Quantum Heisenberg models and random loop representations

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Yerevan, 6 September 2012

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Heisenberg ferromagnet

Heisenberg spin-$\frac{1}{2}$ ferromagnet: $H_{\rm F} = -\sum_{\substack{x,y\in\Lambda \\ \|x-y\|=1}} \big(S_x^1 S_y^1 + S_x^2 S_y^2 + S_x^3 S_y^3 \big)$

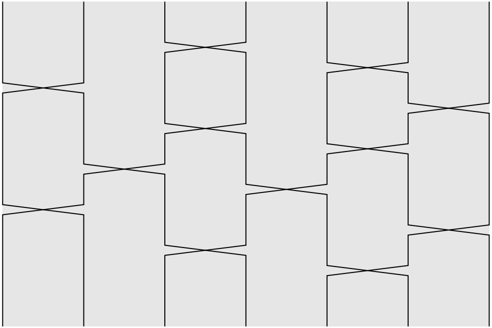
where Λ is a box in $\mathbb{Z}^d$ and $H_{\rm F}$ acts in $\bigotimes_{x\in\Lambda} \mathbb{C}^2$

Partition function: $Z = {\rm Tr}\, {\rm e}^{-\beta H_{\rm F}}$

Two-point correlation function: $\langle S_x^3 S_y^3 \rangle = \frac{1}{Z} {\rm Tr}\, S_x^3 S_y^3 \, {\rm e}^{-\beta H_{\rm F}}$

Tóth's representation of ferromagnet (1993)

Motivation: **Conlon-Solovej**'s work on bounds for the free energy, using random-walk representation (1991)



Independent Poisson point processes on $\times_{\text{edges of }\Lambda}[0,\beta]$

Random interchange model

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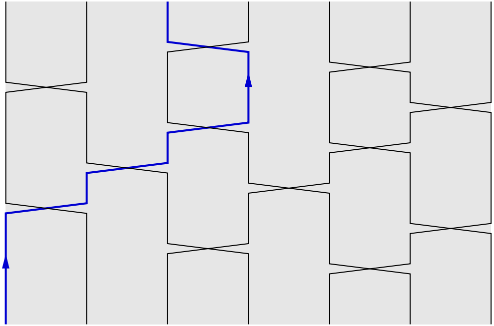

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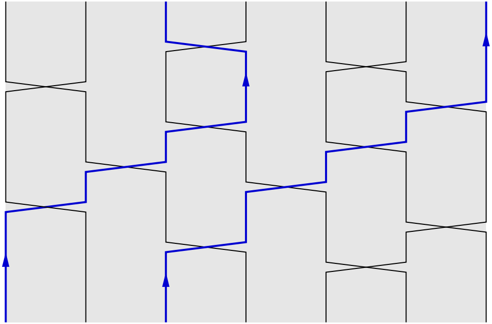

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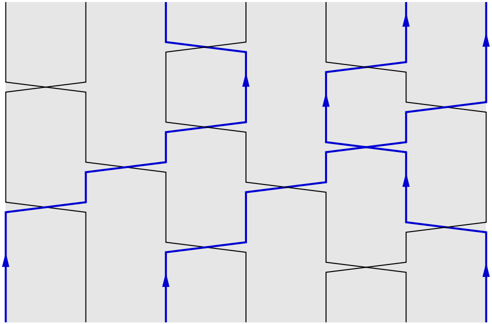

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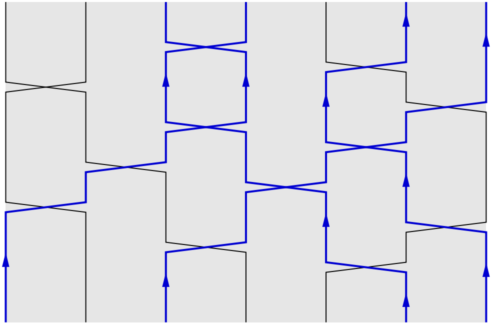

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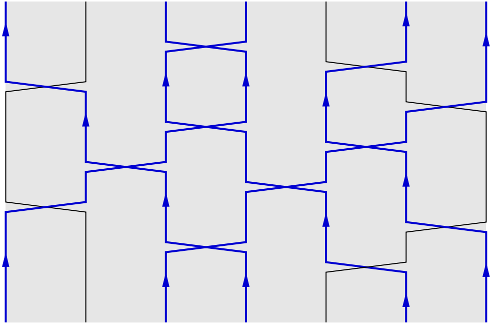

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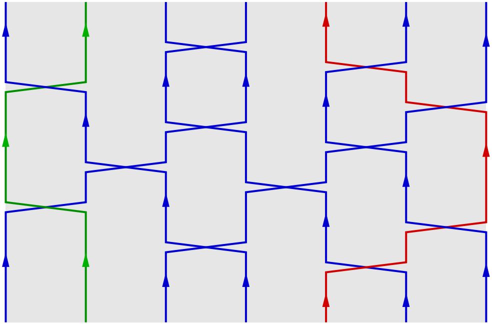

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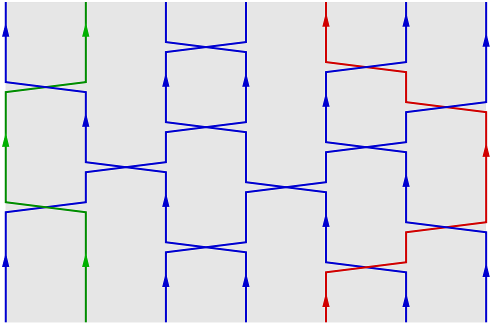

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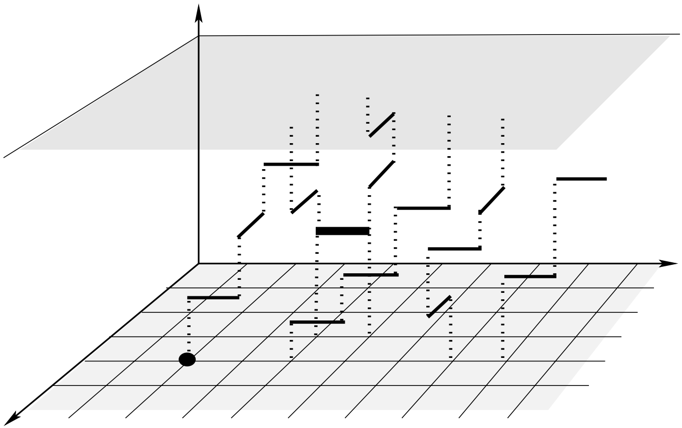
Random interchange model

$Z = \int \mathrm{d}\rho(\omega) 2^{|\mathcal{L}(\omega)|}$

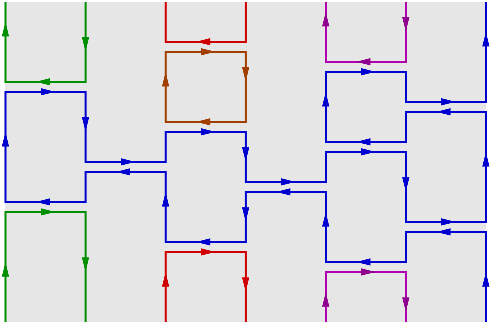
$\langle S_x^3 S_y^3 \rangle = \frac{1}{4}\mathbb{P}\big((x,0) \sim (y,0)\big)$

Picture in 2+1 dimensions



Aizenman-Nachtergaele's repr. of antiferro. (1994)

Context: The representation allows to relate the 1D quantum model to 2D classical random cluster and Potts models



Independent Poisson point processes on $\times_{\text{edges of } \Lambda}[0, \beta]$

$$Z = \int \mathrm{d}\rho(\omega) 2^{|\mathcal{L}(\omega)|}$$

$$\langle S_x^3 S_y^3 \rangle = \frac{(-1)^{\|x-y\|_1}}{4} \times \mathbb{P}\big((x,0) \sim (y,0)\big)$$

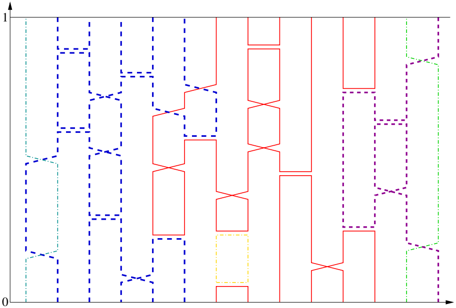
One-parameter family of Heisenberg models

$$H^{(u)} = - \sum_{\substack{x,y \in \Lambda \\ \|x-y\|=1}} \big(S_x^1 S_y^1 + u S_x^2 S_y^2 + S_x^3 S_y^3 \big)$$

- $u = +1$: Heisenberg ferromagnet
- $u = -1$: unitarily equivalent to Heisenberg antiferromagnet, $H_{\rm AF} = U^{-1} H^{(-1)} U$ with $U = \prod_{x \in \Lambda_{\rm B}} {\rm e}^{{\rm i}\pi S_x^2}$
- $u = 0$: quantum XY model, equivalent to hard-core bosons

Partition function: $Z = {\rm Tr}\, {\rm e}^{-\beta H^{(u)}}$

Repr. for one-parameter family of Heisenberg models



Let ρ denote Poisson point processes on $\times_{\text{edges of } \Lambda}[0,\beta]$, where **crossings** occur with intensity $\frac{1+u}{2}$ and **bars** occur with intensity $\frac{1-u}{2}$

One can combine and extend Tóth and Aizenman-Nachtergaele:

Theorem (U, 2012)

$$Z = \int \mathrm{d}\rho(\omega)\, 2^{|\mathcal{L}(\omega)|}$$

$$\langle S_x^3 S_y^3 \rangle = \frac{1}{4} \mathbb{P}\big((x,0) \sim (y,0)\big)$$

Long-range order vs macroscopic loops

Definition of long-range order (spontaneous magnetization):
$\frac{1}{|\Lambda|^2} \sum_{x,y \in \Lambda} \langle S_x^3 S_y^3 \rangle > c > 0$, with c indep. of Λ

Definition of macroscopic loops: $\mathbb{E}\Big(\frac{L_{(0,0)}}{\beta|\Lambda|} \Big) > c > 0$

Using properties of Duhamel two-point function and Falk-Bruch inequality, we have that

$$\frac{4}{|\Lambda|^2} \sum_{x,y \in \Lambda} \langle S_x^3 S_y^3 \rangle - \sqrt{\tfrac{2d(1-u)}{|\Lambda|} \mathbb{E}\big(\tfrac{L_{(0,0)}}{\beta|\Lambda|} \big)} \leqslant \mathbb{E}\Big(\frac{L_{(0,0)}}{\beta|\Lambda|} \Big) \leqslant \frac{4}{|\Lambda|^2} \sum_{x,y \in \Lambda} \langle S_x^3 S_y^3 \rangle$$

Existence of macroscopic loops / spont. magnetization

Theorem (**Dyson, Lieb, Simon** (1978) for $d \geqslant 5$, **Kennedy, Lieb, Shastry** (1988) for $d \geqslant 3$)

Assume $u \in [-1, 0]$. There exists $\beta_0 < \infty$ and $c > 0$ such that for all $\beta > \beta_0$,

$$\mathbb{E}\Big(\frac{L_{(0,0)}}{\beta|\Lambda|}\Big) > c$$

Proved by extension of the reflection positivity method of **Fröhlich, Simon, Spencer** (1976) to quantum systems

Proof of the theorem

Partition function with field

Inner product on $L^2(\Lambda \times [0,\beta])$:

$$(f,g) = \sum_{x\in\Lambda} \int_0^\beta \overline{f_{xt}}\, g_{xt}\, \mathrm{d}t$$

Introduce a partition function with real field $v \in C^2(\Lambda \times [0,\beta])$:

$$Z(v) = \int \mathrm{d}\rho(\omega) \sum_{\sigma:\omega} \exp\Big\{ (\sigma, \Delta v) + (v, \Delta v) + \sum_{x\in\Lambda} \int_0^\beta \mathrm{d}t \Big[a\sigma_{xt} \frac{\partial^2 v_{xt}}{\partial t^2} - b\Big(\frac{\partial v_{xt}}{\partial t}\Big)^2 \Big] \Big\}$$

where the first sum is over spins $\sigma_{xt} = \pm 1$, constant on each loop

Fourier transform in space and time: for $k \in \frac{2\pi}{L}\Lambda$ and $\tau \in \frac{2\pi}{\beta}\mathbb{Z}$,

$$\widehat{\kappa}(k,\tau) = \sum_{x\in\Lambda} \int_0^\beta \mathrm{e}^{-\mathrm{i}kx - \mathrm{i}\tau t}\, \kappa(x,t) \mathrm{d}t$$

Infrared bound

We prove below "Gaussian domination" $\boxed{Z(v) \leqslant Z(0)}$ (for small v), with $b > 8a^2 d(1-u)\kappa(e_1,0)$. Choosing $v_{xt} = h\cos(kx + \tau t)$ with $h \to 0$, we get the infrared bound

$$\widehat{\kappa}(k,\tau) \leqslant 2 \frac{\varepsilon(k) + b\tau^2}{(\varepsilon(k) + a\tau^2)^2}$$

where $\varepsilon(k) = 2\sum_{i=1}^d (1 - \cos k_i)$

Optimizing, we get $\boxed{\widehat{\kappa}(k,\tau) \leqslant \frac{2}{\varepsilon(k) + \frac{\tau^2}{8d(1-u)\kappa(e_1,0)}}}$

Case $\tau = 0$: **Dyson, Lieb, Simon** (1978). Recent similar bound for Ising model in transverse field, **Björnberg** (arXiv:1205.3385)

Consequence: macroscopic loops

Use **KLS** sum rule: $\kappa(e_1,0) = \frac{1}{\beta d|\Lambda|} \sum_{k\in\frac{2\pi}{L}\Lambda} \sum_{\tau\in\frac{2\pi}{\beta}\mathbb{Z}} \Big(\sum_{i=1}^d \cos k_i\Big) \widehat{\kappa}(k,\tau)$

Then

$$\mathbb{E}\Big(\frac{L_{(0,0)}}{\beta|\Lambda|}\Big) = \frac{\widehat{\kappa}(0,0)}{\beta|\Lambda|} \geqslant \kappa(e_1,0) - \frac{1}{\beta|\Lambda|} \sum_{\tau\neq 0} \frac{C}{\tau^2} - \frac{1}{\beta d|\Lambda|} \sum_{k\neq 0} \sum_\tau \frac{2(\sum \cos k_i)_+}{\varepsilon(k) + \frac{\tau^2}{8d(1-u)\kappa(e_1,0)}}$$

Then

$$\lim_{\beta\to\infty} \lim_{|\Lambda|\to\infty} \mathbb{E}\Big(\frac{L_{(0,0)}}{\beta|\Lambda|}\Big) \geqslant \sqrt{\kappa(e_1,0)} \times \Big[\sqrt{\kappa(e_1,0)} - \frac{\sqrt{8d(1-u)}}{d(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{(\sum \cos k_i)_+}{\sqrt{\varepsilon(k)}} \mathrm{d}k \Big]$$

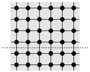
Positive for $d \geqslant 3$. Better bound in **Kennedy, Lieb, Shastry** (1988)

Reflection positivity

The goal now is to prove $Z(v) \leqslant Z(0)$. We prove below "reflection positivity"

$$Z(v_1,v_2)^2 \leqslant Z(v_1, Rv_1) Z(Rv_2, v_2)$$

Suppose (v_1,v_2) is maximiser. Then (v_1, Rv_1) is also maximiser



There is a space-invariant maximiser!

Reflection positivity

We need to cast $Z(v)$ in reflection positive form
(We set $a = b = 0$ for simplicity)

$$Z(v) = \int \mathrm{d}\rho(\omega) \sum_{\sigma:\omega} \exp\Big\{ -\sum_{\{x,y\}} \int_0^\beta \mathrm{d}t \Big[(\sigma_{xt} - \sigma_{yt})(v_{xt} - v_{yt}) + (v_{xt} - v_{yt})^2 \Big] \Big\}$$

$$= \int \mathrm{d}\rho(\omega) \sum_{\sigma:\omega} \exp\Big\{ \sum_{\{x,y\}} \int_0^\beta \mathrm{d}t \Big[-(\tfrac12 \sigma_{xt} - v_{xt} - \tfrac12 \sigma_{yt} + v_{yt})^2 + \tfrac14 (\sigma_{xt} - \sigma_{yt})^2 \Big] \Big\}$$

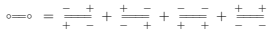
$$= \lim_{N\to\infty} \int \mathrm{d}\rho(\omega) \sum_{\sigma:\omega} \exp\Big\{ \sum_{\{x,y\}} \int_0^\beta \mathrm{d}t \Big[-(\dots)^2 \Big] \Big\} \prod_{\{x,y\}} \prod_{t\in\frac{\beta}{N}\{1,\dots,N\}} \Big(1 + \tfrac{\beta}{4N} (\sigma_{xt} - \sigma_{yt})^2\Big)$$

Reflection positivity

$$Z(v) = \lim_{N\to\infty} \int \mathrm{d}\rho(\omega) \sum_{\sigma:\omega} \exp\Big\{ \sum_{\{x,y\}} \int_0^\beta \mathrm{d}t \Big[-(\dots)^2 \Big] \Big\} \prod_{\{x,y\}} \prod_{t\in\frac{\beta}{N}\{1,\dots,N\}} \Big(1 + \tfrac{\beta}{N} - \tfrac{\beta}{N} \chi_{\sigma_{xt} = \sigma_{yt}}\Big)$$

$$= \int \mathrm{d}\rho'(\omega) \sum_{\sigma:\omega} \exp\Big\{ \sum_{\{x,y\}} \int_0^\beta \mathrm{d}t \Big[-(\tfrac12 \sigma_{xt} - v_{xt} - \tfrac12 \sigma_{yt} + v_{yt})^2 \Big] \Big\}$$

where ρ' is a Poisson point process on $\times_{\text{edges of }\Lambda} [0,\beta]$ where



occurs with intensity $\frac{1+u}{2}$, and  occurs with intensity $-u$
We need $u \in [-1, 0]$

Reflection positivity

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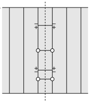
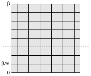


Manifestly reflection positive! (Cf **Fröhlich, Simon, Spencer** 1976)

Reflection positivity in time direction

For v space-invariant,

$$Z(v) = \lim_{N\to\infty} \int \mathrm{d}\rho(\omega) \sum_{\sigma:\omega} \exp\Big\{ -\frac{N}{\beta} \sum_{x\in\Lambda} \sum_{t\in\frac{\beta}{N}\{1,\dots,N\}} \Big[a(\sigma_{x,t+\frac{\beta}{N}} - \sigma_{xt})(v_{t+\frac{\beta}{N}} - v_t) + b(v_{t+\frac{\beta}{N}} - v_t)^2 \Big] \Big\}$$



This gives a **period-2** maximiser
$v_t = (-1)^{Nt/\beta} c$

For $b > 8a^2 d(1-u)\kappa(e_1,0)$, one can show that $Z(v) \leqslant Z(0)$ for v small

Existence of macroscopic loops / spont. magnetization

Theorem (**Dyson, Lieb, Simon** (1978) for $d \geqslant 5$, **Kennedy, Lieb, Shastry** (1988) for $d \geqslant 3$)

Assume $u \in [-1, 0]$. There exists $\beta_0 < \infty$ and $c > 0$ such that for all $\beta > \beta_0$,

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Can we say more about the macroscopic loops? Is there just one, or several loops?

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— Rather surprisingly, one can formulate an exact conjecture for their joint distribution: **Poisson-Dirichlet**!

Random partition of loop lengths

A **partition** of $[0,1]$ is a sequence $(\lambda_1, \lambda_2, \dots)$ of nonnegative, decreasing numbers such that $\sum_i \lambda_i = 1$

At finite volume, the following is a **random partition** of $[0,1]$:

$$\Big(\frac{L_1}{\beta|\Lambda|}, \frac{L_2}{\beta|\Lambda|}, \frac{L_3}{\beta|\Lambda|}, \dots \Big)$$

Goal: understand the limit distribution as $|\Lambda| \to \infty$. The meaning of

$$\big(\tfrac{L_1}{\beta|\Lambda|}, \tfrac{L_2}{\beta|\Lambda|}, \tfrac{L_3}{\beta|\Lambda|}, \dots \big) \xrightarrow{\text{d}} (\lambda_1, \lambda_2, \lambda_3, \dots)$$

is that the **joint distribution** of the **first** k **numbers** of the left side converge (in probability) to the joint distribution of the first k numbers of the right side, for any k

As $|\Lambda| \to \infty$, only macroscopic loops matter, and they converge to a random partition of $[0,\nu]$ where ν turns out to be equal to $3\,\mathbb{E}(\frac{L_{(0,0)}}{\beta|\Lambda|})$

Poisson-Dirichlet distribution

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— Choose λ_1 uniformly in $[0, 1]$
— Choose λ_2 uniformly in $[0, 1 - \lambda_1]$
— Choose λ_3 uniformly in $[0, 1 - \lambda_1 - \lambda_2]$
— Etc...

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Rearranging (λ_i) in decreasing order gives **Poisson-Dirichlet PD(1)**

This is a **one-parameter** family of distributions. For Poisson-Dirichlet PD(θ), choose the λ_js as beta random variables with parameter $\theta > 0$, $\mathbb{P}(\lambda_j > s) = (1 - s)^\theta$ (with rescaling)

Why should we expect Poisson-Dirichlet?

The mechanism is rather **indirect**, but it is **very general**

— Introduce a stochastic process such that the equilibrium measure $2^{|\mathcal{L}(\omega)|} \, \mathrm{d}\rho(\omega)$ is the invariant measure

— Effective split-merge process on partitions

— The invariant measure of the split-merge process is Poisson-Dirichlet (**Mayer-Wolf, Zeitouni, Zerner** 2002)

This is motivated by **Schramm** (2005), who studied compositions of random transpositions, proving a conjecture of **Aldous** about the Poisson-Dirichlet distribution of the lengths of permutation cycles

Invariant measure of stochastic process

- A new edge-time (e, t) appears at rate $2^{\alpha}\mathrm{d}t$ if its appearance causes a loop to split, and at rate $2^{-\alpha}\mathrm{d}t$ if it causes two loops to merge
- An edge-time already present disappears at rate $2^{1-\alpha}$ if its removal causes a loop to split, and at rate $2^{-(1-\alpha)}$ if it causes two loops to merge

By considering all possible cases, we can check the **detailed balance condition**:

$$\rho(\mathrm{d}\omega)2^{|\mathcal{L}(\omega)|}p(\omega, \mathrm{d}\omega') = \rho(\mathrm{d}\omega')2^{|\mathcal{L}(\omega')|}p(\omega', \mathrm{d}\omega)$$

and since the process is ergodic, the measure $\rho(\mathrm{d}\omega)2^{|\mathcal{L}(\omega)|}$ is the unique invariant measure

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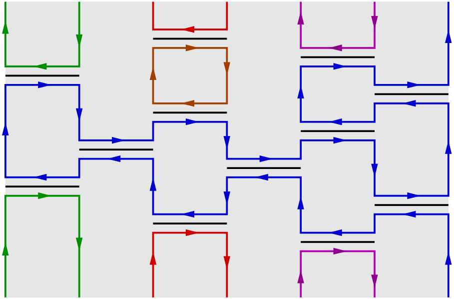
A) Removing an edge between two different loops:



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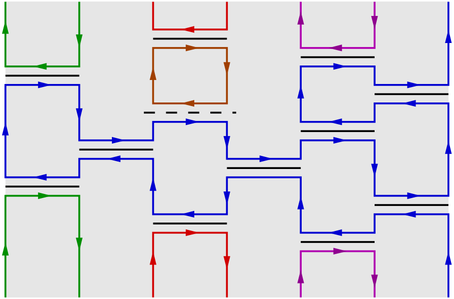

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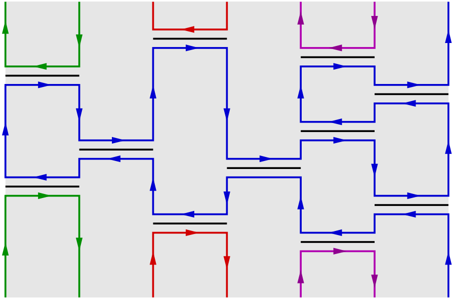

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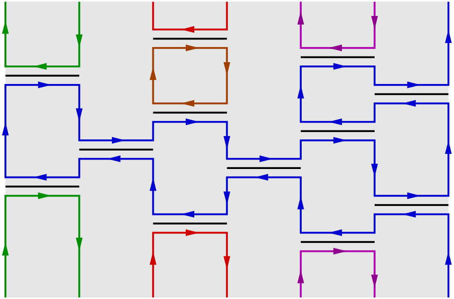
B) Removing an edge within a loop:



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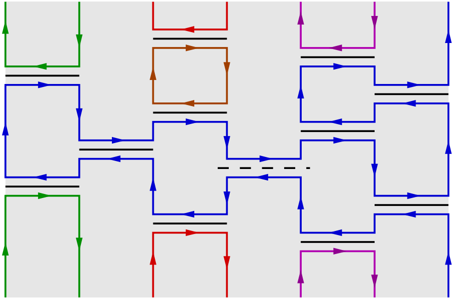

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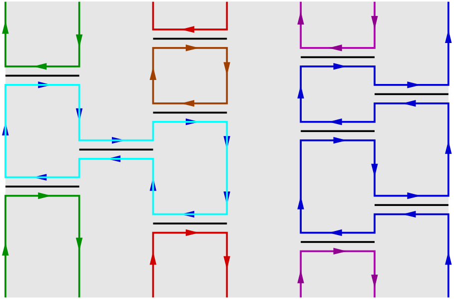

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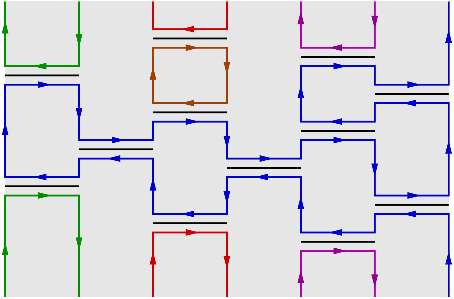
This stochastic process necessarily **splits** a loop or **merges** two loops

C) Adding an edge between two loops:



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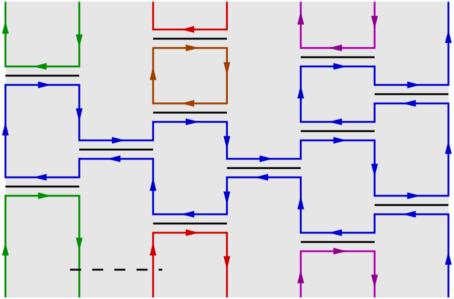

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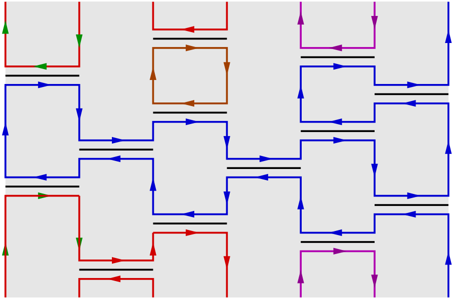

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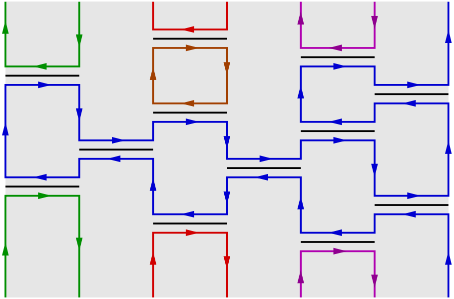
D) Adding an edge within a loop:



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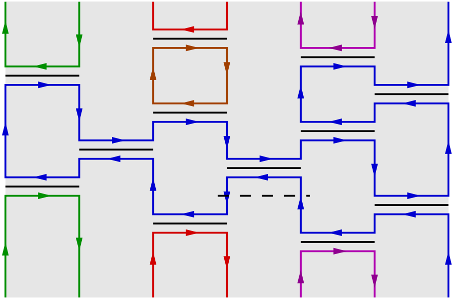

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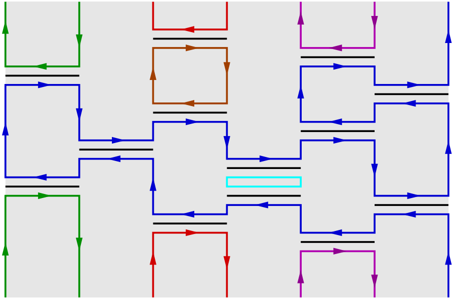


Split-merge process (coagulation-fragmentation)

Stochastic process on partitions of $[0, 1]$

- choose two numbers randomly, independently in $[0, 1]$
- if they fall in two distinct elements of the partition: merge them with probability θ^{-1}
- if they fall in same element of the partition: split it (uniformly)
- rearrange the elements in decreasing order

We expect that **macroscopic loops are spread everywhere**, that there are no correlations between different regions in space

Macroscopic loops merge and split **at the same rates as in the split-merge process**

Numerical evidence in related model

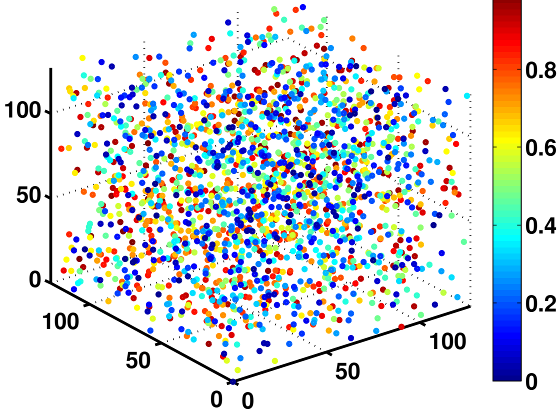


Figure for random lattice permutations, cf **Grosskinsky, Lovisolo, U** (2012)

Related situation: permutation cycles in ideal Bose gas

Motivated by **Feynman** (1953), **Sütő** (1993, 2002)

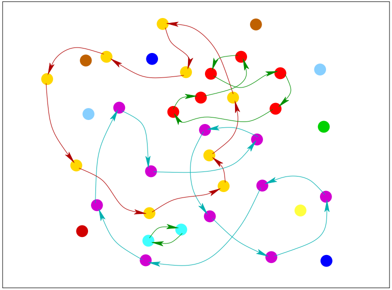
State space: $\Omega_{\Lambda,n} = \Lambda^n \times \mathcal{S}_n$

Classical model with Gibbs measure

$$\frac{1}{Z} \prod_{i=1}^{n} \mathrm{e}^{-\|x_i - x_{\sigma(i)}\|^2} \, \theta^{\#\text{cycles in } \sigma} \mathrm{d}x_1 \dots \mathrm{d}x_n$$

(Ideal Bose gas: $\theta = 1$)

Critical density $\rho_c = \theta \pi^{d/2} \zeta(d/2)$



Let $L_1, L_2, \dots$ the lengths of permutation cycles in decreasing order

Theorem (Betz, U, 2011)
$\left(\frac{L_1}{n}, \frac{L_2}{n}, \dots\right) \xrightarrow{\mathrm{d}} \mathrm{PD}(\theta)$ of $[0, \nu]$ where $\nu = \max\left(0, \frac{\rho - \rho_c}{\rho}\right)$

The spatial structure disappears when considering the Fourier space, so the mechanism here is somewhat different

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- Heuristics supported by numerical results for lattice permutations

THANK YOU!