambda \) is defined by

\[
N_{cc}^\Lambda(\gamma) = \lim_{\Delta \to \mathbb{R}^d} N_{cc}(\gamma_\Lambda) - N_{cc}(\gamma_{\Lambda \setminus \Delta})
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Definition

Let $\Lambda \subset \mathbb{R}^d$. The local modification of $N_{cc}$ in $\Lambda$ is defined by

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Definition (DLR equations)

A probability measure $P$ on the set of infinite configurations is a CRCM if for any bounded $\Lambda \subset \mathbb{R}^d$

$$P(d\gamma_\Lambda | \gamma_{\Lambda^c}) = \frac{1}{Z_\Lambda(\gamma_{\Lambda^c})} q^{N_{cc}^\Lambda(\gamma)} \pi_{\Lambda}^{z,Q}(d\gamma_\Lambda).$$
Existence results

Theorem (Der.-Houdebert)

1. If there exists $R_0 > 0$ such that $Q([0, R_0]) = 1$, then for any $z > 0$ and any $q > 0$ it exists at least one CRCM.

2. If $\int_0^{+\infty} R^dQ(dR) < +\infty$, then for any $z > 0$ and any $q \geq 1$ it exists at least one CRCM.
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**Ingredients for the proof:**

- entropy tightness tool $\mapsto$ accumulation point
- Burton and Keane argument $\mapsto$ uniqueness of the infinite connected component
- fine study of Gibbs kernels $\mapsto$ DLR equations
Phase transition results

**Corollary**

In the setting $\int_{0}^{+\infty} R^d Q(dR) < +\infty$, the Widom-Rowlinson model exhibits a phase transition. There exists $0 < z_0 < z_1 < \infty$, such that

- for $z < z_0$, the WR model is unique.
- for $z > z_1$, the WR model is not unique.
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3. An extreme regime
Which extreme regime?

Assumption: \( \int_0^+ \infty R^d Q(dR) = \infty \)
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The existence of a CRCM is obvious: $\pi^{z,Q}$ is a CRCM

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Question: Is there exist different CRCM from \( \pi^{\mathbb{z}, Q} \)?
Result

Theorem (Der.-Houdebert)

For any integer \( q \geq 2 \), there exists \( z_0 > 0 \) such that for any \( z < z_0 \), there exists a CRCM different from \( \pi^{z,Q} \).
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Conjecture

For any \( q > 1 \), there exists \( z_1 > 0 \) such that for any \( z > z_1 \), there exists an unique CRCM which is simply \( \pi^{z,Q} \).

We proved the conjecture for \( d = 1 \). The general case is in progress...
Sketch of the proof

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- Let $\Lambda_n = [-n, n]^d$ and $P_n$ the WR model on $\Lambda_n$.
- By standard tightness entropy tool, $P_n \rightarrow P$. 

Specific entropy:
$$I(Q) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} I(P_{\Lambda_n} \mid \pi \otimes q_{\Lambda_n})$$

Lemma

For any monochromatic probability measure $Q$, $I(Q) \geq (q-1)z$.

Lemma

For $z$ small enough, $I(P) < (q-1)z$. 

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