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Dynamics of a boson system for repeated interaction

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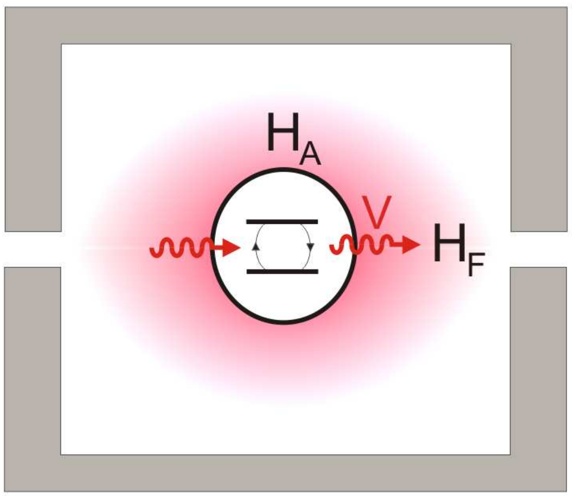
Repeated interaction of a *two-level* atomsbeam with a *one-mode* photon cavity is considered. For *stationary* beam of *randomly* excited atoms this state is **not stationary**. For a **leaky cavity** (Kossakowski-Lindblad *dissipation*) the corresponding **open system** yields **non-equilibrium steady state**.

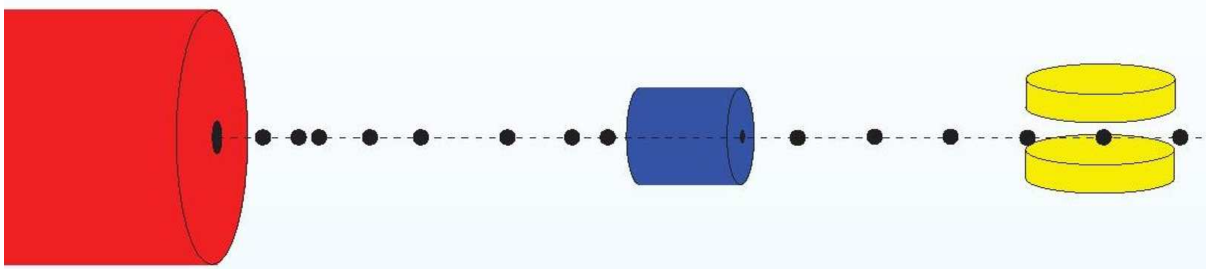
B.Nachtergaele, A.Vershynina and V.A.Z. arXiv:1206.6169v3

1. Motivation: One-Atom Maser



• **Two-Level Atom** H_A, • **Photon-Field** H_F • **Interaction:** V

2. Beam of Atoms and Cavity



Repeated interaction scheme

$$\mathcal{H} = \mathcal{H}_{\mathrm{cavity}} \otimes \mathcal{H}_{\mathrm{beam}}, \qquad \mathcal{H}_{\mathrm{beam}} = \bigotimes_{n\geq 1} \mathcal{H}_{\mathrm{atom}\ n}$$

$H_n = H_{\mathrm{JC}}$ acting on $\mathcal{H}_{\mathrm{cavity}} \otimes \mathcal{H}_{\mathrm{atom}\ n}$

- **Regular Beam**

3. Mathematical Model: Non-Leaky Cavity

• **A system: photon cavity + beam of two-level atoms**. The Hilbert space of the system is a *tensor product*, $\mathcal{H} := \mathcal{H}_C \otimes \mathcal{H}_A$, of **boson** Hilbert space of cavity $\mathcal{H}_C$ and the Hilbert space of **two-level** atoms $\mathcal{H}_A := \otimes_{n\geq 1}\mathcal{H}_{A_n}$, where $\mathcal{H}_{A_n} = \mathbb{C}^2$ for $n \geq 1$.

• The Hamiltonian of the **one-mode cavity** is: $H_C = \epsilon\ b^* b \otimes I$, where the photon (*boson*) creation-annihilation operators verify: $[b, b^*] = I$ and $[b, b] = [b^*, b^*] = 0$.

• The Hamiltonian for the **individual n-th atom** is : $H_{A_n} = I \otimes E\ \eta_n$, where

$$\eta_n = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}\left(I + \sigma^z\right)\ ,\ n \geq 1\ .$$

• *Time-dependent* interaction between the n-**th atom** of the **beam** and the **cavity**:

$$W_n(t) := \chi_{[(n-1)\tau,n\tau)}(t)(\lambda\ \eta_n \otimes (b^* + b))\ ,\ \chi_{[a,b)}(t \in [a,b)) = 1\ .$$

• For a **tuned** homogeneous beam, **exactly one** atom *always* present in the cavity.

• The Hamiltonian of the **model**: the **cavity** + the **beam of atoms**

$$H(t) := H_C + \sum_{n\geq 1}(H_{A_n} + W_n(t)) =$$
$$\epsilon b^* b \otimes I + \sum_{n\geq 1} I \otimes E\eta_n + \sum_{n\geq 1} \chi_{[(n-1)\tau,n\tau)}(t)(\lambda\eta_n \otimes (b^* + b)).$$

4. Hamiltonian Dynamics of State: Chain Rule

• For $t \in [(n-1)\tau, n\tau) =: \Delta_n(\tau)$, **only** the **$n$-th atom** of the **beam** interacts with the **cavity**: a **tuned case**.
• The **evolution** is defined by $H(t)\Big|_{\Delta_n(\tau)} := H_n + I \otimes \sum_{k \neq n} E a_k^* a_k$, where

$$H_n = \epsilon \ b^* b \otimes I + I \otimes E \ \eta_n + \lambda \ (b^* + b) \otimes \eta_n \ .$$

• Since $\chi_{\Delta_n(\tau)}(t < 0) = 0$, the **initial** *density-matrix* **state :** $\rho(t = -0) := \rho_C \otimes \rho_A$ corresponds to the non-interacting system.
• Suppose (for simplicity) that the **initial product-state** of atoms $\rho_A := \bigotimes_{k \geq 1} \rho_k$ **commutes** with η_n for any n. For example, all atoms are in the **Gibbs equilibrium state:**

$$\rho_k = e^{-\beta E \eta_k} / (1 + e^{-\beta E}) \ .$$

- The Hamiltonian dynamics of the state is defined by:

$$\partial_t \rho(t) = -i[H(t), \rho(t)] =: L(t)\rho(t) \ .$$

- Let $L_n := L(t),\ t \in [(n-1)\tau, n\tau)$. Then for the *piece-wise* constant **1st atom-evolution**, when $n = 1$, one obtains:

$$\rho(t) = e^{-itH_1}(\rho_C \otimes \rho_{n=1})e^{itH_1} = e^{tL_1}\rho(t=0) \ .$$

- The **chain of atoms** in the **beam** implies the **chain rule** for the Hamiltonian dynamics. When $n-1$ atoms are **passed through** the cavity and the n-th is **still inside**, one gets:

$$\rho(t) = e^{\nu L_n} e^{\tau L_{n-1}} \ldots e^{\tau L_2} e^{\tau L_1} (\rho_C \otimes \bigotimes_{k=1}^{n} \rho_k) \ ,$$

here $t \in [(n-1)\tau, n\tau)$ and $t = (n-1)\tau + \nu,\ 0 \le \nu < \tau$.

5. Cavity Evolution and Pumping

• The **cavity** density matrix state ρ_C^t at the moment: $t = n\tau$, is the **partial trace** over the first n atoms of the **beam**:

$$\rho_C^{n\tau} := \text{Tr}_{\mathcal{H}_A}\rho(n\tau) = \text{Tr}_{\mathcal{H}_A}[e^{\tau L_n}...e^{\tau L_2}e^{\tau L_1}(\rho_C \otimes \bigotimes_{k=1}^{n} \rho_k)] =$$

$$\text{Tr}_{\mathcal{H}_{A_n}}\{e^{\tau L_n}[\text{Tr}_{\mathcal{H}_{A_{n-1}}}...\text{Tr}_{\mathcal{H}_{A_1}} e^{\tau L_{n-1}}...e^{\tau L_2}e^{\tau L_1}(\rho_C \otimes \bigotimes_{k=1}^{n-1} \rho_k) \otimes \rho_n]\}$$

$$= \text{Tr}_{\mathcal{H}_{A_n}}[e^{\tau L_n}(\rho_C^{(n-1)\tau} \otimes \rho_n)] =: L\ [\rho_C^{(n-1)\tau}] = L^n[\rho_C] \quad .$$

• To study the (reduced) *evolution* of the state ρ_C^t we first look on the variation in the cavity of the **mean photon number:**

$$N(t) := \text{Tr}_{\mathcal{H}_C}(b^* b\ \rho_C^t) \ .$$

• **Theorem 1.** Let ρ_C be a *gauge-invariant* state. Then for a **stationary** beam with $p := \mathrm{Tr}(\eta_n\ \rho_{A_n})$, the mean photon **number** in the cavity for $t = n\tau$ is

$$N(t) = N(0) + n\,p(1-p)\ \frac{2\lambda^2}{\epsilon^2}(1-\cos\epsilon\tau) + p^2\ \frac{2\lambda^2}{\epsilon^2}(1-\cos n\epsilon\tau).$$

• **Remark 1 (Pumping).** Theorem implies that **only** the beam of **randomly exited** atoms ($0 < p < 1$) is able to produce **unlimited pumping** of the cavity, under the following condition: $\{\epsilon\tau \neq 2\pi\, m : m \in \mathbb{N} \cup 0\}$ (*detuning*).

• When **all** atoms are exited ($p = 1$), then the cavity state is **oscillating** (*Rabi oscillations*). Whereas the beam of **non-exciting** atoms ($p = 0$) keeps $N(t) = N(0)$.

6. Energy Production

- **Energy** variation between moments t_{k+1} and $t_k = (k-1)\tau + \nu$:

$$\Delta\mathcal{E}(t_{k+1}, t_k) := \mathrm{Tr}_{\mathcal{H}_C \otimes \mathcal{H}_A}(\rho(t_{k+1})H(t_{k+1})) - \mathrm{Tr}_{\mathcal{H}_C \otimes \mathcal{H}_A}(\rho(t_k)H(t_k))$$
$$= \frac{2\lambda^2}{\epsilon}\{p(1-p)\ [1-\cos(\tau\epsilon)] + p^2[\cos((k-1)\tau\epsilon) - \cos(k\tau\epsilon)]\}$$

- **Theorem 2.** The interaction-jump *external work* for $[t_1, t_{n+1})$:

$$\Delta\mathcal{E}(t_{n+1}, t_1) = \frac{2\lambda^2}{\epsilon}\{n\, p(1-p)[1-\cos(\tau\epsilon)] + p^2[1-\cos n\tau\epsilon]\}\ .$$

- The **Cavity** energy-variation for $t = 0 \rightarrow t = n\tau + \nu$:

$$\Delta\mathcal{E}^C(t) = \epsilon(N(t) - N(0)) = \frac{2\lambda^2}{\epsilon}\{n\, p(1-p)\,(1-\cos\epsilon\tau) + p^2\,(1-\cos n\epsilon\tau)\}.$$

7. Entropy Production

• **Relative Entropy** of the normal state ρ with respect to the normal state ρ_0 can be defined as:

$$\mathsf{Ent}(\rho|\rho_0) := \mathsf{Tr}_{\mathcal{H}}(\rho \ln \rho - \rho \ln \rho_0) \geq 0 \ .$$

• **Entropy Production** is then naturally to define by

$$\Delta S(t) := \mathsf{Ent}(\rho(t)|\rho(t=0)) = \mathsf{Tr}_{\mathcal{H}_C \otimes \mathcal{H}_A}\{\rho(t) \ln \frac{\rho(t)}{\rho_C \otimes \rho_A}\} \ .$$

• Suppose that all atoms of the beam are in the *Gibbs state* with temperature $1/\beta$:

$$\rho_A(\beta) := \bigotimes_{n \geq 1} \rho_{A_n}(\beta) \ , \quad \rho_{A_n}(\beta) := \frac{e^{-\beta H_{A_n}}}{Z(\beta)} \ .$$

• Then (in general) the **Relative Entropy** variation for $t = 0 \to t = n\tau + \nu$ is equal to

$$\Delta S(t) = \mathrm{Tr}_{\mathcal{H}_C}\{[\rho_C - \rho_C^{n\tau}] \ln \rho_C\} +$$
$$\beta \sum_{k=1}^{n} \mathrm{Tr}_{\mathcal{H}_C \otimes \mathcal{H}_{A_k}}\{(\rho_C^{(k-1)\tau} \otimes \rho_k)[e^{i\tau H_k}(I \otimes H_{A_k})e^{-i\tau H_k} - I \otimes H_{A_k}]\} \ .$$

• Since for our model $[H_k, H_{A_k}] = 0$, the last term vanishes and for initial Gibbs cavity state with temperature $1/\beta$

$$\Delta S(t) = \mathrm{Tr}_{\mathcal{H}_C}\{[\rho_C - \rho_C^{n\tau}] \ln \rho_C\} =$$
$$\mathrm{Tr}_{\mathcal{H}_C}\{[\rho_C - \rho_C^{n\tau}](-\beta\, \epsilon\, b^* b - \ln(1 - e^{-\beta\epsilon})\} =$$
$$\beta\, \frac{2\lambda^2}{\epsilon}\, \{n\, p(1-p)\, (1 - \cos \epsilon\tau) + p^2\, (1 - \cos n\epsilon\tau)\} \ .$$

• Let denote by $\Delta\mathcal{E}^C(t) = \epsilon(N(t) - N(0))$ the energy variation of the thermal cavity, which is due to the photon number variation, see **Theorem 1**.

• Then relation:

$$\begin{aligned} \Delta S(t) &= \beta\,\frac{2\lambda^2}{\epsilon}\,\{n\,p(1-p)\,(1-\cos\epsilon\tau) + p^2\,(1-\cos n\epsilon\tau)\} \\ &= \beta\,\epsilon(N(t) - N(0))\ , \end{aligned}$$

expresses the **2nd Law of Thermodynamics:**

$$\Delta S(t) = \beta\Delta\mathcal{E}^C(t)\ ,$$

for **pumping** of the initially thermal cavity.

8. Open Systems and Reduced Dynamics

• Let $\{\mathcal{S}, \mathfrak{H}_S\}$ be an Open Quantum System interacting with the external Reservoir $\{\mathcal{R}, \mathfrak{H}_R\}$. Then:

a. $\mathfrak{H}_S \otimes \mathfrak{H}_R$ is the Hilbert space of the total system.

b. $H_{tot} =: H_S \otimes I + I \otimes H_R + H_{SR}$ is the **total system** Hamiltonian.

c. The **initial state** of the **total system** is $\omega^{t=0} := \omega_S \otimes \omega_R$ and evolution: $\omega^t := \exp(-itH_{tot})(\omega_S \otimes \omega_R) \exp(itH_{tot})$.

d. **Evolution** of the **state** for the **open system** $\{\mathcal{S}, \mathfrak{H}_S\}$ is the mapping: $\omega_S \mapsto \Lambda_t \, \omega_S := \mathrm{Tr}_{\mathfrak{H}_R}(\omega^t) =: \rho_S^t$ (reduced dynamics) .

NB [*Kraus-Kossakowski*] Form of the reduced dynamical map:

$$\Lambda_\tau : \rho_S^t \mapsto \Lambda_\tau \, \rho_S^t = \sum_\alpha C_\alpha(t, \tau) \; W_\alpha \; \rho_S^t \; W_\alpha^* = \rho_S^{t+\tau} \; ,$$

for a *family* of bounded operators $W_\alpha \in \mathfrak{B}(\mathfrak{H}_S)$ and a function $C_\alpha(t, \tau) \geq 0$ satisfying the condition $\sum_\alpha C_\alpha(t, \tau) \; W_\alpha^* W_\alpha = I$.

• The **Heisenberg picture** defines a dual reduced dynamics Λ_t^*:

$$\rho_S^t(A) = \mathrm{Tr}_{\mathfrak{H}_S}((\Lambda_t\, \omega_S) A) =: \mathrm{Tr}_{\mathfrak{H}_S}(\omega_S\ \Lambda_t^* A)\ ,\ \ A \in \mathfrak{B}(\mathfrak{H}_S).$$

NB Dynamics Λ_t^* is *unity-preserving* positive map on $\mathfrak{B}(\mathfrak{H}_S)$.
(i) A (*canonical*) form of the *dual* reduced dynamical map:

$$\Lambda_\tau^*\,(\Lambda_t^* A) = \sum_\alpha C_\alpha(t, \tau)\ W_\alpha^*\ \Lambda_t^* A\ W_\alpha\,,\ \ \Lambda_\tau^* I = I\,,\ \ A \in \mathfrak{B}(\mathfrak{H}_S),\ \ .$$

(ii) The map Λ_t^* is *completely* positive, i.e. the **operators**:

$$(\Lambda_t^*)_n := (\Lambda_t^* \otimes I) : \mathfrak{B}(\mathfrak{H}_S \otimes \mathbb{C}^n) \to \mathfrak{B}(\mathfrak{H}_S \otimes \mathbb{C}^n)$$

are **positive** for all $n = 1, 2, \ldots$, and .
• Generally the function $t \mapsto \Lambda_t$ (or mapping $t \mapsto \Lambda_t^*$) satisfies the **Bogoliubov hierarchy** of *integro-differential* equations.

9. Open Systems and Quantum Dynamical Semigroups

• *Markov Approximation* $\Leftrightarrow$ Quantum *Evolution Operators*:
a. $\left\{\Lambda_{t,0}\right\}_{t\geq 0}$ **reduced dynamics** (for *time*-dependent $H_{SR}(t)$).
b. $\Lambda_{t,s}\Lambda_{s,0} = \Lambda_{t+s,0}$ **Markov property** = *cocycle property*.
c. $\rho_S^t(A) = \mathrm{Tr}_{\mathfrak{H}_S}((\Lambda_{t,0}\,\omega_S)A)$ is a **continuous function** of $t \geq 0$ for any *density* matrix $\omega_S \in \mathrm{TrClass}(\mathfrak{H}_S)$ and $A \in \mathfrak{B}(\mathfrak{H}_S)$.
• Operator family $\{\rho_S^t = \Lambda_{t,0}\,\omega_S\}_{t\geq 0}$ is solution of the quantum Markovian *master* equation with (*unbounded*) **generator** $L(t)$:

$$\frac{d}{dt}\rho_S^t = L(t)\rho_S^t \ , \ \Lambda_{t,s} := I + \int_s^t d\tau L(\tau)\Lambda_{\tau,s}.$$

• **Euler formula** for the Propagator (*Evolution Operator*):

$$\Lambda_{t,s} = \lim_{m\to\infty} \prod_{k=m}^{1} \left[I - \frac{t-s}{m}L(s + k\frac{t-s}{m})\right]^{-1} \ .$$

• For *time-independent* H_{SR} and *Markov Approximation* of completely positive evolution $\Leftrightarrow$ Quantum *Semigroup* :

a. Markov completely positive maps: $\Lambda_t \Lambda_s = \Lambda_{t+s}$.

b. $\left\{\Lambda_t = e^{t\,L}\right\}_{t\geq 0}$ **semigroup** of *reduced quantum evolution* with generator L.

c. $\rho_S^t(A) = \mathrm{Tr}_{\mathfrak{H}_S}((\Lambda_t\,\omega_S)A)$ is a **continuous function** of $t \geq 0$ for any *density* matrix $\omega_S \in \mathrm{TrClass}(\mathfrak{H}_S)$ and $A \in \mathfrak{B}(\mathfrak{H}_S)$.

• Operator family $\{\rho_S^t = \Lambda_t\,\omega_S\}_{t\geq 0}$ is solution of the quantum Markovian *master* equation with (*unbounded*) **generator** L:

$$\frac{d}{dt}\rho_S^t = L\,\rho_S^t \ , \ \Lambda_t\,\rho = \lim_{m\to\infty}[I/(I - tL/m)]^m \rho \ .$$

• [*Kossakowski-Lindblad-Davies*] A *standard* form of Quantum Semigroup **generators** L and L^* is defined by a family of operators $\{V_\alpha \in \mathfrak{B}(\mathfrak{H}_S)\}_\alpha$ with $\sum_\alpha V_\alpha V_\alpha^* \in \mathfrak{B}(\mathfrak{H}_S)$:

$$L\rho = -i[H_S, \rho] + \frac{1}{2}\sum_\alpha \sigma_\alpha\{[V_\alpha \rho, V_\alpha^*] + [V_\alpha, \rho V_\alpha^*]\} \ ,$$

$$L^* A = i[H_S, A] + \frac{1}{2}\sum_\alpha \sigma_\alpha\{V_\alpha^*[A, V_\alpha] + [V_\alpha^*, A]V_\alpha\} \, .$$

• **Example:** Open Leaking/Pumping Cavity. $H_S = \epsilon \ b^* b$ and let $\alpha = -/+$, with leaking/pumping **rates** $\sigma_- \geq \sigma_+ \geq 0$, $V_- = b$ (leaking), $V_+ = b^*$ (pumping).

• *Density matrix* evolution and *adjoint* evolution of observables:

$$\langle A \rangle_t := \mathrm{Tr}_{\mathfrak{H}_S}(\rho^t \, A) = \mathrm{Tr}_{\mathfrak{H}_S}((e^{tL}\rho) \, A) = \mathrm{Tr}_{\mathfrak{H}_S}(\rho \, (e^{tL^*} A)) \ .$$

- *Adjoint* evolution of observables in the leaking/pumping cavity:

$$\partial_t(e^{tL^*}A) = e^{tL^*}L^*A = e^{tL^*}\{i[\epsilon\ b^*b, A] + \frac{1}{2}\sigma_-(b^*[A,b] + [b^*,A]b) + \frac{1}{2}\sigma_+(b[A,b^*] + [b,A]b^*)\}\ .$$

- Weyl operators and the C^*-algebra of CCR $\subseteq \mathfrak{B}(\mathfrak{H}_S)$ (N.B.).

$$\mathfrak{W}(\mathbb{C}) := \left\{W(\zeta) = \exp\left[\frac{i}{\sqrt{2}}\ (\bar{\zeta}\, b + \zeta\, b^*)\right]\right\}_{\zeta\in\mathbb{C}}\ .$$

- Adjoint evolution is a completely positive *quasi-free* *-automorphism, which is continuous in the weak*-topology on $\overline{\mathfrak{W}(\mathbb{C})}$:

$$e^{tL^*}\, W(\zeta) = e^{-\Omega_t(\zeta)}\ W(\zeta(t))\ ,\ \zeta(t) := \zeta\ e^{i\,\epsilon t-(\sigma_- - \sigma_+)t/2}\ .$$

$$\Omega_t(\zeta) := \frac{|\zeta|^2}{4}\frac{(\sigma_- + \sigma_+)}{(\sigma_- - \sigma_+)}\left\{1 - e^{-(\sigma_- - \sigma_+)t}\right\}\ .$$

- *Limit* cavity state $\rho_\infty := \lim_{t\to\infty} e^{tL}\rho$:

$$\lim_{t\to\infty} \mathrm{Tr}_{\mathfrak{H}_S}(\rho\,(e^{tL^*}W(\zeta))) = \lim_{t\to\infty} \mathrm{Tr}_{\mathfrak{H}_S}((e^{tL}\rho)\,W(\zeta))$$
$$= \mathrm{Tr}_{\mathfrak{H}_S}(\rho_\infty\,W(\zeta)) = \exp\left\{-\frac{|\zeta|^2}{4}\,\frac{(\sigma_- + \sigma_+)}{(\sigma_- - \sigma_+)}\right\} .$$

- For $\sigma_- > \sigma_+ \geq 0$ *any* initial cavity state ρ converges to the *quasi-free* Gibbs state:

$$\rho_\infty = \frac{e^{-\beta_{cav}\ \epsilon\ b^*b}}{(1-e^{-\beta_{cav}\ \epsilon})^{-1}} , \qquad \beta_{cav} := \frac{1}{\epsilon}\ln\frac{\sigma_-}{\sigma_+} .$$

- Evolution of the photon-number operator:

$$e^{tL^*}b^*b = e^{-(\sigma_- - \sigma_+)t}\ b^*b + \frac{\sigma_+}{\sigma_- - \sigma_+}\left\{1 - e^{-(\sigma_- - \sigma_+)t}\right\} .$$

Here $\sigma_+/(\sigma_- - \sigma_+) = (e^{\beta_{cav}\ \epsilon} - 1)^{-1}$.

10. Atoms in a Leaky Cavity

• *Repeated* interaction = *Markovian* time-partition of Hamiltonian dynamics over intervals $t \in \Delta_n(\tau) = [(n-1)\tau, n\tau)$:

$$H(t)\Big|_{\Delta_n(\tau)} := H_n + I \otimes \sum_{k \neq n} E\ \eta_k$$

• For $H_n = \epsilon\ b^* b \otimes I + I \otimes E\ \eta_n + \lambda\ (b^* + b) \otimes \eta_n$ the *Kossakowski-Lindblad* generator $L_{\sigma,n}$ has the form ($\sigma_- \geq \sigma_+ \geq 0$):

$$L_{\sigma,n}\, \rho_C \otimes \rho_A := -i[H_n, \rho_C \otimes \rho_A] +$$
$$\frac{1}{2}\sigma_-\ \{[b\, \rho_C \otimes \rho_A, b^*] + [b, \rho_C \otimes \rho_A\, b^*]\} +$$
$$\frac{1}{2}\sigma_+\ \{[b^*\, \rho_C \otimes \rho_A, b] + [b^*, \rho_C \otimes \rho_A\, b]\}.$$

• **Theorem 3.**

(a) For repeated interaction in the Leaking/Pumping cavity, $\sigma_- \geq \sigma_+ \geq 0$, the evolution of the Weyl is a convex **combination** of quasi-free completely positive maps.

(b) A regular normal *limiting* cavity state $\omega_{\mathcal{C},\sigma}(\cdot)$ exists:

$$\rho_\sigma^\infty := \|\cdot\|_1 - \lim_{n\to\infty} (\mathcal{L}_{\sigma,\tau})^n(\rho_C).$$

(c) It is an *infinite* convex combination of *non-gauge-invariant* quasi-free states on the Weyl algebra $\mathfrak{W}(\mathbb{C})$.

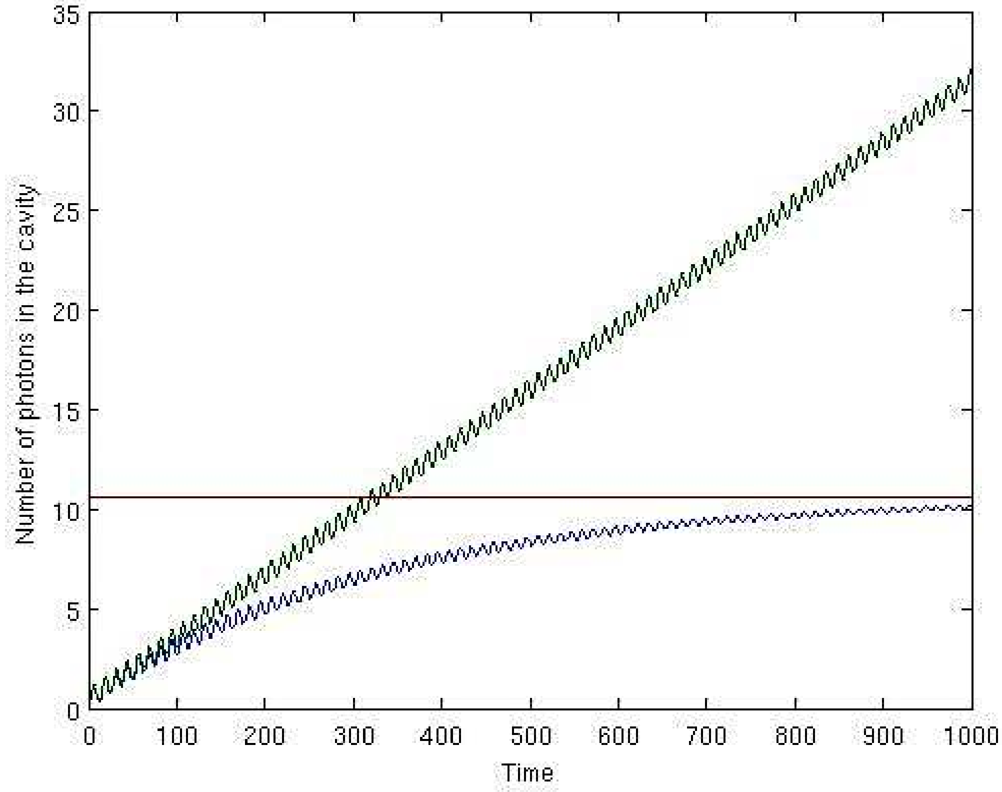
• **Theorem 4.** For repeated interaction in a leaky cavity with $\sigma_- \geq \sigma_+ \geq 0$, the mean-value of photon number in the cavity is **bounded**:

$$\lim_{n\to\infty} N_\sigma(t = n\tau) = \lim_{n\to\infty} \mathrm{Tr}_{\mathcal{H}_C}(b^*b\ \rho^{n\tau}_{\sigma,C}) =$$
$$\frac{\lambda^2}{|\mu|^2}\ \frac{p}{1-e^{-(\sigma_- - \sigma_+)\tau}} \times$$
$$\times\{1 + e^{-(\sigma_- - \sigma_+)\tau}(1-2p) - 2e^{-(\sigma_- - \sigma_+)\tau/2}(1-p)\cos\epsilon\tau\}$$
$$+\ \sigma_+/(\sigma_- - \sigma_+)\ ,$$

for any gauge-invariant initial state ρ_C.

• If $\sigma_+ = 0$, then for $\epsilon\tau \neq 2\pi s$, $s \in \mathbb{Z}$, $\lim_{\sigma_-\to 0} \omega_{\mathcal{C},\sigma}(b^*b) = +\infty$.

• If $\sigma_- \to \sigma_+$, then $\lim_{\sigma_-\to\sigma_+} \omega_{\mathcal{C},\sigma}(b^*b) = +\infty$. For the leaking and pumping of the same rate, the limiting state is the infinite-temperature Gibbs state.


Number of photons in the cavity
Time
0
5
10
15
20
25
30
35
0
100
200
300
400
500
600
700
800
900
1000

THANK YOU FOR YOUR ATTENTION !