Universal Bounds for Large Determinants from Non–Commutative Hölder Inequalities in Fermionic Constructive QFT

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Based on the following paper:

"Universal Bounds for Large Determinants from Non Commutative Hölder Inequalities in Fermionic Constructive QFT"
(Submitted 2016)

KMS States and Time–Ordered Green Functions

★ Let $\mathcal{U}$ be a C^*–algebra, $\tau = \{\tau_t\}_{t\in\mathbb{R}}$ a group of automorphisms of $\mathcal{U}$ and $\rho \in \mathcal{U}^*$ be a (τ, β)–KMS state for some $\beta \in \mathbb{R}^+$. This means the following:

- Let $\mathfrak{D} \doteq \{z \in \mathbb{C} : 0 < \mathrm{Im}z < \beta\}$. Then, for any $\mathbf{A} = (A_1, A_2) \in \mathcal{U}^2$, there is a continuous and bounded function $F_{\mathbf{A}}$ on $\overline{\mathfrak{D}}$ which is analytic on $\mathfrak{D}$ and such that

$$F_{\mathbf{A}}(t) = \rho(A_1\tau_t(A_2)), \qquad F_{\mathbf{A}}(t + i\beta) = \rho(\tau_t(A_2)A_1), \qquad t \in \mathbb{R}.$$

- More generally, for any $\mathbf{A} = (A_1, \ldots, A_n) \in \mathcal{U}^n$ $(n \geq 2)$, there is a continuous and bounded function $F_{\mathbf{A}}$ on $\overline{\mathfrak{D}^{n-1}}$, that is analytic on $\mathfrak{D}^{n-1}$ and satisfies

$$F_{\mathbf{A}}(t_2 - t_1, t_3 - t_2, \ldots, t_n - t_{n-1}) = \rho(\tau_{t_1}(A_1)\cdots\tau_{t_n}(A_n)), \qquad t_1, \ldots, t_n \in \mathbb{R}.$$

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★ For $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ so that $\alpha_j \leq \alpha_{j+1}$, we define the time–ordered Green functions

$$G_{\mathbf{A}}(\alpha_1, \ldots, \alpha_n) \doteq F_{\mathbf{A}}(i(\alpha_2 - \alpha_1), \ldots, i(\alpha_n - \alpha_{n-1})) \ .$$

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$\Longrightarrow$ The aim is to show the existence and uniqueness of KMS states and evaluate (truncated) Green functions.

Example: Fermionic Dynamics on Lattices

- Let $d \in \mathbb{N}$ and $\mathcal{U}_0$ be the normed $*$–algebra generated by the identity $\mathbf{1}$ and the creation / annihilation operators $\{a_x^*, a_x\}_{x\in\mathbb{Z}^d}$ satisfying the CAR. This means that elements of $\mathcal{U}_0$ are finite sums of monomials of $\{a_x^*, a_x\}_{x\in\mathbb{Z}^d}$.

- The CAR C^*–algebra $\mathcal{U} = \mathrm{CAR}(\ell^2(\mathbb{Z}^d))$ of the infinite system is by definition the completion of the normed $*$–algebra $\mathcal{U}_0$.

- Let $[A, B] \doteq AB - BA$ and $h, v : \mathbb{R}_0^+ \to \mathbb{R}$ be such that for some fixed $\varsigma, C \in \mathbb{R}^+$,

$$\max\{|h(r)|, |v(r)|\} \leq C(1+r)^{-(d+\varsigma)}, \qquad r \in \mathbb{R}_0^+ .$$

  The fermionic dynamics is given for $\lambda \in \mathbb{R}$ by the group $\tau^{(\lambda)} = \{\tau_t^{(\lambda)}\}_{t\in\mathbb{R}}$ of automorphisms of $\mathcal{U}$ with generator defined by

$$\delta^{(\lambda)}(A) \doteq i \sum_{x,y\in\mathbb{Z}^d} h(|x-y|)\,[a_x^* a_y, A] + \lambda v(|x-y|)\left[a_x^* a_x a_y^* a_y, A\right], \qquad A \in \mathcal{U}_0 .$$

  The case $\lambda = 0$ is the free dynamics and we want to study $G_{\mathbf{A}}^{(\lambda)}$, starting from $G_{\mathbf{A}}^{(0)}$.

Spaces of Antiperiodic Functions on Discrete Tori

- Fix $\beta \in \mathbb{R}^+$, an *even* integer $n \in 2\mathbb{N}$ and let

$$\mathbb{T}_n \doteq \left\{ -\beta + kn^{-1}\beta : k \in \{1, 2, \ldots, 2n\} \right\} \subset (-\beta, \beta]$$

- Pick any separable Hilbert space $\mathfrak{h}$ and let $\ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h})$ be the Hilbert space of functions from $\mathbb{T}_n$ to $\mathfrak{h}$ which are *antiperiodic*. That is, for $f \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h})$,

$$f(\alpha + \beta) = -f(\alpha) \ , \qquad \alpha \in \mathbb{T}_n \ .$$

  The scalar product on $\ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h})$ is then defined to be

$$\langle f_1, f_2 \rangle_{\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h})} \doteq n^{-1}\beta \sum_{\alpha \in \mathbb{T}_n} \langle f_1(\alpha), f_2(\alpha) \rangle_{\mathfrak{h}} \ , \qquad f_1, f_2 \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h}) \ .$$

- Vectors φ of $\mathfrak{h}$ are viewed as antiperiodic functions $\hat{\varphi}$ of $\ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h})$ via the definition

$$\hat{\varphi}(\alpha) \doteq (\beta^{-1}n/2)\delta_{0,\alpha}\varphi \ , \qquad \alpha \in (-\beta, 0] \cap \mathbb{T}_n \ .$$

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Example: $\mathfrak{h}$ is the one–particle Hilbert space $\ell^2(\mathbb{Z}^d)$ with ONB $\{\mathfrak{e}_x\}_{x \in \mathbb{Z}^d}$ defined by

$$\mathfrak{e}_x(y) := \delta_{x,y}, \qquad x, y \in \mathbb{Z}^d.$$

Discrete Time Covariance

- Any operator H acting on $\mathfrak{h}$ with domain $\mathrm{dom}(H)$ is viewed as an operator $\hat{H}$ with domain $\ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathrm{dom}(\hat{H})) \subset \ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h})$ by the definition

$$[\hat{H}f]\,(\alpha) \doteq H\,(f\,(\alpha))\ , \qquad f \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathrm{dom}(\hat{H})),\ \alpha \in \mathbb{T}_n\ .$$

- The discrete time derivative $\partial \in \mathcal{B}(\ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h}))$ is the normal invertible operator defined by

$$\partial f\,(\alpha) \doteq \beta^{-1} n \left(f \left(\alpha + n^{-1}\beta \right) - f\,(\alpha) \right)\ , \qquad f \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h}),\ \alpha \in \mathbb{T}_n\ .$$

- The *discrete time covariance* is thus defined for any $H = H^*$ to be

$$C_H \doteq -2 \left(\partial + \hat{H} \right)^{-1} \in \mathcal{B}(\ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h}))\ .$$

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Example: $H = H^* \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$ is the one–particle Hamiltonian such that

$$h(|x-y|) = \langle \mathfrak{e}_x, H\mathfrak{e}_y \rangle_{\ell^2(\mathbb{Z}^d)} \ , \qquad x, y \in \mathbb{Z}^d.$$

Determinant Bounds in Constructive QFT

The convergence of perturbation expansions of all correlation functions (cf. $G_{\mathbf{A}}^{(\lambda)}$) in non–relativistic fermionic constructive QFT at weak coupling λ is ensured if the interaction and the covariance are summable and if certain determinants arising in the expansion can be bounded efficiently:

Definition (Determinant bounds)

Let $\mathfrak{h}$ be a separable Hilbert space with ONB $\{\varphi_{\mathrm{i}}\}_{\mathrm{i}\in\mathbb{I}}$, $\mathbb{I}$ being countable.
$\gamma_H \in \mathbb{R}^+$ is a **determinant bound** of $H = H^*$ if, for any $\beta \in \mathbb{R}^+$, $n \in 2\mathbb{N}$, $m, N \in \mathbb{N}$, $\mathfrak{M} \in \mathrm{Mat}\,(m, \mathbb{R})$ with $\mathfrak{M} \geq 0$, and all parameters

$$\{(\alpha_q, \mathrm{i}_q, j_q)\}_{q=1}^{2N} \subset \mathbb{T}_n \cap [0, \beta) \times \mathbb{I} \times \{1, \ldots, m\} \ ,$$

the following bound holds true:

$$\left| \det \left[\mathfrak{M}_{j_k, j_{N+l}} \left\langle \varphi_{\mathrm{i}_{N+l}}, (C_H \hat{\varphi}_{\mathrm{i}_k})(\alpha_k - \alpha_{N+l}) \right\rangle_{\mathfrak{h}} \right]_{k,l=1}^{N} \right| \leq \gamma_H^{2N} \prod_{q=1}^{2N} \mathfrak{M}_{j_q, j_q}^{1/2} \ .$$

Example: Correlation Functions of Lattice Fermion System

- For $\lambda \in \mathbb{R}$, $\tau^{(\lambda)} = \{\tau_t^{(\lambda)}\}_{t\in\mathbb{R}}$ is the group of automorphisms of the CAR algebra $\mathcal{U} = \mathrm{CAR}(\ell^2(\mathbb{Z}^d))$ with generator defined by

$$\delta^{(\lambda)}(A) \doteq i \sum_{x,y\in\mathbb{Z}^d} \langle \mathfrak{e}_x, H\mathfrak{e}_y\rangle_{\ell^2(\mathbb{Z}^d)} [a_x^* a_y, A] + \lambda v(|x-y|) \left[a_x^* a_x a_y^* a_y, A\right] , \qquad A \in \mathcal{U}_0 .$$

- Let

$$\omega_H \doteq \limsup_{n\to\infty} \sup_{x\in\mathbb{Z}^d} \left\{ n^{-1}\beta \sum_{\vartheta\in\mathbb{T}_n} \sum_{y\in\mathbb{Z}^d} \left| \langle \mathfrak{e}_y, (C_H\hat{\mathfrak{e}}_x)(\vartheta)\rangle_{\ell^2(\mathbb{Z}^d)} \right| \right\} .$$

- If $\omega_H \gamma_H^2 |\lambda| \ll 1$, then one gets:

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- If $\omega_H \gamma_H^2 |\lambda| \ll 1$, then one gets:

1. Existence of a unique *translation invariant* $(\tau^{(\lambda)}, \beta)$–KMS state.

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- If $\omega_H \gamma_H^2 |\lambda| \ll 1$, then one gets:

1. Existence of a unique *translation invariant* $(\tau^{(\lambda)}, \beta)$–KMS state.
2. Perturbation expansion of *all correlation functions* in terms of powers of λ converges absolutely. More precisely, all correlation functions are analytic functions of the coupling λ at $\lambda = 0$ with analyticity radius of order $\omega_H^{-1}\gamma_H^{-2}$.

Example: Correlation Functions of Lattice Fermion System

- For $j \in \{1,\ldots,n\}$, let $X_j \doteq (\alpha_j,\nu_j,x_j) \in \mathbb{R}\times\{+,-\}\times\mathbb{Z}^d$. If $\alpha_j \leq \alpha_{j+1}$ then

$$G^{(n)}(X_1,\ldots,X_n) \doteq G_{\left(a_{x_1}^{\nu_1},\ldots,a_{x_n}^{\nu_n}\right)}(\alpha_1,\ldots,\alpha_n) \quad \text{with} \quad a_x^+ \doteq a_x^* \text{ and } a_x^- \doteq a_x \ .$$

  $G^{(n)}$ is extended to all $\alpha_j \in \mathbb{R}$ so that it is antisymmetric w.r.t. permutations $\pi \in \mathcal{S}_n$ of its n arguments and periodic with period 2β w.r.t. to each α_j.

- Truncated Green functions $G_T^{(n)}(X_1,\ldots,X_n)$ are recursively defined by

$$\sum_{k=1}^{n}\sum_{\pi\in\mathcal{S}_n}\frac{(-1)^\pi}{k!(n-k)!}G_T^{(k)}\left(X_{\pi(1)},\ldots,X_{\pi(k)}\right)G_T^{(n-k)}\left(X_{\pi(k+1)},\ldots,X_{\pi(n-k)}\right) = G^{(n)}(X_1,\ldots,X_n)\ .$$

Theorem (Pedra, 2005 and many others)

For $\omega_H\gamma_H^2|\lambda|$ sufficiently small, there are a unique translation invariant $(\tau^{(\lambda)},\beta)$–KMS state, constants $C_1,C_2,\ldots$ and a radius $R=\mathcal{O}(\omega_H^{-1}\gamma_H^{-2})>0$ so that

$$\left|\frac{\partial^k}{\partial\lambda^k}G_T^{(n,\lambda)}(X_1,\ldots,X_n)\right| \leq C_n k! R^{-k}, \quad k\in\mathbb{N}_0,\ n\in\mathbb{N},\ X_1,\ldots,X_n\in\mathbb{R}\times\{+,-\}\times\mathbb{Z}^d.$$

Gram Bound for Determinants

Lemma (Gram bound)

Let $\mathcal{H}$ be an Hilbert space and $C \in \mathcal{B}(\mathcal{H})$. Then, for any $N \in \mathbb{N}$ and $u_1, \ldots, u_{2N} \in \mathcal{H}$,

$$\left|\det \left[\langle u_{N+l}, Cu_k\rangle_{\mathcal{H}}\right]_{k,l=1}^{N}\right| \leq \|C\|_{\mathcal{B}(\mathcal{H})}^{N} \|u_1\|_{\mathcal{H}} \cdots \|u_{2N}\|_{\mathcal{H}} \ .$$

Therefore, for any $\beta \in \mathbb{R}^+$, $n \in 2\mathbb{N}$, $m, N \in \mathbb{N}$, $\mathfrak{M} \in \mathrm{Mat}\,(m, \mathbb{R})$ with $\mathfrak{M} \geq 0$, and

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Problem: In general, the norm of C_H **diverges**, as $n \to \infty$. This problem appears already for bounded $H \in \mathcal{B}(\mathfrak{h})$ when $0 \in \mathrm{spec}(H)$. In this case,

$$\|C_H\|_{\mathcal{B}(\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h}))}^{1/2} = \mathcal{O}\left(\sqrt{n}\right) \quad \text{and} \quad \|\hat{\varphi}_{\mathrm{i}_q}\|_{\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h})} = \mathcal{O}\left(\sqrt{n}\right) \ .$$

Gram Bound and Multiscale Analyses

One uses the *Gram bound* for some regularized covariances $C_H\hat{\varkappa}_L(\hat{H}, i\partial)$ at every $L \in \mathbb{N}$. Here, for $L \in \mathbb{N}$, $\hat{\varkappa}_L : \mathbb{R}^2 \to [0,1]$ defines a family of measurable functions so that

$$\sum_{L=1}^{\infty} \hat{\varkappa}_L (x, y) = \mathbf{1} \ , \qquad x, y \in \mathbb{R} \ .$$

This decomposition can be chosen such that there are constants $\hat{\gamma}_L \in \mathbb{R}^+$, $L \in \mathbb{N}$, which at least do *not* depend on $n \in 2\mathbb{N}$ and meanwhile satisfy

$$\left| \det \left[\mathfrak{M}_{j_k, j_{N+l}} \left\langle \varphi_{\mathfrak{i}_{N+l}}, \left(C_H\hat{\varkappa}_L(\hat{H}, i\partial)\hat{\varphi}_{\mathfrak{i}_k} \right) (\alpha_k - \alpha_{N+l}) \right\rangle_{\mathfrak{h}} \right]_{k,l=1}^{N} \right| \leq \hat{\gamma}_L^{2N} \prod_{q=1}^{2N} \mathfrak{M}_{j_q, j_q}^{1/2} \ .$$

See, e.g., [Benfatto-Gallavotti-Procacci-Scoppola, '94], [Giuliani-Mastropietro, '10], [Giuliani-Mastropietro-Porta, '16] and in many others works.

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Observation: [Pedra-Salmhofer, '08] shows that this multiscale analysis for the so–called Matsubara UV problem is *not* necessary, by proving a new bound for determinants that generalizes the original Gram bound. Avoiding this kind of procedure brings various technical benefits.

Universal Determinant Bound

- We also show that the UV regularization of the Matsubara frequency is not necessary, but, in contrast with [Pedra-Salmhofer, '08], the given covariance does *not* need to be written as a chronological sum to obtain determinant bounds.

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$\mathfrak{x} \doteq \sup \{\inf \gamma_H : H = H^*$ *acting on a separable Hilbert space* $\mathfrak{h}$ *with ONB* $\{\varphi_\mathfrak{i}\}_{\mathfrak{i}\in\mathbb{I}}\}$, *named the universal determinant bound, is equal to* $\mathfrak{x} = 1$.

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In other words, for any $\beta \in \mathbb{R}^+$, $n \in 2\mathbb{N}$, $m, N \in \mathbb{N}$, $\mathfrak{M} \in \mathrm{Mat}\,(m, \mathbb{R})$ with $\mathfrak{M} \geq 0$, and

$$\{(\alpha_q, \mathrm{i}_q, j_q)\}_{q=1}^{2N} \subset \mathbb{T}_n \cap [0, \beta) \times \mathbb{I} \times \{1, \ldots, m\} \ ,$$

the following bound holds true:

$$\left| \det \left[\mathfrak{M}_{j_k, j_{N+l}} \langle \varphi_{N+l}, (C_H \hat{\varphi}_k)(\alpha_k - \alpha_{N+l}) \rangle_{\mathfrak{h}} \right]_{k,l=1}^{N} \right| \leq \prod_{q=1}^{2N} \|\varphi_q\|_{\mathfrak{h}} \, \mathfrak{M}_{j_q, j_q}^{1/2} \ .$$

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- [Pedra-Salmhofer,'08] gives $\gamma_H = 2$ for the class of *bounded* operators H it applies.
- Thus, the convergence of perturbation series at $\lambda = 0$ of any non–relativistic fermionic QFT (possibly in the continuum) is *only* ensured by the smallness of

$$\omega_H \doteq \limsup_{n\to\infty} \sup_{x\in\mathbb{Z}^d} \left\{ n^{-1}\beta \sum_{\vartheta\in\mathbb{T}_n} \sum_{y\in\mathbb{Z}^d} \left| \left\langle \mathfrak{e}_y, \left(C_H \hat{\mathfrak{e}}_x\right)(\vartheta)\right\rangle_{\ell^2(\mathbb{Z}^d)} \right| \right\} .$$

Heuristic of the Proof: Quasi–Free States

Let $\mathcal{H}$ be any (separable) Hilbert space and $\mathcal{U} = \mathrm{CAR}(\mathcal{H})$ the C^*–algebra generated by the unit $\mathbf{1}$ and the family $\{a(\Psi)\}_{\Psi \in \mathcal{H}}$ of elements satisfying the CAR.

Definition (Quasi–free states with one-particle Hamiltonian $\mathrm{H} = \mathrm{H}^*$)

Quasi–free states are positive linear functionals $\rho \in \mathrm{CAR}(\mathcal{H})^*$ such that $\rho(\mathbf{1}) = 1$ and, for all $N \in \mathbb{N}$ and $\Psi_1, \ldots, \Psi_{2N} \in \mathcal{H}$,

$$\rho\left(a^+(\Psi_1) \cdots a^+(\Psi_N) a(\Psi_{2N}) \cdots a(\Psi_{N+1})\right) = \det\left[\left\langle \Psi_k, \frac{1}{1+\mathrm{e}^{\mathrm{H}}} \Psi_{N+l} \right\rangle_{\mathcal{H}}\right]_{k,l=1}^{N} .$$

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Compare with the determinant we have to estimate:

$$\det\left[\mathfrak{M}_{j_k,j_{N+l}}\left\langle \varphi_{N+l}, \left(C_H\hat{\varphi}_k\right)\left(\alpha_k - \alpha_{N+l}\right)\right\rangle_{\mathfrak{h}}\right]_{k,l=1}^{N} .$$

Actually, we construct a one-particle Hamiltonian $\mathrm{H} = \mathrm{H}^* \in \mathcal{B}(\mathcal{H})$ such that

$$\mathfrak{M}_{j_k,j_{N+l}}\left\langle \varphi_{N+l}, \left(C_H\hat{\varphi}_k\right)\left(\alpha_k - \alpha_{N+l}\right)\right\rangle_{\mathfrak{h}} = \pm\left\langle \Psi_k, \frac{1}{1+\mathrm{e}^{\mathrm{H}}}\Psi_{N+l}\right\rangle_{\mathcal{H}} = \pm\rho\left(a^+(\Psi_k)a(\Psi_{N+l})\right) .$$

Heuristic of the Proof: Quasi–Free States

To simplify, fix $n \in 2\mathbb{N}$, $\beta = 1 = \mathfrak{M} \in \mathrm{Mat}\,(1, \mathbb{R})$, and the one-particle hamiltonian $H = H^* \in \mathcal{B}(\mathfrak{h})$.

Lemma

For $m \in \mathbb{N}$, $\alpha_1, \alpha_2 \in \mathbb{T}_n \cap [0, 1)$, *and* $\varphi_1, \varphi_2 \in \mathfrak{h}$, *there are* $\psi_1, \psi_2 \in \mathfrak{h}$ *with* $\|\psi_{1,2}\|_{\mathfrak{h}} \leq \|\varphi_{1,2}\|_{\mathfrak{h}}$, *quasi–free states* ρ_ν, $\nu \in \mathbb{R}^+$, *with one-particle hamiltonian*

$$\mathrm{H}_\nu \simeq -n \ln \left|1 - n^{-1} H\right|, \quad i.e., \quad \mathrm{e}^{\mp \mathrm{H}_\nu} = \left(1 - n^{-1} H\right)^{\pm n} \simeq \mathrm{e}^{\mp H} + o\,(1)\,,$$

such that for $\alpha_1 \leq \alpha_2$,

$$\langle \varphi_2, (C_H \hat{\varphi}_1)\,(\alpha_1 - \alpha_2) \rangle_{\mathfrak{h}} = \lim_{\nu \to \infty} \rho_\nu \left(a^+ \left((\mathrm{e}^{-\alpha_1 \mathrm{H}_\nu} \psi_1) \right) a \left((\mathrm{e}^{(\alpha_2 + n^{-1}) \mathrm{H}_\nu} \psi_2) \right) \right)$$

while for $\alpha_1 > \alpha_2$,

$$\langle \varphi_2, (C_H \hat{\varphi}_1)\,(\alpha_1 - \alpha_2) \rangle_{\mathfrak{h}} = - \lim_{\nu \to \infty} \rho_\nu \left(a \left((\mathrm{e}^{(\alpha_2 + n^{-1}) \mathrm{H}_\nu} \psi_2) \right) a^+ \left((\mathrm{e}^{-\alpha_1 \mathrm{H}_\nu} \psi_1) \right) \right).$$

Heuristic of the Proof: Hölder Inequalities

- Assume that $\mathfrak{h}$ is a finite dimensional Hilbert space. Then, $\mathrm{CAR}(\mathfrak{h}) \sim \mathcal{B}(\wedge\mathfrak{h})$ with $\wedge\mathfrak{h}$ being the fermionic Fock space and, for $\nu \in \mathbb{R}^+$,

$$\rho_\nu(A) \doteq \mathrm{Tr}_{\wedge\mathfrak{h}}(A\mathrm{D}_\nu) \doteq \mathrm{Tr}_{\wedge\mathfrak{h}}\left(A\mathrm{e}^{-\mathrm{H}_\nu}\right), \qquad A \in \mathcal{B}(\wedge\mathfrak{h}) .$$

- For $q \in \{1, \ldots, 2N\}$ and $\alpha_q \in \mathbb{T}_n \cap [0,1)$ such that $\vartheta_q \doteq \alpha_q - \alpha_{q-1} \geq 0$ for $q \geq 2$,

$$\left|\det\left[\langle \varphi_{N+l}, (C_H \hat{\varphi}_k)(\alpha_k - \alpha_{N+l})\rangle_{\mathfrak{h}}\right]_{k,l=1}^{N}\right| \simeq \lim_{\nu\to\infty} |\mathrm{Tr}_{\wedge\mathfrak{h}}(\mathbf{X}_\nu)| \leq \lim_{\nu\to\infty} \|\mathbf{X}_\nu\|_1$$

with

$$\mathbf{X}_\nu \doteq \mathrm{D}_\nu^{\vartheta_2} a^+(\psi_2) \cdots a^+(\psi_N) \mathrm{D}_\nu^{\vartheta_{N+1}} a(\psi_{N+1}) \cdots \mathrm{D}_\nu^{\vartheta_{2N}} a(\psi_{2N}) \mathrm{D}_\nu^{1-(\vartheta_2+\cdots+\vartheta_{2N})} a^+(\psi_1)$$

and

$$\|A\|_s \doteq \left(\mathrm{Tr}_{\wedge\mathfrak{h}}(|A|^s)\right)^{\frac{1}{s}}, \qquad A \in \mathcal{B}(\wedge\mathfrak{h}) , \ s \geq 1 .$$

Heuristic of the Proof: Hölder Inequalities

- Hölder inequalities for Schatten norms: For $n \in \mathbb{N}$, $r, s_1, \ldots, s_n \in [1, \infty]$ such that $\sum_{j=1}^{n} 1/s_j = 1/r$, and all operators $A_1, \ldots, A_n \in \mathcal{B}(\wedge\mathfrak{h})$,

$$\|A_1 \cdots A_n\|_r \leq \prod_{j=1}^{n} \|A_j\|_{s_j} \ .$$

- Then, using Hölder inequalities for $r = 1$,

$$\begin{aligned} \|\mathbf{X}_\nu\|_1 &= \left\| \mathrm{D}_\nu^{\vartheta_2} a^+(\psi_2) \cdots \mathrm{D}_\nu^{\vartheta_{2N}} a(\psi_{2N}) \mathrm{D}_\nu^{1-(\vartheta_2+\cdots+\vartheta_{2N})} a^+(\psi_1) \right\|_1 \\ &\leq \left\| \mathrm{D}_\nu^{\vartheta_2} \right\|_{\frac{1}{\vartheta_2}} \cdots \left\| \mathrm{D}_\nu^{\vartheta_{2N}} \right\|_{\frac{1}{\vartheta_{2N}}} \left\| \mathrm{D}_\nu^{1-(\vartheta_2+\cdots+\vartheta_{2N})} \right\|_{\frac{1}{1-(\vartheta_2+\cdots+\vartheta_{2N})}} \prod_{q=1}^{p-1} \|a(\psi_q)\|_\infty \ . \end{aligned}$$

- Since $\|\mathrm{D}_\nu\|_1 = 1$ and for $q \in \{1, \ldots, 2N\}$,

$$\|a(\psi_q)\|_\infty = \|a(\psi_q)\|_{\mathrm{CAR}(\mathfrak{h})} = \|\psi_q\|_{\mathfrak{h}} \leq \|\varphi_q\|_{\mathfrak{h}} \ ,$$

it follows that

$$\left| \det \left[\langle \varphi_{N+l}, (C_H \hat{\varphi}_k)(\alpha_k - \alpha_{N+l}) \rangle_{\mathfrak{h}} \right]_{k,l=1}^{N} \right| \leq \lim_{\nu \to \infty} \|\mathbf{X}_\nu\|_1 \leq \prod_{q=1}^{2N} \|\varphi_q\|_{\mathfrak{h}} \ .$$