Stability for system of fermions with zero-range interactions

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- Finco, T., *Rep. Math. Phys.* (2012)
- Correggi, Dell'Antonio, Finco, Michelangeli, T., *Rev. Math. Phys.* (2012)

Introduction

System of n quantum particles in $\mathbb{R}^d$, $d = 1, 2, 3$, interacting via a zero-range, two-body interaction. Formally

$$\mathcal{H} = -\sum_{i=1}^{n} \frac{1}{2m_i}\Delta_{\mathbf{x}_i} + \sum_{\substack{i,j=1 \\ i<j}}^{n} \mu_{ij}\,\delta(\mathbf{x}_i - \mathbf{x}_j),$$

where $\mathbf{x}_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, m_i is the mass, $\Delta_{\mathbf{x}_i}$ is the Laplacian relative to $\mathbf{x}_i$, and $\mu_{ij} \in \mathbb{R}$. We set $\hbar = 1$.

Original physical motivation in Nuclear Physics. More recently, relevant in the analysis of ultra-cold quantum gases.
Here it is "experimentally realized" the unitary limit

$$a \longrightarrow \infty, \qquad r_0 \longrightarrow 0$$

a scattering length, r_0 effective range

General problem: rigorous construction of $\mathcal{H}$ as self-adjoint and, possibly, bounded from below operator in $L^2(\mathbb{R}^{nd})$

1. Definition

Notice: $\mathcal{H}\psi = \mathcal{H}_0\psi$ if ψ vanishes on each hyperplane $\{\mathbf{x}_i = \mathbf{x}_j\}$

Then consider

$$\dot{\mathcal{H}}_0 = -\sum_{i=1}^n \frac{1}{2m_i}\Delta_{\mathbf{x}_i}, \qquad D(\dot{\mathcal{H}}_0) = C_0^\infty(\mathbb{R}^{nd} \setminus \cup_{i<j}\{\mathbf{x}_i = \mathbf{x}_j\})$$

$\dot{\mathcal{H}}_0$ is symmetric but not self-adjoint. One (trivial) self-adjoint extension is the free Hamiltonian.

By definition, any other self-adjoint extension of $\dot{\mathcal{H}}_0$ is a Hamiltonian of n quantum particles in $\mathbb{R}^d$ with zero-range interactions.

2. Explicit construction

More delicate and strongly dependent on the dimension d.
Each self-adjoint extension of $\dot{\mathcal{H}}_0$ is characterized by a (generalized) boundary condition on each hyperplane $\{\mathbf{x}_i = \mathbf{x}_j\}$.

For $d = 3, n = 2$, in the relative coordinate $\mathbf{x}$

$$\mathcal{H} = \frac{1}{2m}\Delta_{\mathbf{x}} + \delta(\mathbf{x})$$

the entire class of self-adjoint extensions of $\dot{\mathcal{H}}_0$ can be constructed. The domain of each extension consists of functions $\psi \in H^2(\mathbb{R}^3 \setminus \{0\})$ satisfying the boundary condition at the origin

$$\psi(\mathbf{x}) = \frac{q}{|\mathbf{x}|} + \alpha q + o(1), \qquad \text{for } |\mathbf{x}| \to 0, \quad q \in \mathbb{C}, \alpha \in \mathbb{R}$$

For $d = 3, n > 2$, by analogy, one takes the Skornyakov-Ter-Martirosyan (STM) extension H_α, defined on $H^2(\mathbb{R}^{3n} \setminus \cup_{i<j}\{\mathbf{x}_i = \mathbf{x}_j\})$ and s.t.

$$\psi(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \frac{q_{ij}}{|\mathbf{x}_i - \mathbf{x}_j|} + \alpha q_{ij} + o(1), \qquad \text{for } |\mathbf{x}_i - \mathbf{x}_j| \to 0, \quad \alpha \in \mathbb{R}$$

q_{ij} functions on $\{\mathbf{x}_i = \mathbf{x}_j\}$ and α strength of the interaction

As a matter of fact, in general
STM extension H_α is symmetric but not self-adjoint.

For three identical bosons and for three different particles, any self-adjoint extension of STM is unbounded from below (sequence of eigenvalues $E_k \rightarrow -\infty$)
Faddeev, Minlos (1962); Melnikov, Minlos (1991)

Instability known as Thomas effect

Remark: Thomas effect can be avoided introducing "non local" boundary conditions (e.g. choosing α as integral operator)
Albeverio, Hoegh-Krohn, Streit (1977); Frank, Seiringer (2012)

System of fermions

Thomas effect could not occur if the Hilbert space is suitably restricted (symmetry constraint)

Indeed, for a system of identical fermions the zero-range interaction is ineffective

Open problem: stability or occurrence of Thomas effect for a system composed by a mixture of two different species of fermions. The answer depends on m (mass ratio), N, M (number of fermions).

Partial results only in the case of N fermions (of mass $= 1$) plus one particle of mass m

Physical literature

- case $2+1$ (Efimov (1972),(1973); Petrov (2003))
There is a critical mass $m^*(2) \simeq 0.0735$ s.t.
$m < m^*(2)$ three-body bound states of energy $\to -\infty$
$m > m^*(2)$ boundedness from below

- case $3+1$ (Castin, Mora, Pricoupenko (2010))
$m^*(2) < m < 0.0747$ four-body bound states of energy $\to -\infty$

Rigorous results

- case $2+1$ (Shermatov (2003); Minlos (2011))
For $m < m^*(2)$ STM extension (restricted to $l = 1$) is not self-adjoint and all its self-adjoint extensions are unbounded from below (Thomas)

- case $N+1$ (Minlos (2011))
For $N < 5$ and m suff. large STM extension is self-adjoint

Results

We consider a system of N fermions (of mass $= 1$) plus a different particle (of mass m),

we construct, via renormalization procedure (Dell'Antonio, Figari,T. (1994)), the quadratic form $\mathcal{F}_\alpha$, naturally associated with the STM extension H_α

$$\mathcal{F}_\alpha(u) = (u, H_\alpha u) \qquad \text{for} \;\; u \in D(H_\alpha),$$

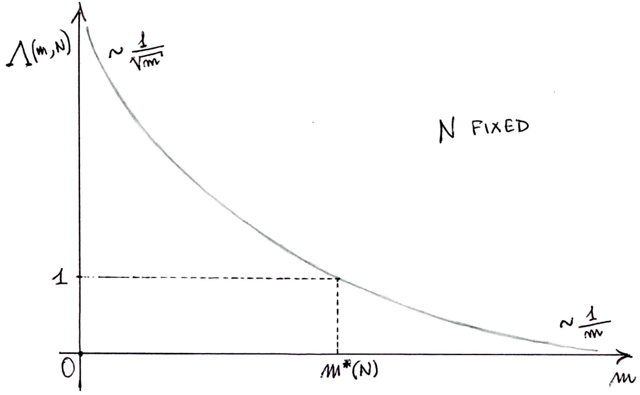
we define $m^*(N)$ as the unique solution (for N fixed) of

$$\Lambda(m,N) := 2\pi^{-1}(N-1)(m+1)^2\left[\frac{1}{\sqrt{m(m+2)}} - \arcsin\left(\frac{1}{m+1}\right)\right] = 1$$

$\Lambda(m, N) > 0$, decreases with m,
$\lim_{m\to 0} \Lambda(m, N) = \infty$, $\lim_{m\to\infty} \Lambda(m, N) = 0$

therefore

$m^*(N) > 0$, increases with N, for $N = 2$ reduces to $m^*(2)$ and
$\Lambda(m, N) < 1$ iff $m > m^*(N)$



and we prove

Th. 1 (stability)

If $N \geq 2$ and $m > m^*(N)$ then $\mathcal{F}_\alpha$ is closed and bounded from below. In particular $\mathcal{F}_\alpha$ is
- positive for $\alpha \geq 0$ and
- bounded below by $-\frac{\alpha^2}{4\pi^4(1-\Lambda(m,N))^2}$ for $\alpha < 0$.

This implies:

If $N \geq 2$ and $m > m^*(N)$ then the STM extension H_α is self-adjoint and bounded from below. In particular H_α is
- positive for $\alpha \geq 0$ and
- $\inf \sigma(H_\alpha) \geq -\frac{\alpha^2}{4\pi^4(1-\Lambda(m,N))^2}$ for $\alpha < 0$.

Th. 2 (instability)

If $N \geq 2$ and $m < m^*(2)$ then $\mathcal{F}_\alpha$ is unbounded from below for any $\alpha \in \mathbb{R}$.

This suggests that Thomas effect occurs.

Sequence of trial functions s.t. $\mathcal{F}_\alpha(u_n) \to -\infty$ for $N = 2$,

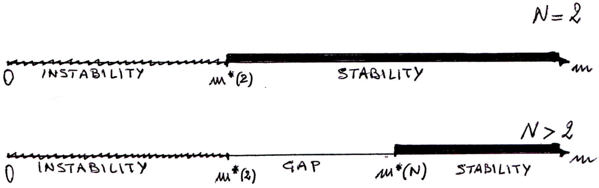
then the sequence is adapted to the case $N > 2$, with $N - 2$ fermions sufficiently far away.

Optimal result for $N = 2$

Partial result for $N > 2$

e.g. by numerical simul., the case $m = 1$ seems stable for any N.

The role of the antisymmetry must be more carefully taken into account



Case $2+1$: the quadratic form $\mathcal{F}_\alpha$

Coord. $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_0$, C.M. frame, rel. coord. $\mathbf{y}_1 = \mathbf{x}_1 - \mathbf{x}_0$, $\mathbf{y}_2 = \mathbf{x}_2 - \mathbf{x}_0$ ($\to \mathbf{k}_1, \mathbf{k}_2$), Hilbert space $L^2_f(\mathbb{R}^6)$.

The interaction is not a small form perturbation of the free Ham.

Formal energy form + renorm. procedure $\to \mathcal{F}_\alpha(u) = (u, H_\alpha u)$.

Domain strictly larger than $H^1_f(\mathbb{R}^6)$. One has to add

$$\widehat{\mathcal{G}^\lambda \xi}(\mathbf{k}_1, \mathbf{k}_2) = \frac{\hat{\xi}(\mathbf{k}_1) - \hat{\xi}(\mathbf{k}_2)}{\mathbf{k}_1^2 + \mathbf{k}_2^2 + \frac{2}{m+1}\mathbf{k}_1 \cdot \mathbf{k}_2 + \lambda} \qquad \xi \in H^{1/2}_f(\mathbb{R}^3)$$

Def.

$$D(\mathcal{F}_\alpha) = \left\{ u \in L^2_f(\mathbb{R}^6) \,|\, \exists \xi \in H^{1/2}_f(\mathbb{R}^3) \; s.t. \; u - \mathcal{G}^\lambda \xi \in H^1_f(\mathbb{R}^6) \right\}$$

$$\mathcal{F}_\alpha(u) = F^\lambda_0(u - \mathcal{G}^\lambda \xi) - \lambda \|u\|^2 + 2\left(\Phi^\lambda_D(\xi) + \Phi^\lambda_N(\xi) + \alpha \|\xi\|^2 \right)$$

where

$$\lambda > 0$$

$$F_0^\lambda(u-\mathcal{G}^\lambda\xi) = \int d\mathbf{k}_1 d\mathbf{k}_2 \left(\mathbf{k}_1^2 + \mathbf{k}_2^2 + \frac{2}{m+1}\mathbf{k}_1\cdot\mathbf{k}_2 + \lambda\right)|\widehat{u - \mathcal{G}^\lambda\xi}|^2$$

$$\Phi_D^\lambda(\xi) = 2\pi^2 \int d\mathbf{p} \sqrt{\frac{m(m+2)}{(m+1)^2}\mathbf{p}^2 + \lambda}\; |\hat{\xi}(\mathbf{p})|^2$$

$$\Phi_N^\lambda(\xi) = \int d\mathbf{p} d\mathbf{q}\, \frac{\hat{\xi}^*(\mathbf{p})\xi(\mathbf{q})}{\mathbf{p}^2 + \mathbf{q}^2 + \frac{2}{m+1}\mathbf{p}\cdot\mathbf{q} + \lambda}$$

$\alpha \in \mathbb{R}$ strength of interaction

F_0^λ contribution of the reg. part $u - \mathcal{G}^\lambda\xi$ (positive)
Φ_D^λ diag. contribution of the sing. part $\mathcal{G}^\lambda\xi$ (positive)
Φ_N^λ non-diag. contribution of the sing. part $\mathcal{G}^\lambda\xi$ (not definite in sign)

Lower bound for $\mathcal{F}_\alpha$

Crucial point, also for closure of $\mathcal{F}_\alpha$

$$\mathcal{F}_\alpha(u) = F_0^\lambda(u - \mathcal{G}^\lambda \xi) - \lambda\|u\|^2 + 2\Big(\Phi_D^\lambda(\xi) + \Phi_N^\lambda(\xi) + \alpha\|\xi\|^2\Big)$$
$$\geq -\lambda\|u\|^2 + 2\Big(\Phi_D^\lambda(\xi) + \Phi_N^\lambda(\xi) + \alpha\|\xi\|^2\Big)$$

If $\Phi_N^\lambda(\xi) \geq -\Lambda\, \Phi_D^\lambda(\xi)$, with $0 < \Lambda < 1$, one easily obtains

$$\geq -\lambda\|u\|^2 + 2\Big((1-\Lambda)\Phi_D^\lambda(\xi) + \alpha\|\xi\|^2\Big)$$
$$\equiv -\lambda\|u\|^2 + 2\int d\mathbf{p}\left((1-\Lambda)\, 2\pi^2\sqrt{\frac{m(m+2)}{(m+1)^2}\mathbf{p}^2 + \lambda} + \alpha\right)|\hat{\xi}(\mathbf{p})|^2$$

Therefore

If $\alpha \geq 0$ then lower bound $= 0$
If $\alpha < 0$ then lower bound $= -\frac{\alpha^2}{4\pi^4(1-\Lambda)^2}$

Estimate $\Phi_N^\lambda(\xi) \geq -\Lambda\, \Phi_D^\lambda(\xi)$, with $\Lambda < 1$ if $m > m^*(2)$

Main steps of the (elementary) proof

$$\Phi_N^\lambda(\xi) = \int d\mathbf{p} d\mathbf{q} \; \frac{\hat{\xi}^*(\mathbf{p})\xi(\mathbf{q})}{\mathbf{p}^2 + \mathbf{q}^2 + \frac{2}{m+1}\mathbf{p}\cdot\mathbf{q} + \lambda}$$

exp. in spherical harmonics $\hat{\xi}(\mathbf{p}) = \sum_{lm} \xi_{lm}(p) Y_l^m(\theta,\phi)$

$$= 2\pi \sum_{lm} \int_0^\infty dp \int_0^\infty dq \; p^2 \xi_{lm}^*(p) q^2 \xi_{lm}(q) \int_{-1}^1 dy \; \frac{P_l(y)}{p^2 + q^2 + \frac{2}{m+1} pq\, y + \lambda}$$

positivity for l even (fermions)

$$\geq 2\pi \sum_{lm, l\ odd} \int_0^\infty dp \int_0^\infty dq \; p^2 \xi_{lm}^*(p) q^2 \xi_{lm}(q) \int_{-1}^1 dy \; \frac{P_l(y)}{p^2 + q^2 + \frac{2}{m+1} pq\, y + \lambda}$$

terms with l odd increasing in λ

$$\geq 2\pi \sum_{lm, l\ odd} \int_0^\infty dp \int_0^\infty dq \; p^2 \xi_{lm}^*(p) q^2 \xi_{lm}(q) \int_{-1}^1 dy \; \frac{P_l(y)}{p^2 + q^2 + \frac{2}{m+1} pq\, y}$$

diagon. via Mellin transform

$$=\sum_{lm, l\, odd}\int dk\, S_l(k)|\xi^\sharp_{lm}(k)|^2, \quad \xi^\sharp_{lm}(k)=\frac{1}{\sqrt{2\pi}}\int dx\, e^{-ikx}e^{2x}\xi_{lm}(e^x)$$

for any l odd $\quad S_l(k)\geq S_1(0)$, and $S_1(0)<0$

$$\geq -|S_1(0)|\sum_{lm, l\, odd}\int dk\, |\xi^\sharp_{lm}(k)|^2$$

inverse transform

$$=-|S_1(0)|\sum_{lm, l\, odd}\int_0^\infty dp\, p^3|\xi_{lm}(p)|^2$$

def. $\Lambda=|S_1(0)|(m+1)\left(2\pi^2\sqrt{m(m+2)}\right)^{-1}$ and add λ

$$\geq -\Lambda\sum_{lm, l\, odd}2\pi^2\int_0^\infty dp\, p^2\sqrt{\frac{m(m+2)}{(m+1)^2}p^2+\lambda}\,|\xi_{lm}(p)|^2$$

add terms with l even and sum on l, m

$$\geq -\Lambda\,\Phi^\lambda_D(\xi), \quad \text{and } \Lambda<1 \text{ if } m>m^*(2)$$