A construction of some new interacting, non-classical point processes

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Abstract

It is shown that recent results of Suren Poghosyan and Daniel Ültschi [9] combined with those of Benjamin Nehring [2] yield a construction of limiting interacting point processes for the Ginibre Bose of Brownian Loops and the dissolution in $\mathbb{R}^d$ of Ginibre's Fermi-Dirac gas of interacting polygonal loops (cf. [4, 3]). The latter has classical as well as quantum behaviour, according to the structure of their defining measures which are given by

$$\varrho_k(\mathrm{d}\,x_1 \dots \mathrm{d}\,x_k) = \mathrm{e}^{-E_\phi(\delta_{x_1}+\dots+\delta_{x_k})}\,\mathcal{J}_K(x_1,\dots,x_k)\;\mathrm{d}\,x_1 \dots \mathrm{d}\,x_k \quad ,$$

where $\mathfrak{J}$ denotes the immanant of some nice underlying kernel K

Nehring's general cluster expansion method ([2])

A point process is a random mechanism realizing configurations of particles in space. Our approach to design such a mechanism uses the cluster expansion method, which, in the words of Dobrushin, *traces back to the deeps of theoretical physics.*

Formally we are given a *bounded phase space*

$$(X, \mathcal{B}, \mathcal{B}_0) \quad ,$$

which we assume to be *Polish.* Here $\mathcal{B}$ denotes the Borel $\sigma-$field in X, and $\mathcal{B}_0$ the subset of bounded, i.e. relatively compact, sets. The configurations are *Radon point measures* μ on X. Its collection is denoted by

$$\mathcal{M}^{\cdot\cdot} = \mathcal{M}^{\cdot\cdot}(X) \quad .$$

This again is a Polish space with respect to the vague topology. And finally, on a third level, a *point process in* X is a law P on $\mathcal{M}^{\cdot\cdot}$. Its collection is denoted by

$$\mathcal{P}\mathcal{M}^{\cdot\cdot} \quad .$$

Our main examples of phase spaces are the Euclidean space $E = \mathbb{R}^d$, the space $\mathfrak{X} = \mathcal{M}^{\cdot\cdot}_f$ of *finite* configurations, and the space of Brownian loops in E.

The following extension of the *cluster expansion method* is due to Nehring. First of all, we are given a signed Radon measure

$$\varrho$$

on X; as well as a family of signed Radon measures

$$\mathcal{B}_m^x(\mathrm{d}\,x_2 \dots \mathrm{d}\,x_m)$$

on X^{m-1}. These two ingredients define the measures

$$\Theta_m(\mathrm{d}\,x_1 \dots \mathrm{d}\,x_m) = \mathcal{B}_m^{x_1}(\mathrm{d}\,x_2 \dots \mathrm{d}\,x_m)\,\varrho(\mathrm{d}\,x_1)$$

on X^m.

Then these measures define the socalled *cluster measures*

$$L(\varphi) = \sum_{m \geq 1} \frac{1}{m} \int_{X^m} \varphi(\delta_{x_1} + \dots + \delta_{x_m})\ \Theta_m(\mathrm{d}\,x_1 \dots \mathrm{d}\,x_m) \quad , \varphi \in F.$$

F denotes the space of non-negative, measurable variables. Moreover, they determine the cluster representation in terms of the measures $(\Theta_m)_m$ by means of

$$\varrho_k(f_1 \otimes \dots \otimes f_k) = \sum_{\sigma \in \mathcal{S}_k} \prod_{\omega \in \sigma} \Theta_{\ell(\omega)}(\otimes_{j \in \omega} f_j), f_j \in F_0.$$

Here the product is taken over all cycles of the cycle decomposition of the permutation σ, $\ell(\omega)$ is the length of the cycle ω, and F_0 the space of functions from F, which are bounded with bounded support.

We then assume the following two conditions:

$(\mathcal{A}_1)$ the variation $|L|$ of L is of first order;

$(\mathcal{A}_2)$ all measures ϱ_k are positive.

Here $(\mathcal{A}_1)$ means that

$$\int_{\mathfrak{X}} \nu(f)\,|L|(\mathrm{d}\,\nu) < \infty \quad , f \in F_0.$$

These conditions enables us to define locally in $G \in \mathcal{B}_0$ a finite point process

$$Q_G(\varphi) = \frac{1}{\Xi_G} \sum_{k \geq 0} \frac{1}{k!} \int_{X^k} \varphi(\delta_{x_1} + \dots + \delta_{x_k})\,\varrho_k(\mathrm{d}\,x_1 \dots \mathrm{d}\,x_k),\ \varphi \in F,$$

which converges weakly, as $G \uparrow X$, to some point process

$$P = \Im_L \quad ,$$

having *Lévy-measure* L. This means that the Laplace transform of P is of the form

$$\mathcal{L}_P(f) = \exp(-L(1 - \mathrm{e}^{-\zeta_f})) \quad , f \in F_0.$$

Moreover, the process P solves the following two equations:

$$\Im_{L^+} = \Im_{L^-} * P \quad ;$$

$$\mathcal{C}_P = \mathcal{C}_L \star P \quad .$$

Here $*$ is usual convolution of point processes, and $\star$ is some kind of convolution operation between the Campbell measure $\mathcal{C}_L$ and the process P. The first equation says that P is the convolution quotient of the infinitely divisible processes $\Im_{L^+}$ and $\Im_{L^-}$. The second equation allows the computation of all moment measures of P in terms of the cluster measure L.

The estimate of Poghosyan and Ültschi ([9])

In addition we now are given a measurable symmetric pair potential

$$u : X \times X \longrightarrow \mathbb{R} \cup \{\infty\} \quad ,$$

which is stable, regular and integrable in the sense of [9].

Given some signed Radon measure ϱ on X we consider from now on the above construction for the measures

$$(1) \qquad \Theta_m(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_m) = \frac{1}{(m-1)!} U_u(x_1, \ldots, x_m)\, \varrho(\mathrm{d}\, x_1) \ldots \varrho(\mathrm{d}\, x_m) \quad ,$$

where U_u denotes the Ursell function built on u.

In this situation, i.e. under the conditions made on u, Poghosyan and Ültschi can show that the cluster measure L has a variation of first order.

To apply Nehring's construction we thus have to guarantee condition $(\mathcal{A}_2)$. This will be considered for the quantum gases of Ginibre [4].

Ginibre's Bose and Fermi gas

Note first that the cluster representation (1) of the measures Θ_m, now defined by means of the Ursell functions, are given by Ruelle's exponent, i.e.

$$(2) \qquad \varrho_k(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_k) = \exp(-E_u(\delta_{x_1} + \cdots + \delta_{x_k}))\, \varrho(\mathrm{d}\, x_1) \ldots \varrho(\mathrm{d}\, x_k) \quad .$$

This immediately shows: If ϱ is positive then condition $(\mathcal{A}_2)$ is satisfied; and Nehring's construction above shows that under the conditions of Poghosyan and Ültschi on the potential u there exists a point process in X having the cluster measure L, now built on $(\Theta_m)_m$, as a Lévy-measure.

The Ginibre Bose gas ([4])

Consider the space X of Brownian loops in $E = (\mathbb{R}^d, \mathrm{d}\, a)$. The measure ϱ is defined by means of some nice pair potential ϕ in E. Given ϕ define a self-potential υ in X and a pair potential u in X as Ginibre does in [4]. Then for parameters $z, \beta > 0$ let

$$(3) \qquad \varrho(f) = \sum_{m \geq 1} \frac{1}{m} \cdot z^m \int_E \int_X f(x)\, \mathrm{e}^{-\upsilon(x)}\ P^a_{m\beta}(\mathrm{d}\, x)\, \mathrm{d}\, a \quad , f \in F.$$

Here $P^a_{m\beta}(\mathrm{d}\, x)$ is the Brownian loop measure of loops of length $m\beta$. This defines a positive measure on the loop space X; and under natural conditions on the pairpotential ϕ the variation $|L|$ is of first order and one obtains the *Ginibre Bose gas.*

A Gibbs modification of a determinantal process

We now replace in the definition of ϱ the term z^m by $(-1)^{m-1}z^m$ and, to be more modest, $P^a_{m\beta}$ by

$$b^a_m(\mathrm{d}\,a_2 \ldots \mathrm{d}\,a_m) = K(a,a_2)K(a_2,a_3)\cdots K(a_m,a)\ \mathrm{d}\,a_2 \ldots \mathrm{d}\,a_m \quad .$$

Here K is a nice kernel, e.g. a centered Gaußian kernel.

The positive measure ϱ in (3) is then replaced by

(4)

$$\varrho(f) = \sum_{m\geq 1} \frac{1}{m}(-1)^{m-1}z^m \int_{E^m} f(\delta_{a_1}+..+\delta_{a_m})\,\mathrm{e}^{-v(\delta_{a_1}+..+\delta_{a_m})}\ b^{a_1}(\mathrm{d}\,a_2..\,\mathrm{d}\,a_m)\,\mathrm{d}\,a_1.$$

$(f \in F.)$ This is a signed measure on $\mathfrak{X}$.

We do not know whether in this situation the measures ϱ_k, as defined by (2), are positive. In case of positivity one would obtain a point process on configurations of *polygonal loops* $\delta_{a_1} + .. + \delta_{a_m}$.

The idea now is to represent ϱ_k in terms of the underlying Lebesgue measure. Let

$$\text{(5)} \qquad \mathcal{W}_\zeta = \sum_{n\geq 0} \frac{1}{n!}\zeta^n \quad , \text{ where}$$

$$\text{(6)} \qquad \zeta(f) = \sum_{m\geq 1} \frac{1}{m}(-1)^{m-1}z^m \int_{E^m} f(a,a_2,\ldots,a_m)\, b^a(\mathrm{d}\,a_2..\,\mathrm{d}\,a_m)\,\mathrm{d}\,a, f \in F.$$

This is a signed measure on $\mathfrak{E} = \sum_{m\geq 0} E^m$.

We consider also the following *immanantal measure* on $\mathfrak{E}$:

$$\text{(7)} \qquad \mathcal{J} = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\mathcal{J}_K(a_1,\ldots,a_\ell)\ \mathrm{d}\,a_1 \ldots \mathrm{d}\,a_\ell,$$

$$\text{(8)} \qquad \mathcal{J}_K(a_1,\ldots,a_\ell) = \sum_{\sigma\in\mathcal{S}_\ell} (-1)^{m-1}z^m \prod_{j=1}^{\ell} K(a_j,a_{\sigma(j)}).$$

Here $\mathcal{J}_K$ is the immanant built on the kernel K. (Note that $\mathcal{J}_K$ depends also on z.) It is well known by a result of Schur [10] that the immanantal measure $\mathcal{J}$ is positive if K is positive-definite which we'll assume in the sequel.

On the other hand we have the basic observation of Ginibre [4] which states that the measures $\mathcal{W}_\zeta$ and $\mathcal{J}$ coincide on the subspace of all symmetric non-negative, measurable functions on $\mathfrak{E}$.

This implies with $f \in F$

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1}{k!} \varrho_k(f \circ \xi) = \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\mathfrak{X}^k} f(x_1 + \cdots + x_k)\, \mathrm{e}^{E_u(\delta_{x_1} + \cdots + \delta_{x_k})}\ \varrho(\mathrm{d}\, x_1) \ldots \varrho(\mathrm{d}\, x_k) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\mathfrak{X}^k} f(\nu_{a^{(1)}} + .. + \nu_{a^{(k)}})\, \mathrm{e}^{-E_\phi(\nu_{a^{(1)}} + .. + \nu_{a^{(k)}})}\ \zeta(\mathrm{d}\, a^{(1)}) .. \zeta(\mathrm{d}\, a^{(k)}) \\
&= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \int_{E^\ell} f(\delta_{a_1} + .. + \delta_{a_\ell})\, \mathrm{e}^{-E_\phi(\delta_{a_1} + .. + \delta_{a_\ell})}\ \mathcal{J}_K(a_1, .., a_\ell)\ \mathrm{d}\, a_1 .. \mathrm{d}\, a_\ell .
\end{aligned}
$$

Here $\nu_{(a_1,..,a_k)} = \delta_{a_1} + .. + \delta_{a_k}$; and Ginibre's representation has been used in the last step.

This shows that the image of Q_G under the dissolution mapping

$$
\xi : \mathbb{X} = \sum_{k=0}^{\infty} \mathfrak{X}^k \longrightarrow \mathcal{M}^{\cdot\cdot}(E), \quad \delta_{x_1} + \delta_{x_2} + \ldots \longmapsto x_1 + x_2 + \ldots,
$$

coincides with the finite point process in G defined by the *positive* measures

$$
\varrho'(\mathrm{d}\, a_1 \ldots \mathrm{d}\, a_\ell) = \mathrm{e}^{-E_\phi(\delta_{a_1} + .. + \delta_{a_\ell})}\ \mathcal{J}_K(a_1, .., a_\ell)\ \mathrm{d}\, a_1 .. \mathrm{d}\, a_\ell \quad .
$$

To summarize we obtained the following

Theorem 1 *Let ϕ be a stable pair potential in E with stability constant $B \geq 0$ satisfying the condition*

$$
C_\phi = \sup_{b \in E} \int_E |\phi(a, b)|\ \mathrm{d}\, a < \infty \quad .
$$

Assume also that K is some bounded positive-definite kernel on E with

$$
C_K = \sup_{b \in E} \int_E |K(a, b)|\ \mathrm{d}\, a < \infty \quad .
$$

Then, for $z > 0$ small enough such that there exists a constant $0 < c < +\infty$ with

$$
\|K\|_\infty C_\phi \sum_{m \geq 1} z^m\, \mathrm{e}^{(c+B)m}\, C_K^{m-1} \leq c \quad ,
$$

then there exists a point process P' in E for the measures

$$
\varrho'(\mathrm{d}\, a_1 \ldots \mathrm{d}\, a_\ell) = \mathrm{e}^{-E_\phi(\delta_{a_1} + .. + \delta_{a_\ell})}\ \mathcal{J}_K(a_1, .., a_\ell)\ \mathrm{d}\, a_1 .. \mathrm{d}\, a_\ell \quad .
$$

Its signed Lévy measure is given by

$$
L'(\varphi) = \sum_{m \geq 1} \frac{1}{m} \int_{\mathfrak{X}} \varphi(x_1 + .. + x_m) \frac{1}{(m-1)!} U_u(x_1, .., x_m)\, \varrho(\mathrm{d}\, x_1) .. \varrho(\mathrm{d}\, x_m) \quad .
$$

Some comments are in order here. It seems that the process does not exist as a point process on the level of cluster configurations, but only if the clusters are dissolved into its particles. It would be interesting to know whether the theorem remains true if the measures b_m^a are replaced by $P_{m\beta}^a$.

In case of the Bose gas of Ginibre, which exists on the level of cluster configurations, one can dissolve the Brownian loops into its graphs, i.e. into its geometric representation in E; in this case one obtains a random closed set in the sense of Matheron [5], which, independently of the context given here, is of great mathematical interest.

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