Two- and Multi-phase Quadrature Domains

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Stochastic and Analytic Methods in Mathematical Physics,
4-11 September, 2016, Yerevan, Armenia

Outline

Mean value property for harmonic functions

The well-known mean value property for harmonic functions reads:

$$\int_{B_r(x_0)} h(x)dx = |B_r|h(x_0), \qquad \forall h \in HL^1(B_r(x_0)),$$

where $HL^1(B_r(x_0))$ denotes L^1 harmonic functions on $B_r(x_0)$.

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where $HL^1(B_r(x_0))$ denotes L^1 harmonic functions on $B_r(x_0)$.

Let $d\mu = |B_r|\delta_{x_0}(x)dx$, then the above identity can be written as

$$\int_{B_r(x_0)} h(x)dx = \int h(x)d\mu \qquad \forall h \in HL^1(B_r(x_0)).$$

Definition of Quadrature domain

Definition (Quadrature domain)

Suppose that we are given a finite, positive measure μ with compact support in $\mathbb{R}^N$, and a class of functions T. We say Ω is a **Quadrature domain** with respect to μ and a class T if support of μ is subset of Ω and

$$\int_\Omega h\,dx = (\geq) \int h\,d\mu, \quad \forall h \in T. \tag{1}$$

We denote this $\Omega \in QD(\mu, T)$.

Examples of QD

Let

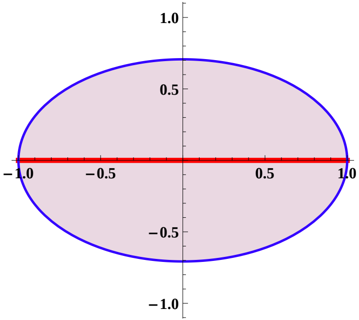
$$E = \{(x, y) \text{ s.t. } x^2/a^2 + y^2/b^2 < 1\},$$

and

$$d\mu = 2ab\sqrt{1 - x^2}\chi_{[-1,1]}dx.$$

Then

$$\int_E hdxdy = \int hd\mu, \qquad \forall h \in HL^1(E).$$



Examples of QD

Let

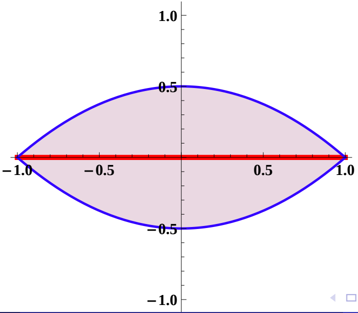
$$D = \left\{ (x, y) \text{ s.t. } |x| < 1,\ |y| < \frac{(1 - x^2)}{2} \right\},$$

and

$$d\mu = 2(1 - \sqrt{|x|})\chi_{[-1,1]}dx.$$

Then

$$\int_D h dxdy = \int h d\mu, \qquad \forall h \in HL^1(D).$$



Newtonian potentials

Let

$$G(x) = \begin{cases} \frac{\omega_N}{N}|x|^{2-N} \text{ for } N \geq 3, \\ -\frac{1}{2\pi} \ln |x| \text{ for } N = 2, \end{cases}$$

denotes the Fundamental solution to the Laplace operator. Here ω_N are chosen such that $-\Delta G = \delta_0$.

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Then for $\Omega \in QD(HL^1, \mu)$ and $\forall x \in \Omega^c$ we have

$$U^\mu(x) = \int_{\mathbb{R}^N} G(x-y)d\mu(y) = \int_\Omega G(x-y)dy = U^{\chi_\Omega}(x).$$

A PDE formulation

Define $u = U^{\mu}(x) - U^{\chi_\Omega}(x)$. Then

$$\begin{cases} -\Delta u = \mu - \chi_\Omega & \text{in } \mathbb{R}^N, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

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If in addition we assume $\Omega \in QD(SL^1, \mu)$, then we will apparently have $u \geq 0$, for all $x \in \mathbb{R}^N$. This will give us the obstacle problem

$$\begin{cases} -\Delta u = \mu - 1 & \text{in } \{u > 0\}, \\ u = |\nabla u| = 0 & \text{on } \partial\{u > 0\}. \end{cases}$$

Motivation

Recently, Shahgholian, Emamizadeh, and Prajapat introduced the notion of the Two-phase quadrature domains by linking it to the so-called Two-phase obstacle problem.

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Definition (Two phase QD)

Suppose we are given constants $\lambda_\pm > 0$, bounded nonnegative measures $\mu_\pm$, and disjoint domains $\Omega_\pm$ such that $supp(\mu_\pm) \subset \Omega_\pm$. If for every integrable harmonic function h on $\Omega_+ \cup \Omega_-$, that also has continuous extension to $\partial\Omega_+ \cap \partial\Omega_-$, the following integral identity holds:

$$\int_{\Omega_+} \lambda_+ h dx - \int_{\Omega_-} \lambda_- h dx = \int h d(\mu_+ - \mu_-), \qquad (2)$$

then we call $\Omega = \Omega_+ \cup \Omega_-$ a *Two-phase quadrature domain* with respect to $\mu_\pm$, and $\lambda_\pm$.

A PDE counterpart

In terms of partial differential equation above definition gives rise the following free boundary problem:

$$\begin{cases} \Delta u = \lambda_+ \chi_{\Omega_+} - \lambda_- \chi_{\Omega_-} - (\mu^+ - \mu^-) & \text{in } \mathbb{R}^N, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

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If in addition we enlarge the class to $\forall h \in SubL^1(\Omega_+) \cap SuperL^1(\Omega_-)$, then using the same Newtonian potential argument as before we arrive at

$$\Delta u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} - (\mu^+ - \mu^-) \text{ in } \mathbb{R}^N.$$

Plot of Two phase QD

Here we take $\mu^- = \delta_{-1/4}, \mu^+ = 2\delta_{1/4}$.

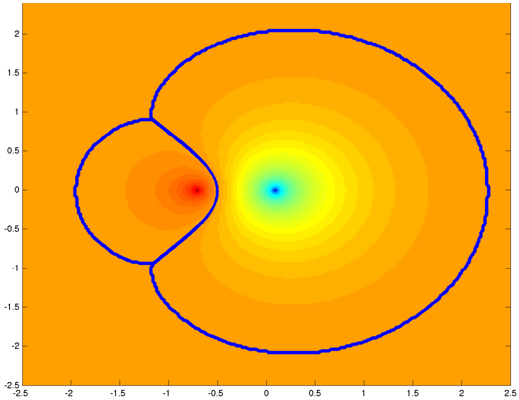

Figure: Courtesy of F. Bozorgnia

The model equation

The setting of the problem in terms of partial differential equation is as follows: Given are m positive measures μ_i and constants $\lambda_i, (i = 1, \ldots, m)$. We want to find functions $u_i \geq 0, (i = 1, \ldots, m)$, with $\Omega_i \cap \Omega_j = \emptyset$ ($i \neq j$ and $\Omega_i = \{u_i > 0\}$) and such that

$$\Delta(u_i - u_j) = (\lambda_i \chi_{\Omega_i} - \lambda_j \chi_{\Omega_j}) - (\mu_i - \mu_j) \text{ in } \mathbb{R}^N \setminus \cup_{k \neq i,j} \overline{\Omega}_k. \tag{3}$$

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This, in other words, means that for each pair (i, j) with $i \neq j$ the function $u_i - u_j$ solves a two-phase problem outside the union of the supports of the other functions.

Definition of Multi-phase QD

Definition (Multi-phase Quadrature domain)

Suppose we are given m bounded positive measures μ_i, and disjoint domains Ω_i such that $supp(\mu_i) \subset \Omega_i$. For each $i \neq j$ let $h \in HL^1(\Omega_i \cup \Omega_j)$, h is continuous across $\partial\Omega_i \cap \partial\Omega_j$, and $h = 0$ on $\cup_{k \neq i,j} \partial\Omega_k$. If for each $i \neq j$ the above class of harmonic functions admit the following QI

$$\int_{\Omega_i} h\lambda_i dx - \int_{\Omega_j} h\lambda_j dx = \int hd(\mu_i - \mu_j), \tag{4}$$

then we call $\Omega = \{\Omega_i\}_{i=1}^m$ an *m-phase QD* with respect to the measure $\{\mu_i\}_{i=1}^m$, and the positive constants $\{\lambda_i\}_{i=1}^m$. (In general λ_i can be taken to be strictly positive functions.)

Triple junction points

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Theorem (A.A, H.Shahgholian, 2016)

Multi-phase Quadrature Domains with triple junction point, which does not touch the support of the measure, do exist.

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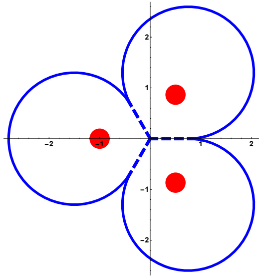

Figure: Three-phase QD with junction point at $(0, 0)$.

Quadruple junction points

Quadruple junction points

Let the origin be a quadruple junction point, for a given multi-phase QD, and that for some $r > 0$ we have $B_r \cap \text{supp}(\mu) = \emptyset$, where μ is the measure corresponding to that multi-phase QD. Let further u_i $(i = 1, \cdots, 4)$ be the corresponding function for each phase Ω_i. Suppose we have the following simple geometry

$$(B_r \cap \partial\Omega_1 \cap \partial\Omega_3) \setminus \{0\} = \emptyset, \qquad (B_r \cap \partial\Omega_2 \cap \partial\Omega_4) \setminus \{0\} = \emptyset. \tag{5}$$

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Theorem (A.A, H.Shahgholian, 2016)

Multi-phase Quadrature Domains with a quadruple junction point, and the geometry described in (5), *do not exist.*

Quadruple junction points

If we allow the junction point to hit the support of the measure, one can actually create a quadruple point as follows.
Take the following Quadrature domain D

$$D = \left\{ (x, y) \text{ s.t. } 0 < x < 2,\ |y| < \frac{(1 - (x-1)^2)}{2} \right\},$$

with respect to the measure μ defined as follows

$$d\mu = 2(1 - \sqrt{|x-1|})\chi_{[0,2]} dx.$$

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Now, we rotate the domain D clockwise with respect to the origin, by angle $\pi/2$. Repeating this process three times we will end up with a picture as in the next figure.

Quadruple junction points

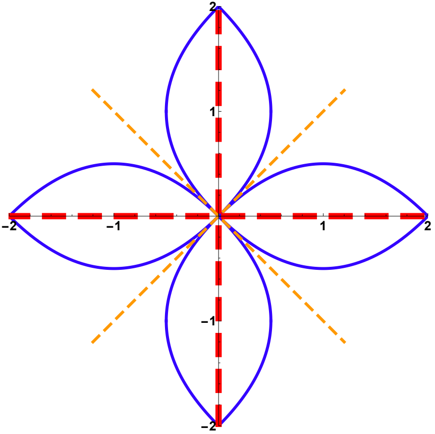

Figure: Quadruple junction in the support of the measure.

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