Universal Bounds for Large Determinants from Non–Commutative Hölder Inequalities in Fermionic Constructive QFT

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Based on the following paper:

“Universal Bounds for Large Determinants from Non Commutative Hölder Inequalities in Fermionic Constructive QFT”

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Let $\mathcal{U}$ be a $C^*$–algebra, $\tau = \{\tau_t\}_{t \in \mathbb{R}}$ a group of automorphisms of $\mathcal{U}$ and $\rho \in \mathcal{U}^*$ be a $(\tau, \beta)$–KMS state for some $\beta \in \mathbb{R}^+$. This means the following:

- Let $\mathcal{D} = \{z \in \mathbb{C} : 0 < \text{Im} z < \beta\}$. Then, for any $A = (A_1, A_2) \in \mathcal{U}^2$, there is a continuous and bounded function $F_A$ on $\mathcal{D}$ which is analytic on $\mathcal{D}$ and such that

  $$F_A(t) = \rho(A_1 \tau_t(A_2)), \quad F_A(t + i\beta) = \rho(\tau_t(A_2)A_1), \quad t \in \mathbb{R}.$$

- More generally, for any $A = (A_1, \ldots, A_n) \in \mathcal{U}^n$ ($n \geq 2$), there is a continuous and bounded function $F_A$ on $\mathcal{D}^{n-1}$, that is analytic on $\mathcal{D}^{n-1}$ and satisfies

  $$F_A(t_2 - t_1, t_3 - t_2, \ldots, t_n - t_{n-1}) = \rho(\tau_{t_1}(A_1) \cdots \tau_{t_n}(A_n)), \quad t_1, \ldots, t_n \in \mathbb{R}.$$
KMS States and Time–Ordered Green Functions

★ Let $\mathcal{U}$ be a $C^*$–algebra, $\tau = \{\tau_t\}_{t \in \mathbb{R}}$ a group of automorphisms of $\mathcal{U}$ and $\rho \in \mathcal{U}^*$ be a $(\tau, \beta)$–KMS state for some $\beta \in \mathbb{R}^+$. This means the following:

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- More generally, for any $A = (A_1, \ldots, A_n) \in \mathcal{U}^n$ ($n \geq 2$), there is a continuous and bounded function $F_A$ on $\mathcal{D}^{n-1}$, that is analytic on $\mathcal{D}^{n-1}$ and satisfies
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★ For $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ so that $\alpha_j \leq \alpha_{j+1}$, we define the time–ordered Green functions
  \[ G_A(\alpha_1, \ldots, \alpha_n) \doteq F_A(i(\alpha_2 - \alpha_1), \ldots, i(\alpha_n - \alpha_{n-1})). \]
Let $\mathcal{U}$ be a $C^*$–algebra, $\tau = \{\tau_t\}_{t \in \mathbb{R}}$ a group of automorphisms of $\mathcal{U}$ and $\rho \in \mathcal{U}^*$ be a $(\tau, \beta)$–KMS state for some $\beta \in \mathbb{R}^+$. This means the following:

- Let $\mathcal{D} = \{ z \in \mathbb{C} : 0 < \text{Im} z < \beta \}$. Then, for any $A = (A_1, A_2) \in \mathcal{U}^2$, there is a continuous and bounded function $F_A$ on $\mathcal{D}$ which is analytic on $\mathcal{D}$ and such that

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- More generally, for any $A = (A_1, \ldots, A_n) \in \mathcal{U}^n$ ($n \geq 2$), there is a continuous and bounded function $F_A$ on $\mathcal{D}^{n-1}$, that is analytic on $\mathcal{D}^{n-1}$ and satisfies

  $$F_A(t_2 - t_1, t_3 - t_2, \ldots, t_n - t_{n-1}) = \rho (\tau_{t_1}(A_1) \cdots \tau_{t_n}(A_n)), \quad t_1, \ldots, t_n \in \mathbb{R}.$$  

For $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ so that $\alpha_j \leq \alpha_{j+1}$, we define the time–ordered Green functions

$$G_A(\alpha_1, \ldots, \alpha_n) \equiv F_A(i(\alpha_2 - \alpha_1), \ldots, i(\alpha_n - \alpha_{n-1})).$$

The aim is to show the existence and uniqueness of KMS states and evaluate (truncated) Green functions.
Example: Fermionic Dynamics on Lattices

- Let $d \in \mathbb{N}$ and $\mathcal{U}_0$ be the normed $*$-algebra generated by the identity $1$ and the creation / annihilation operators $\{a^*_x, a_x\}_{x \in \mathbb{Z}^d}$ satisfying the CAR. This means that elements of $\mathcal{U}_0$ are finite sums of monomials of $\{a^*_x, a_x\}_{x \in \mathbb{Z}^d}$.

- The CAR $C^*$-algebra $\mathcal{U} = CAR(\ell^2(\mathbb{Z}^d))$ of the infinite system is by definition the completion of the normed $*$-algebra $\mathcal{U}_0$.

- Let $[A, B] = AB - BA$ and $h, v : \mathbb{R}_0^+ \to \mathbb{R}$ be such that for some fixed $\varsigma, C \in \mathbb{R}^+$,
  \[
  \max \{|h(r)|, |v(r)|\} \leq C (1 + r)^{-(d+\varsigma)} , \quad r \in \mathbb{R}_0^+ .
  \]
  The fermionic dynamics is given for $\lambda \in \mathbb{R}$ by the group $\tau^{(\lambda)} = \{\tau_t^{(\lambda)}\}_{t \in \mathbb{R}}$ of automorphisms of $\mathcal{U}$ with generator defined by
  \[
  \delta^{(\lambda)}(A) = i \sum_{x, y \in \mathbb{Z}^d} h (|x - y|) [a^*_x a_y, A] + \lambda v (|x - y|) [a^*_x a_x a^*_y a_y, A] , \quad A \in \mathcal{U}_0 .
  \]
  The case $\lambda = 0$ is the free dynamics and we want to study $G_A^{(\lambda)}$, starting from $G_A^{(0)}$. 
Spaces of Antiperiodic Functions on Discrete Tori

- Fix $\beta \in \mathbb{R}^+$, an **even** integer $n \in 2\mathbb{N}$ and let
  $$\mathbb{T}_n \doteq \left\{-\beta + kn^{-1} \beta : k \in \{1, 2, \ldots, 2n\}\right\} \subset (-\beta, \beta]$$

- Pick any separable Hilbert space $\mathfrak{h}$ and let $\ell^2_{ap}(\mathbb{T}_n; \mathfrak{h})$ be the Hilbert space of functions from $\mathbb{T}_n$ to $\mathfrak{h}$ which are **antiperiodic**. That is, for $f \in \ell^2_{ap}(\mathbb{T}_n; \mathfrak{h})$,
  $$f(\alpha + \beta) = -f(\alpha) \; , \quad \alpha \in \mathbb{T}_n .$$
  The scalar product on $\ell^2_{ap}(\mathbb{T}_n; \mathfrak{h})$ is then defined to be
  $$\langle f_1, f_2 \rangle_{\ell^2_{ap}(\mathbb{T}_n; \mathfrak{h})} \doteq n^{-1} \beta \sum_{\alpha \in \mathbb{T}_n} \langle f_1(\alpha) , f_2(\alpha) \rangle_{\mathfrak{h}} , \quad f_1, f_2 \in \ell^2_{ap}(\mathbb{T}_n; \mathfrak{h}) .$$

- Vectors $\varphi$ of $\mathfrak{h}$ are viewed as antiperiodic functions $\hat{\varphi}$ of $\ell^2_{ap}(\mathbb{T}_n; \mathfrak{h})$ via the definition
  $$\hat{\varphi}(\alpha) \doteq (\beta^{-1} n/2) \delta_{0,\alpha} \varphi \; , \quad \alpha \in (-\beta, 0] \cap \mathbb{T}_n .$$
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  The scalar product on $\ell^2_{\text{ap}}(\mathbb{T}_n; \mathfrak{h})$ is then defined to be
  $$\langle f_1, f_2 \rangle_{\ell^2_{\text{ap}}(\mathbb{T}_n; \mathfrak{h})} \doteq n^{-1}\beta \sum_{\alpha \in \mathbb{T}_n} \langle f_1(\alpha), f_2(\alpha) \rangle_{\mathfrak{h}}, \quad f_1, f_2 \in \ell^2_{\text{ap}}(\mathbb{T}_n; \mathfrak{h}).$$

- Vectors $\varphi$ of $\mathfrak{h}$ are viewed as antiperiodic functions $\hat{\varphi}$ of $\ell^2_{\text{ap}}(\mathbb{T}_n; \mathfrak{h})$ via the definition
  $$\hat{\varphi}(\alpha) \doteq (\beta^{-1}n/2)\delta_{0,\alpha}\varphi, \quad \alpha \in (-\beta, 0] \cap \mathbb{T}_n.$$

Example: $\mathfrak{h}$ is the one–particle Hilbert space $\ell^2(\mathbb{Z}^d)$ with ONB $\{\varepsilon_x\}_{x \in \mathbb{Z}^d}$ defined by
  $$\varepsilon_x(y) \doteq \delta_{x,y}, \quad x, y \in \mathbb{Z}^d.$$
Any operator $H$ acting on $\mathfrak{h}$ with domain $\text{dom}(H)$ is viewed as an operator $\hat{H}$ with domain $\ell^2_{\text{ap}}(\mathbb{T}_n; \text{dom}(\hat{H})) \subset \ell^2_{\text{ap}}(\mathbb{T}_n; \mathfrak{h})$ by the definition

$$[\hat{H}f](\alpha) = H(f(\alpha)) , \quad f \in \ell^2_{\text{ap}}(\mathbb{T}_n; \text{dom}(\hat{H})), \alpha \in \mathbb{T}_n .$$

The discrete time derivative $\partial \in \mathcal{B}(\ell^2_{\text{ap}}(\mathbb{T}_n; \mathfrak{h}))$ is the normal invertible operator defined by

$$\partial f(\alpha) = \beta^{-1}n \left( f(\alpha + n^{-1}\beta) - f(\alpha) \right) , \quad f \in \ell^2_{\text{ap}}(\mathbb{T}_n; \mathfrak{h}), \alpha \in \mathbb{T}_n .$$

The *discrete time covariance* is thus defined for any $H = H^*$ to be

$$C_H \doteq -2 \left( \partial + \hat{H} \right)^{-1} \in \mathcal{B}(\ell^2_{\text{ap}}(\mathbb{T}_n; \mathfrak{h})) .$$
Discrete Time Covariance

- Any operator $H$ acting on $\mathfrak{h}$ with domain $\text{dom}(H)$ is viewed as an operator $\hat{H}$ with domain $\ell^2_{ap}(\mathbb{T}_n; \text{dom}(\hat{H})) \subset \ell^2_{ap}(\mathbb{T}_n; \mathfrak{h})$ by the definition
  $$[\hat{H}f](\alpha) \doteq H(f(\alpha)),$$  
  $f \in \ell^2_{ap}(\mathbb{T}_n; \text{dom}(\hat{H})), \alpha \in \mathbb{T}_n.$

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  $$\partial f(\alpha) \doteq \beta^{-1}n\left(f\left(\alpha + \frac{1}{n}\beta\right) - f(\alpha)\right),$$  
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- The discrete time covariance is thus defined for any $H = H^*$ to be
  $$C_H \doteq -2\left(\partial + \hat{H}\right)^{-1} \in \mathcal{B}(\ell^2_{ap}(\mathbb{T}_n; \mathfrak{h})).$$

Example: $H = H^* \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$ is the one–particle Hamiltonian such that
  $$h(|x - y|) = \langle \mathbf{e}_x, H \mathbf{e}_y \rangle_{\ell^2(\mathbb{Z}^d)}, \quad x, y \in \mathbb{Z}^d.$$
The convergence of perturbation expansions of all correlation functions (cf. $G^{(\lambda)}_A$) in non–relativistic fermionic constructive QFT at weak coupling $\lambda$ is ensured if the interaction and the covariance are summable and if certain determinants arising in the expansion can be bounded efficiently:

**Definition (Determinant bounds)**

Let $\mathfrak{h}$ be a separable Hilbert space with ONB $\{\varphi_i\}_{i \in \mathbb{I}}$, $\mathbb{I}$ being countable. $\gamma_H \in \mathbb{R}^+$ is a **determinant bound** of $H = H^*$ if, for any $\beta \in \mathbb{R}^+$, $n \in 2\mathbb{N}$, $m, N \in \mathbb{N}$, $\mathcal{M} \in \text{Mat}(m, \mathbb{R})$ with $\mathcal{M} \geq 0$, and all parameters

$$\{(\alpha_q, i_q, j_q)\}_{q=1}^{2N} \subset \mathbb{T}_n \cap [0, \beta) \times \mathbb{I} \times \{1, \ldots, m\},$$

the following bound holds true:

$$\left| \det \left[ \mathcal{M}_{j_k,j_{N+l}} \langle \varphi_{i_{N+l}}, (C_H \hat{\varphi}_i) (\alpha_k - \alpha_{N+l}) \rangle_{\mathfrak{h}} \right]_{k,l=1}^N \right| \leq \gamma_H^{2N} \prod_{q=1}^{2N} \mathcal{M}_{j_q,j_q}^{1/2}.$$
For $\lambda \in \mathbb{R}$, $\tau^{(\lambda)} = \{\tau^{(\lambda)}_t\}_{t \in \mathbb{R}}$ is the group of automorphisms of the CAR algebra $\mathcal{U} = \text{CAR}(\ell^2(\mathbb{Z}^d))$ with generator defined by

$$
\delta^{(\lambda)}(A) \doteq i \sum_{x, y \in \mathbb{Z}^d} \langle e_x, H e_y \rangle_{\ell^2(\mathbb{Z}^d)} [a^*_x a_y, A] + \lambda v(|x - y|) [a^*_x a_x a^*_y a_y, A], \quad A \in \mathcal{U}_0.
$$

Let

$$
\omega_H \doteq \limsup_{n \to \infty} \sup_{x \in \mathbb{Z}^d} \left\{ n^{-1} \beta \sum_{\vartheta \in \mathcal{T}_n} \sum_{y \in \mathbb{Z}^d} \left| \langle e_y, (C_H \hat{e}_x)(\vartheta) \rangle_{\ell^2(\mathbb{Z}^d)} \right| \right\}.
$$

If $\omega_H \gamma_H^2 |\lambda| \ll 1$, then one gets:
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If $\omega_H \gamma^2_H |\lambda| \ll 1$, then one gets:

1. Existence of a unique translation invariant $(\tau^{(\lambda)}, \beta)$–KMS state.
Example: Correlation Functions of Lattice Fermion System

- For $\lambda \in \mathbb{R}$, $\tau^{(\lambda)} = \{\tau_t^{(\lambda)}\}_{t \in \mathbb{R}}$ is the group of automorphisms of the CAR algebra $\mathcal{U} = \text{CAR}(\ell^2(\mathbb{Z}^d))$ with generator defined by

$$\delta^{(\lambda)}(A) \doteq i \sum_{x,y \in \mathbb{Z}^d} \langle e_x, H e_y \rangle_{\ell^2(\mathbb{Z}^d)} [a_x^* a_y, A] + \lambda \nu (|x - y|) \left[ a_x^* a_x a_y^* a_y, A \right], \quad A \in \mathcal{U}_0.$$

- Let

$$\omega_H \doteq \limsup_{n \to \infty} \sup_{x \in \mathbb{Z}^d} \left\{ n^{-1} \beta \sum_{\vartheta \in T_n} \sum_{y \in \mathbb{Z}^d} \left| \langle e_y, (C_H \hat{e}_x)(\vartheta) \rangle_{\ell^2(\mathbb{Z}^d)} \right| \right\}.$$

- If $\omega_H \gamma_H^2 |\lambda| \ll 1$, then one gets:

1. Existence of a unique \textit{translation invariant} $(\tau^{(\lambda)}, \beta)$–KMS state.

2. Perturbation expansion of \textit{all correlation functions} in terms of powers of $\lambda$ converges absolutely. More precisely, all correlation functions are analytic functions of the coupling $\lambda$ at $\lambda = 0$ with analyticity radius of order $\omega_H^{-1} \gamma_H^{-2}$.
Example: Correlation Functions of Lattice Fermion System

- For \( j \in \{1, \ldots, n\} \), let \( X_j = (\alpha_j, \nu_j, x_j) \in \mathbb{R} \times \{+, -\} \times \mathbb{Z}^d \). If \( \alpha_j \leq \alpha_{j+1} \) then
  \[
  G^{(n)} (X_1, \ldots, X_n) = G_{(a_{\alpha_1}, \ldots, a_{\alpha_n})} (\alpha_1, \ldots, \alpha_n) \quad \text{with} \quad a_x^+ \overset{=}{\doteq} a_x^* \text{ and } a_x^- \overset{=}{\doteq} a_x.
  \]
  
  \( G^{(n)} \) is extended to all \( \alpha_j \in \mathbb{R} \) so that it is antisymmetric w.r.t. permutations \( \pi \in S_n \) of its \( n \) arguments and periodic with period \( 2\beta \) w.r.t. to each \( \alpha_j \).

- Truncated Green functions \( G_T^{(n)} (X_1, \ldots, X_n) \) are recursively defined by
  \[
  \sum_{k=1}^{n} \sum_{\pi \in S_n} \frac{(-1)^\pi}{k!(n-k)!} G_T^{(k)} (X_{\pi(1)}, \ldots, X_{\pi(k)}) G_T^{(n-k)} (X_{\pi(k+1)}, \ldots, X_{\pi(n-k)}) = G^{(n)} (X_1, \ldots, X_n).
  \]

Theorem (Pedra, 2005 and many others)
For \( \omega H \gamma_H^2 |\lambda| \) sufficiently small, there are a unique translation invariant \( (\tau^{(\lambda)}, \beta) \)-KMS state, constants \( C_1, C_2, \ldots \) and a radius \( R = O(\omega_H^{-1} \gamma_H^{-2}) > 0 \) so that

\[
\left| \frac{\partial^k}{\partial \lambda^k} G_T^{(n, \lambda)} (X_1, \ldots, X_n) \right| \leq C_n k! R^{-k}, \quad k \in \mathbb{N}_0, \ n \in \mathbb{N}, \ X_1, \ldots, X_n \in \mathbb{R} \times \{+, -\} \times \mathbb{Z}^d.
\]
Lemma (Gram bound)

Let $\mathcal{H}$ be an Hilbert space and $C \in \mathcal{B}(\mathcal{H})$. Then, for any $N \in \mathbb{N}$ and $u_1, \ldots, u_{2N} \in \mathcal{H}$,

$$
\left| \det \left[ \langle u_{N+l}, Cu_k \rangle_{\mathcal{H}} \right]_{k,l=1}^N \right| \leq \|C\|_{\mathcal{B}(\mathcal{H})}^N \|u_1\|_{\mathcal{H}} \cdots \|u_{2N}\|_{\mathcal{H}}.
$$

Therefore, for any $\beta \in \mathbb{R}^+$, $n \in 2\mathbb{N}$, $m, N \in \mathbb{N}$, $\mathcal{M} \in \text{Mat}(m, \mathbb{R})$ with $\mathcal{M} \geq 0$, and

$$
\{(\alpha_q, i_q, j_q) \}_{q=1}^{2N} \subset \mathbb{T}_n \cap [0, \beta) \times \mathbb{I} \times \{1, \ldots, m\},
$$

the following bound holds true:

$$
\left| \det \left[ \mathcal{M}_{j_k,j_{N+l}} \langle \varphi_{i_{N+l}}, (C_H \varphi_i) (\alpha_k - \alpha_{N+l}) \rangle_{\mathcal{H}} \right]_{k,l=1}^N \right| \leq \|C_H\|_{\mathcal{B}(\ell^2_{ap}(\mathbb{T}_n;\mathcal{H}))}^N \prod_{q=1}^{2N} \|\varphi_{i_q}\|_{\ell^2_{ap}(\mathbb{T}_n;\mathcal{H})} \mathcal{M}_{j_q,j_q}^{1/2}.
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Lemma (Gram bound)

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$$\left| \det \left[ \langle u_{N+l}, Cu_k \rangle_{\mathcal{H}} \right]_{k,l=1}^N \right| \leq \| C \|_{\mathcal{B}(\mathcal{H})}^N \| u_1 \|_{\mathcal{H}} \cdots \| u_{2N} \|_{\mathcal{H}} .$$

Therefore, for any $\beta \in \mathbb{R}^+$, $n \in 2\mathbb{N}$, $m, N \in \mathbb{N}$, $M \in \text{Mat} (m, \mathbb{R})$ with $M \geq 0$, and

$$\{(\alpha_q, i_q, j_q)\}_{q=1}^{2N} \subset \mathbb{T}_n \cap [0, \beta) \times I \times \{1, \ldots, m\} ,$$

the following bound holds true:

$$\left| \det \left[ M_{j_q,j_{N+l}} \langle \varphi_{i_{N+l}}, (C_H \hat{\varphi}_{i_k})(\alpha_k - \alpha_{N+l}) \rangle_{\mathfrak{h}} \right]_{k,l=1}^N \right| \leq \| C_H \|_{\mathcal{B}(\ell^2_{ap}(\mathbb{T}_n;\mathfrak{h}))}^N \prod_{q=1}^{2N} \| \hat{\varphi}_{i_q} \|_{\ell^2_{ap}(\mathbb{T}_n;\mathfrak{h})} \frac{M^{1/2}_{j_q,j_q}}{q} .$$

**Problem:** In general, the norm of $C_H$ diverges, as $n \to \infty$. This problem appears already for bounded $H \in \mathcal{B}(\mathfrak{h})$ when $0 \in \text{spec}(H)$. In this case,

$$\| C_H \|_{\mathcal{B}(\ell^2_{ap}(\mathbb{T}_n;\mathfrak{h}))}^{1/2} = \mathcal{O} (\sqrt{n}) \quad \text{and} \quad \| \hat{\varphi}_{i_q} \|_{\ell^2_{ap}(\mathbb{T}_n;\mathfrak{h})} = \mathcal{O} (\sqrt{n}) .$$
Gram Bound and Multiscale Analyses

One uses the *Gram bound* for some regularized covariances $C_H \hat{\kappa}_L (\hat{H}, i\partial)$ at every $L \in \mathbb{N}$. Here, for $L \in \mathbb{N}$, $\hat{\kappa}_L : \mathbb{R}^2 \to [0, 1]$ defines a family of measurable functions so that

$$\sum_{L=1}^{\infty} \hat{\kappa}_L (x, y) = 1, \quad x, y \in \mathbb{R}.$$ 

This decomposition can be chosen such that there are constants $\hat{\gamma}_L \in \mathbb{R}^+$, $L \in \mathbb{N}$, which at least do not depend on $n \in 2\mathbb{N}$ and meanwhile satisfy

$$\left| \det \begin{bmatrix} m_{j_k j_{N+l}} & \left\langle \varphi_{i_{N+l}}, \left( C_H \hat{\kappa}_L (\hat{H}, i\partial) \hat{\varphi}_{i_k} \right) (\alpha_k - \alpha_{N+l}) \right\rangle_{\hbar} \end{bmatrix}_{k,l=1}^N \right| \leq \hat{\gamma}_L^{2N} \prod_{q=1}^{2N} m_{j_q j_q}^{1/2}.$$ 

See, e.g., [Benfatto-Gallavotti-Procacci-Scoppola, '94], [Giuliani-Mastropietro, '10], [Giuliani-Mastropietro-Porta, '16] and in many others works.
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This decomposition can be chosen such that there are constants $\hat{\gamma}_L \in \mathbb{R}^+, L \in \mathbb{N}$, which at least do not depend on $n \in 2\mathbb{N}$ and meanwhile satisfy

$$\left| \det \left[ M_{j_k, i_{N+l}} \left( \varphi_{i_{N+l}}, \left( C_H \hat{\kappa}_L (\hat{H}, i\partial) \varphi_{i_k} \right) (\alpha_k - \alpha_{N+l}) \right)_\hbar \right]_{k, l=1}^{N} \right| \leq \hat{\gamma}_L^{2N} \prod_{q=1}^{2N} M_{j_q, j_q}^{1/2}.$$ 

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**Observation:** [Pedra-Salmhofer, '08] shows that this multiscale analysis for the so-called Matsubara UV problem is not necessary, by proving a new bound for determinants that generalizes the original Gram bound. Avoiding this kind of procedure brings various technical benefits.
We also show that the UV regularization of the Matsubara frequency is not necessary, but, in contrast with [Pedra-Salmhofer, '08], the given covariance does not need to be written as a chronological sum to obtain determinant bounds.
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Our estimates are sharp and hold true for all (possibly unbounded, the latter not being limited to semibounded) one–particle Hamiltonians $H$:
Universal Determinant Bound

- We also show that the UV regularization of the Matsubara frequency is not necessary, but, in contrast with [Pedra-Salmhofer, ’08], the given covariance does not need to be written as a chronological sum to obtain determinant bounds.

- Our estimates are sharp and hold true for all (possibly unbounded, the latter not being limited to semibounded) one–particle Hamiltonians $H$:

**Theorem (B-Pedra, ’16)**

\[ \hat{\tau} \doteq \sup \{ \inf \gamma_H : H = H^* \text{ acting on a separable Hilbert space } \mathfrak{h} \text{ with ONB } \{ \varphi_i \}_{i \in \mathbb{I}} \}, \]

named the universal determinant bound, is equal to $\hat{\tau} = 1$. 
We also show that the UV regularization of the Matsubara frequency is not necessary, but, in contrast with [Pedra-Salmhofer, '08], the given covariance does \textit{not} need to be written as a chronological sum to obtain determinant bounds.

Our estimates are \textit{sharp} and hold true for \textit{all} (possibly unbounded, the latter not being limited to semibounded) one–particle Hamiltonians $H$:

\begin{theorem}[B-Pedra, '16]
\end{theorem}

\begin{align*}
\tau \doteq \sup \left\{ \inf \gamma_H : H = H^* \text{ acting on a separable Hilbert space } \mathfrak{h} \text{ with ONB } \{ \varphi_i \}_{i \in I} \right\}, \\
\text{named the universal determinant bound, is equal to } \tau = 1.
\end{align*}

In other words, for any $\beta \in \mathbb{R}^+$, $n \in 2\mathbb{N}$, $m, N \in \mathbb{N}$, $\mathcal{M} \in \text{Mat}(m, \mathbb{R})$ with $\mathcal{M} \geq 0$, and

\[\{(\alpha_q, i_q, j_q)\}_{q=1}^{2N} \subset \mathbb{T}_n \cap [0, \beta) \times I \times \{1, \ldots, m\},\]

the following bound holds true:

\[
\left| \det \left[ \mathcal{M}_{j_k j_{N+l}} \langle \varphi_{N+l}, (C_H \hat{\varphi}_k) (\alpha_k - \alpha_{N+l}) \rangle_{\mathfrak{h}} \right]_{k,l=1}^N \right| \leq \prod_{q=1}^{2N} \| \varphi_q \|_{\mathfrak{h}} \mathcal{M}_{j_q j_q}^{1/2}.
\]
We also show that the UV regularization of the Matsubara frequency is not necessary, but, in contrast with [Pedra-Salmhofer, '08], the given covariance does not need to be written as a chronological sum to obtain determinant bounds.

Our estimates are sharp and hold true for all (possibly unbounded, the latter not being limited to semibounded) one–particle Hamiltonians $H$:

**Theorem (B–Pedra, '16)**

$$\tilde{\tau} \equiv \sup \left\{ \inf \gamma_H : H = H^* \text{ acting on a separable Hilbert space } \mathcal{H} \text{ with ONB } \{ \varphi_i \}_{i \in I} \right\},$$

named the universal determinant bound, is equal to $\tilde{\tau} = 1$.

[Pedra-Salmhofer,'08] gives $\gamma_H = 2$ for the class of bounded operators $H$ it applies.

Thus, the convergence of perturbation series at $\lambda = 0$ of any non–relativistic fermionic QFT (possibly in the continuum) is only ensured by the smallness of

$$\omega_H \equiv \limsup_{n \to \infty} \sup_{x \in \mathbb{Z}^d} \left\{ n^{-1} \beta \sum_{\vartheta \in T_n} \sum_{y \in \mathbb{Z}^d} \left| \langle e_y, (C_H \hat{e}_x) (\vartheta) \rangle \right|_{\ell^2(\mathbb{Z}^d)} \right\}.$$
Heuristic of the Proof: Quasi–Free States

Let $\mathcal{H}$ be any (separable) Hilbert space and $\mathcal{U} = \text{CAR}(\mathcal{H})$ the $C^*$–algebra generated by the unit $1$ and the family $\{a(\Psi)\}_{\Psi \in \mathcal{H}}$ of elements satisfying the CAR.

**Definition (Quasi–free states with one-particle Hamiltonian $\hat{H} = \hat{H}^*$)**

Quasi–free states are positive linear functionals $\rho \in \text{CAR}(\mathcal{H})^*$ such that $\rho(1) = 1$ and, for all $N \in \mathbb{N}$ and $\Psi_1, \ldots, \Psi_{2N} \in \mathcal{H}$,

$$\rho\left(a^+(\Psi_1) \cdots a^+(\Psi_N)a(\Psi_{2N}) \cdots a(\Psi_{N+1})\right) = \det \left[ \left\langle \psi, \frac{1}{1 + e^{\hat{H}}} \psi_{N+l} \right\rangle_{\mathcal{H}} \right]_{k,l=1}^N.$$
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Compare with the determinant we have to estimate:

$$\det \left[ M_{j_k,j_{N+l}} \langle \varphi_{N+l}, (C_H\hat{\varphi}_k)(\alpha_k - \alpha_{N+l}) \rangle_{\mathcal{H}} \right]_{k,l=1}^N.$$

Actually, we construct a one-particle Hamiltonian $H = H^* \in \mathcal{B}(\mathcal{H})$ such that

$$M_{j_k,j_{N+l}} \langle \varphi_{N+l}, (C_H\hat{\varphi}_k)(\alpha_k - \alpha_{N+l}) \rangle_{\mathcal{H}} = \pm \left\langle \psi_k, \frac{1}{1 + e^H} \psi_{N+l} \right\rangle_{\mathcal{H}} = \pm \rho \left( a^+(\psi_k)a(\psi_{N+l}) \right).$$
To simplify, fix \( n \in 2\mathbb{N} \), \( \beta = 1 = \mathcal{M} \in \text{Mat}(1, \mathbb{R}) \), and the one-particle hamiltonian \( H = H^* \in \mathcal{B}(\mathfrak{h}) \).

**Lemma**

For \( m \in \mathbb{N} \), \( \alpha_1, \alpha_2 \in \mathbb{T}_n \cap [0, 1) \), and \( \varphi_1, \varphi_2 \in \mathfrak{h} \), there are \( \psi_1, \psi_2 \in \mathfrak{h} \) with \( \|\psi_{1,2}\|_{\mathfrak{h}} \leq \|\varphi_{1,2}\|_{\mathfrak{h}} \), quasi–free states \( \rho_\nu, \nu \in \mathbb{R}^+ \), with one-particle hamiltonian

\[
H_\nu \simeq -n \ln \left| 1 - n^{-1}H \right|, \quad \text{i.e.,} \quad e^{\mp H_\nu} = \left(1 - n^{-1}H\right)^{\pm n} \simeq e^{\mp H} + o(1),
\]

such that for \( \alpha_1 \leq \alpha_2 \),

\[
\langle \varphi_2, (C_H \hat{\varphi}_1)(\alpha_1 - \alpha_2) \rangle_{\mathfrak{h}} = \lim_{\nu \to \infty} \rho_\nu \left( a^+ \left( e^{-\alpha_1 H_\nu} \psi_1 \right) a \left( e^{(\alpha_2 + n^{-1})H_\nu} \psi_2 \right) \right)
\]

while for \( \alpha_1 > \alpha_2 \),

\[
\langle \varphi_2, (C_H \hat{\varphi}_1)(\alpha_1 - \alpha_2) \rangle_{\mathfrak{h}} = -\lim_{\nu \to \infty} \rho_\nu \left( a \left( e^{(\alpha_2 + n^{-1})H_\nu} \psi_2 \right) a^+ \left( e^{-\alpha_1 H_\nu} \psi_1 \right) \right).
\]
Assume that $\mathcal{H}$ is a finite dimensional Hilbert space. Then, $\text{CAR}(\mathcal{H}) \sim \mathcal{B}(\Lambda H)$ with $\Lambda H$ being the fermionic Fock space and, for $\nu \in \mathbb{R}^+$,

$$\rho_\nu (A) \doteq \text{Tr}_{\Lambda H} (A D_\nu) \doteq \text{Tr}_{\Lambda H} \left( A e^{-H_\nu} \right) , \quad A \in \mathcal{B}(\Lambda H) .$$

For $q \in \{1, \ldots, 2N\}$ and $\alpha_q \in \mathbb{T}_N \cap [0, 1)$ such that $\vartheta_q \doteq \alpha_q - \alpha_{q-1} \geq 0$ for $q \geq 2$,

$$\left| \det \left[ \langle \phi_{N+1}, (C_H \hat{\phi}_k) (\alpha_k - \alpha_{N+1}) \rangle_{\mathcal{H}} \right]_{k, l=1}^N \right| \simeq \lim_{\nu \to \infty} |\text{Tr}_{\Lambda H} (X_\nu)| \leq \lim_{\nu \to \infty} \|X_\nu\|_1$$

with

$$X_\nu \doteq D_\nu^{\vartheta_2} a^+(\psi_2) \cdots a^+(\psi_N) D_\nu^{\vartheta_{N+1}} a(\psi_{N+1}) \cdots D_\nu^{\vartheta_{2N}} a(\psi_{2N}) D_\nu^{1-(\vartheta_2+\cdots+\vartheta_{2N})} a^+(\psi_1)$$

and

$$\|A\|_s \doteq \left( \text{Tr}_{\Lambda H} (|A|^s) \right)^{\frac{1}{s}} , \quad A \in \mathcal{B}(\Lambda H) , \ s \geq 1 .$$
Heuristic of the Proof: Hölder Inequalities

- Hölder inequalities for Schatten norms: For \( n \in \mathbb{N}, r, s_1, \ldots, s_n \in [1, \infty] \) such that \( \sum_{j=1}^{n} 1/s_j = 1/r \), and all operators \( A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H}) \),

\[
\|A_1 \cdots A_n\|_r \leq \prod_{j=1}^{n} \|A_j\|_{s_j}.
\]

- Then, using Hölder inequalities for \( r = 1 \),

\[
\|X_\nu\|_1 = \left\| D_{\nu}^{\partial_2} a^+(\psi_2) \cdots D_{\nu}^{\partial_{2N}} a(\psi_{2N}) D_{\nu}^{1-(\partial_2+\cdots+\partial_{2N})} a^+(\psi_1) \right\|_1
\]

\[
\leq \left\| D_{\nu}^{\partial_2} \right\|_{1/\partial_2} \cdots \left\| D_{\nu}^{\partial_{2N}} \right\|_{1/\partial_{2N}} \left\| D_{\nu}^{1-(\partial_2+\cdots+\partial_{2N})} \right\|_{1-(\partial_2+\cdots+\partial_{2N})} \prod_{q=1}^{2N} \|a(\psi_q)\|_{\infty}.
\]

- Since \( \|D_\nu\|_1 = 1 \) and for \( q \in \{1, \ldots, 2N\} \),

\[
\|a(\psi_q)\|_{\infty} = \|a(\psi_q)\|_{\text{CAR}(\mathcal{H})} = \|\psi_q\|_{\mathcal{H}} \leq \|\varphi_q\|_{\mathcal{H}},
\]

it follows that

\[
\left| \det \left[ \langle \varphi_{N+1}, (\mathcal{C}_H \hat{\varphi}_k)(\alpha_k - \alpha_{N+1}) \rangle_{\mathcal{H}} \right]_{k,l=1}^{N} \right| \leq \lim_{\nu \to \infty} \|X_\nu\|_1 \leq \prod_{q=1}^{2N} \|\varphi_q\|_{\mathcal{H}}.
\]