The Variable Discrete Asymptotics on Manifolds with Edge

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Stochastic and Analytic Methods in Mathematical Physics

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Edge-degenerate Differential Operators

Let $X^\Delta \times \Omega$ be a wedge with edge $\Omega \subseteq \mathbb{R}^q$ and model cone X^Δ with base X, defined as $X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$.

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The stretched wedge is defined as $X^\wedge \times \Omega := \mathbb{R}_+ \times X \times \Omega$, considered in the splitting of variables (r, x, y) where $X^\wedge = \mathbb{R}_+ \times X \ni (r, x)$.

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A differential operator of order μ on $X^\wedge \times \Omega$ is said to be edge-degenerate if it has the form

$$A = r^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(r, y)(-r\partial_r)^j (rD_y)^\alpha$$

for coefficients $a_{j\alpha}(r, y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X))$.

Weighted Edge-Sobolev Spaces

Definition

A Hilbert space H (throughout this consideration assumed to be separable) is said to be endowed with a group action $\kappa = \{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ if

(i) $\kappa_\lambda : H \longrightarrow H$, $\lambda \in \mathbb{R}_+$, *is a family of isomorphisms, where* $\kappa_\lambda \kappa_\nu = \kappa_{\lambda\nu}$ *for all* $\lambda, \nu \in \mathbb{R}_+$,

(ii) $\lambda \mapsto \kappa_\lambda h$, $\lambda \in \mathbb{R}_+$, *defines a function in* $C(\mathbb{R}_+, H)$ *for every* $h \in H$ *(i.e.,* κ *is strongly continuous).*

By

$$\mathcal{W}^s(\mathbb{R}^q, H), \quad s \in \mathbb{R},$$

we denote the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^q, H)$ with respect to the norm

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} := \left\{ \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} \hat{u}(\eta)\|^2_H d\eta \right\}^{1/2}$$

where $\langle\eta\rangle := (1 + |\eta|^2)^{1/2}$.

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We apply this definition to spaces that are typical for edge singularities.

Let $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^l)$ be the space of all classical parameter-dependent pseudo-differential operators on X of order $\mu \in \mathbb{R}$, where $\lambda \in \mathbb{R}^l$, $l \in \mathbb{N}$, is the parameter.

For closed smooth X we often employ the fact that for every $\mu \in \mathbb{R}$ there is a parameter-dependent elliptic element $R^\mu(\lambda) \in L^\mu_{\text{cl}}(X; \mathbb{R}^l)$ which induces isomorphisms

$$R^\mu(\lambda) : H^s(X) \to H^{s-\mu}(X)$$

for all $s \in \mathbb{R}, \lambda \in \mathbb{R}^l$. Any such R^μ will be called an order reducing family.

Let $Mu(z) := \int_0^\infty r^{z-1}u(r)dr$ be the Mellin transform, first for $u \in C_0^\infty(\mathbb{R}_+)$; then $Mu(z)$ is an entire function in $z \in \mathbb{C}$. We will extend M also to vector-valued distributions. Let

$$\Gamma_\beta := \{z \in \mathbb{C} : \mathrm{Re}\, z = \beta\},$$

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The space $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s, \gamma \in \mathbb{R}$ is defined to be the completion of $C_0^\infty(X^\wedge)$ with respect to the norm

$$\Big\{ \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \| R^s(\mathrm{Im}\, z) Mu(z) \|^2_{L^2(X)} \, đz \Big\}^{\frac{1}{2}}$$

$đz := (2\pi i)^{-1} dz$, $n := \dim X$, where $R^s(\lambda) \in L^s_{\mathrm{cl}}(X; \mathbb{R}_\lambda)$ is an order reducing family.

Let $X^{\asymp} := \mathbb{R} \times X \ni (r, x)$ be interpreted as a manifold with conical exits to infinity $r \to \pm\infty$ (which motivates the notation $X^{\asymp}$). The space $H^s_{\mathrm{cone}}(X^{\asymp})$ is defined to be the completion of $C_0^\infty(X^{\asymp})$ with respect to the norm

$$\left\{ \int_{-\infty}^{\infty} \| \, [r]^{-s+\frac{n}{2}} \mathcal{F}^{-1}_{\rho \to r} R^s([r]\rho)(\mathcal{F}_{r \to \rho} u)(r) \, \|^2_{L^2(X)} \, dr \right\}^{\frac{1}{2}}$$

where $R^s(\tilde{\rho}) \in L^s_{\mathrm{cl}}(X; \mathbb{R}_{\tilde{\rho}})$ is an order reducing family, and $r \to [r]$ is a function in $C^\infty(\mathbb{R})$, $[r] > 0$, with $[r] = |r|$ for $|r| \geq c$ for some $c > 0$.

Note that in the case $X := S^n$ the space $H^s_{\text{cone}}((S^n)^{\asymp})|_{[1,+\infty)\times S^n}$ is the same as $H^s(\mathbb{R}^{n+1})|_{[1,+\infty)\times S^n}$ where $\{\tilde{x} \in \mathbb{R}^{n+1} : |\tilde{x}| \geq 1\}$ is identified with $[1, +\infty) \times S^n$ via polar coordinates $\tilde{x} \longrightarrow (r, x) \in \mathbb{R}_+ \times S^n$ in $\mathbb{R}^{n+1} \setminus \{0\}$. This gives rise to a simple equivalent definition of $H^s_{\text{cone}}(X^{\asymp})$ for arbitrary X via a localisation over sets $\mathbb{R}_+ \times U$ for coordinate neighbourhoods U on X and an identification with conical sets $\mathbb{R}_+ \times V, V \subset S^n, V \simeq U$.

Moreover, we set $H^s_{\mathrm{cone}}(X^\wedge) := H^s_{\mathrm{cone}}(X^\asymp)|_{X^\wedge}$.

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Definition

The space $\mathcal{K}^{s,\gamma}(X^\wedge)$ is defined as

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{\omega u_0 + (1-\omega)u_\infty : u_0 \in \mathcal{H}^{s,\gamma}(X^\wedge), u_\infty \in H^s_{\mathrm{cone}}(X^\wedge)\}$$

for any cut-off function $\omega(r)$,

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for any cut-off function $\omega(r)$*, and we set*

$$\mathcal{K}^{s,\gamma;e}(X^\wedge) := \langle r\rangle^{-e}\mathcal{K}^{s,\gamma}(X^\wedge)$$

for every $s, \gamma, e \in \mathbb{R}$ *where* $\langle r\rangle = (1+r^2)^{1/2}$*.*

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Observe that $\mathcal{K}^{0,0}(X^\wedge) = r^{-n/2}L^2(X^\wedge)$.

Concerning more details, cf.

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B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, North-Holland, Amsterdam, 1991.

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Ju.V. Egorov and B.-W. Schulze, *Pseudo-differential operators, singularities, applications*, Oper. Theory: Adv. Appl. **93**, Birkhäuser Verlag, Basel, 1997.

The Singular Functions of Discrete Edge Asymptotics

Definition

A sequence

$$\mathcal{P} = \{(p_j, m_j)\}_{j=0,\dots,N}$$

of pairs $(p_j, m_j) \in \mathbb{C} \times \mathbb{N}$, *for* $N = N(\mathcal{P}) \in \mathbb{N} \cup \{\infty\}$, *is said to be a discrete asymptotic type, associated with the weight data* (γ, Θ), *with a weight* $\gamma \in \mathbb{R}$ *and a (half-open) weight interval* $\Theta = (\vartheta, 0]$, $-\infty \leq \vartheta < 0$, *if the set* $\pi_{\mathbb{C}}\mathcal{P} := \{p_j\}_{j=0,\dots,N} \subset \mathbb{C}$ *is contained in* $\left\{\frac{n+1}{2} - \gamma + \vartheta < \mathrm{Re}\, z < \frac{n+1}{2} - \gamma\right\}$, *where* $n = \dim X$, *furthermore* $N(\mathcal{P}) < \infty$ *for* $\vartheta > -\infty$, *and* $\mathrm{Re}\, p_j \to -\infty$ *for* $j \to \infty$ *in the case* $\vartheta = -\infty$ *and* $N(\mathcal{P}) = \infty$.

If $\mathcal{P}$ is a discrete asymptotic type associated with (γ, Θ), $\vartheta > -\infty$, and ω a fixed cut-off function, we set

$$\mathcal{E}_{\mathcal{P}}(X^{\wedge}) := \omega(r)\Big\{ \sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk}(x) r^{-p_j} \log^k r : c_{jk} \in C^{\infty}(X) \Big\}.$$

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This space is isomorphic to a direct sum of $\sum_{j=0}^{N}(m_j+1)$ copies of the space $C^{\infty}(X)$ and as such a Fréchet space.

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This space is isomorphic to a direct sum of $\sum_{j=0}^{N}(m_j+1)$ copies of the space $C^\infty(X)$ and as such a Fréchet space. Observe that if we set $\mathcal{E}_{\mathcal{P}}(\mathbb{R}_+) := \omega(r)\{\sum_{j,k} c_{jk} r^{-p_j} \log^k r : c_{jk} \in \mathbb{C}\}$, which is of finite dimension $\sum_{j=0}^{N}(m_j+1)$, we can also write

$$\mathcal{E}_{\mathcal{P}}(X^\wedge) = C^\infty(X, \mathcal{E}_{\mathcal{P}}(\mathbb{R}_+)).$$

Remark

We have

$$\mathcal{E}_{\mathcal{P}}(X^\wedge) \subset \mathcal{K}^{s,\gamma;\infty}(X^\wedge)$$

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Let us define the space of flat functions $\mathcal{K}_{\Theta}^{s,\gamma;g}(X^\wedge)$ (of flatness $-\vartheta - 0$ relative to the weight γ) as the projective limit of the spaces $\mathcal{K}^{s,\gamma-\vartheta-1/(m+1);g}(X^\wedge)$ over $m \in \mathbb{N}$, in the Fréchet topology of the projective limit.

Definition

(i) *Let* $\mathcal{P}$ *be a discrete asymptotic type associated with* (γ, Θ), Θ *finite; we set*

$$\mathcal{K}_{\mathcal{P}}^{s,\gamma;g}(X^{\wedge}) := \mathcal{K}_{\Theta}^{s,\gamma;g}(X^{\wedge}) + \mathcal{E}_{\mathcal{P}}(X^{\wedge}) \qquad (1)$$

(*which is a direct sum*);

(ii) *if* $\mathcal{P}$ *is a discrete asymptotic type,* Θ *infinite, we form*

$$\mathcal{P}_l := \{(p, m) \in \mathcal{P} : \operatorname{Re} p > (n+1)/2 - \gamma - (l+1)\},$$

$l \in \mathbb{N}$, *which is associated with* $\Theta_l = (-(l+1), 0]$, *and we define*

$$\mathcal{K}_{\mathcal{P}}^{s,\gamma;g}(X^{\wedge})$$

to be the projective limit of the spaces $\mathcal{K}_{\mathcal{P}_l}^{s,\gamma;g}(X^{\wedge})$ *over* $l \in \mathbb{N}$.

Analogously as before we will write $\mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge)$ and $\mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)$ when $g = 0$.

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Moreover, we define the space $\mathcal{S}^{\gamma}_{\mathcal{P}}(X^\wedge)$ to be the projective limit of the spaces $\mathcal{K}^{N,\gamma;N}_{\mathcal{P}}(X^\wedge)$ over $N \in \mathbb{N}$.

Theorem

Let $\mathcal{P}$ be a discrete asymptotic type associated with the weight data (γ, Θ) and χ be a $\pi_{\mathbb{C}}\mathcal{P}$-excision function.

(i) *Let $u(r,x) \in \mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge)$ and $\omega(r)$ a cut-off function. Then*

$$M_{\gamma-n/2,r\to z}(\omega u)(z,x)$$

extends from $\Gamma_{(n+1)/2-\gamma}$ to an $H^s(X)$-valued meromorphic function $f(z,x)$ in z, $(n+1)/2-\gamma+\vartheta < \mathrm{Re}\, z < (n+1)/2-\gamma$, with poles at the points p_j of multiplicity m_j+1 and Laurent coefficients at $(z-p_j)^{-(k+1)}$, $0 \le k \le m_j$, belonging to $C^\infty(X)$ such that $\chi(z)f(z,x)|_{\Gamma_\beta} \in \hat{H}^s(\Gamma_\beta \times X)$ for every $(n+1)/2-\gamma+\vartheta < \beta \le (n+1)/2-\gamma$, uniformly in compact β-intervals.

Theorem

(ii) *Let* $f(z,x) \in \hat{H}^s(\Gamma_{(n+1)/2-\gamma} \times X)$ *be a function that extends to an* $H^s(X)$*-valued function which is meromorphic as in* (i), *where* $\chi(z) f(z,x)|_{\Gamma_\beta} \in \hat{H}^s(\Gamma_\beta \times X)$ *for every* $(n+1)/2 - \gamma + \vartheta < \beta \leq (n+1)/2 - \gamma$, *uniformly in compact* β*-intervals. Then for every cut-off function* $\omega(r)$ *we have*

$$\omega(r)(M_{\gamma-n/2, r\to z} f(r,x)) \in \mathcal{K}_{\mathcal{P}}^{s,\gamma}(X^\wedge).$$

For a discrete asymptotic type $\mathcal{P}$, associated with (γ, Θ), $\Theta = (\vartheta, 0]$, $-\infty \leq \vartheta < 0$, the spaces $\mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)$ are subspaces of $\mathcal{K}^{s,\gamma}(X^\wedge)$, $s \in \mathbb{R}$. Those are Fréchet spaces with group action $\kappa = \{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$,

$$(\kappa_\lambda u)(r, x) := \lambda^{(n+1)/2} u(\lambda r, x),$$

$n := \dim X$. Thus we can form the associated wedge space

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)) =: \mathcal{W}^{s,\gamma}_{\mathcal{P}}(X^\wedge \times \mathbb{R}^q).$$

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Theorem

We have

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge)) = \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{\infty,\gamma}_{\mathcal{P}}(X^\wedge)) + \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge))$$

as a non-direct sum of Fréchet spaces, for every $s \in \mathbb{R}$.

Remark

Modulo $\mathcal{W}^\infty(\mathbb{R}^q, \mathcal{K}_\Theta^{\infty,\gamma}(X^\wedge))$ *the elements of* $\mathcal{W}^\infty(\mathbb{R}^q, \mathcal{K}_\mathcal{P}^{\infty,\gamma}(X^\wedge))$ *are of the form*

$$\sum_{j=0}^{N} \sum_{0 \le k \le m_j} \omega(r) r^{-p_j} \log^k r \, v_{jk}(y,x)$$

$v_{jk}(y,x) \in H^\infty(\mathbb{R}^q_y, C^\infty(X))$, *for any cut-off function* $\omega(r)$.

Theorem

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$$\sum_{j=0}^{N} \sum_{0\le k\le m_j} \omega(r) \int e^{iy\eta} \left\{ [\eta]^{(n+1)/2} (r[\eta])^{-p_j} \log^k (r[\eta])\, \hat{v}_{jk}(\eta, x) \right\} d\eta,$$

$\hat{v}_{jk}(\eta, x) \in \hat{H}^s(\mathbb{R}^q_\eta, C^\infty(X))$, *for any cut-off function* $\omega(r)$.

Operator-valued Analytic Functionals

Let

$$A = r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r,y)(-r\partial_r)^j (rD_y)^\alpha,$$

$a_{j\alpha}(r,y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X))$, be an elliptic edge-degenerate elliptic operator on $X^\Delta \times \mathbb{R}^q \ni (r,x,y)$, and assume for simplicity $a_{j\alpha}(r,y)$ to be independent of y for large $|y|$. Solutions $u \in \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$ for $s=\infty$ are expected to have an asymptotic behaviour of the form

$$\sum_{j=0}^{N(y)} \sum_{0\le k\le m_j(y)} \omega(r) r^{-p_j(y)} \log^k r \, v_{jk}(y,x).$$

The involved asymptotic data $\{(p_j(y), m_j(y))\}$ as well as the coefficients $v_{jk} \in C^\infty(X)$ may depend on y. In particular, poles and multiplicities of the meromorphic families arising under Mellin transform, and also the Laurent coefficients depend on y and may be variable and branching.

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It is an enormous challenge to understand the phenomena in detail. Another problem is to describe the functional analytic structure of the singular functions of variable branching asymptotics for arbitrary smoothness $s \in \mathbb{R}$.

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It is an enormous challenge to understand the phenomena in detail. Another problem is to describe the functional analytic structure of the singular functions of variable branching asymptotics for arbitrary smoothness $s \in \mathbb{R}$.

A suitable approach is the concept of continuous asymptotics.

For an open set $U \subseteq \mathbb{C}$ by $\mathcal{A}(U)$ we denote the space of holomorphic functions in U (Fréchet in the topology of uniform convergence on compact subsets). More generally, $\mathcal{A}(U, E)$ will denote the space of holomorphic functions with values in a Fréchet space E. Finally let $\mathcal{A}'(K, E)$ be the space of E-valued analytic functionals carried by the compact set $K \subset \mathbb{C}$. If E is nuclear Fréchet, then so is $\mathcal{A}'(K, E)$.

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Let $K \subset \mathbb{C}$ be compact, and fix any compact counter-clockwise oriented (say, smooth) curve C surrounding K such that the winding number of C with respect to every $z \in K$ is equal to 1.

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Let $K \subset \mathbb{C}$ be compact, and fix any compact counter-clockwise oriented (say, smooth) curve C surrounding K such that the winding number of C with respect to every $z \in K$ is equal to 1.

In the sequel we always impose these assumptions when a curve C is surrounding K.

Remark

Every $f(y,z) \in C^\infty(\Omega, \mathcal{A}(\mathbb{C} \setminus K, E))$, $\Omega \subseteq \mathbb{R}^q$ *open, gives rise to a function* $\zeta_f(y) \in C^\infty(\Omega, \mathcal{A}'(K,E))$ *via*

$$\langle \zeta_f(y), h \rangle := \int_C f(y,z)h(z)đz, \quad h \in \mathcal{A}(\mathbb{C}).$$

Conversely, for any $\zeta(y) \in C^\infty(\Omega, \mathcal{A}'(K,E))$ *we can recover an* $f(y,z) \in C^\infty(\Omega, \mathcal{A}(\mathbb{C} \setminus K, E))$ *such that* $\zeta = \zeta_f$ *by*

$$f(y,z) := M_{r \to z}\big(\omega(r)\langle \zeta(y)_w, r^{-w}\rangle\big).$$

In particular, we may employ this construction to recover constant discrete asymptotics. Let $f(z) \in \mathcal{A}(\mathbb{C} \setminus K, C^\infty(X)))$ and assume that f extends to a meromorphic function across K with poles $p_j \in K$ of multiplicity $m_j + 1$, $j = 1, \dots, N$. Then we have

$$\langle \zeta_f, r^{-z} \rangle = \sum_{j=0}^{N} \sum_{0 \leq k \leq m_j} c_{jk}(x) r^{-p_j} \log^k r,$$

for coefficients $c_{jk}(x) \in C^\infty(X)$ that are in ono-to-one correspondence with the Laurent coefficients of f at $(z - p_j)^{-(k+1)}$, $0 \leq k \leq m_j$.

The Singular Functions of Continuous Edge Asymptotics

Theorem

Modulo $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_\Theta(X^\wedge)) + \mathcal{W}^\infty(\mathbb{R}^q, \mathcal{K}^{\infty,\gamma}_\mathcal{P}(X^\wedge))$ *the elements of* $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_\mathcal{P}(X^\wedge))$ *are of the form*

$$\omega(r) \int e^{iy\eta} \{[\eta]^{(n+1)/2} \langle \hat{\zeta}(\eta)_z, (r[\eta])^{-z} \rangle d\eta,$$

$\hat{\zeta}(\eta) \in \mathcal{A}'(K, \hat{H}^s(\mathbb{R}^q_\eta, C^\infty(X)))$, *for any cut-off function* $\omega(r)$.

The Edge Algebra and Parametrices

There is a pseudo-differential algebra, called the edge algebra, that contains the edge-degenerate differential operators together with the parametrices of elliptic elements.

Variable Discrete Asymptotic Types

Definition

A variable discrete asymptotic type $\mathcal{P}$ over an open set $\Omega \subseteq \mathbb{R}^q$ associated with weight data $(\gamma, \Theta), \Theta = (\vartheta, 0], -\infty \leq \vartheta < 0$, is a system of sequences of pairs

$$\mathcal{P}(y) = \{(p_j(y), m_j(y))\}_{j=0,1,\dots,J(y)}$$

for $J(y) \in \mathbb{N}$, $y \in \Omega$, such that

(i)

$$\pi_{\mathbb{C}}\mathcal{P}(y) \subseteq \{(n+1)/2 - \gamma + \vartheta < \mathrm{Re}\, z < (n+1)/2 - \gamma\},$$

for all $y \in \Omega$,

Definition

Let $\mathcal{U}(\Omega)$ we denote the set of all open $U \subset \Omega$ such that $\overline{U}$ is compact and $\overline{U} \subset \Omega$.

(ii) *for every $b = (c, U)$,*

$$(n+1)/2 - \gamma + \vartheta < c < (n+1)/2 - \gamma,$$

$U \in \mathcal{U}(\Omega)$, there are sets $\{U_i\}_{0 \le i \le N}$, $\{K_i\}_{0 \le i \le N}$, for $N = N(b)$, where $U_i \in \mathcal{U}(\Omega), 0 \le i \le N$, form an open covering of $\overline{U}$,

Definition

(iii) *moreover,*

$$K_i \Subset \mathbb{C}, K_i \subset \{c - \varepsilon_i < \mathrm{Re}\, z < (n+1)/2 - \gamma\},$$

$$\pi_{\mathbb{C}} \mathcal{P}(y) \bigcap \{c - \varepsilon_i < \mathrm{Re}\, z < (n+1)/2 - \gamma\} \subset K_i$$

for all $y \in U_i$ *and for fixed* i

$$\sup_{y \in U_i} \sum_j (1 + m_j(y)) < \infty,$$

with the supremum over those $0 \leq j \leq J(y)$ *such that* $p_j(y) \in K_i$.

The Singular Functions of Variable Discrete Edge Asymptotics

Let $\pi_{\mathbb{C}}\mathcal{P}(y) \subset K$ for some compact set $K \subset \{(n+1)/2 - \gamma + \vartheta < \mathrm{Re}\, z < (n+1)/2 - \gamma\}$ for all $y \in \mathbb{R}^q$. According to the above-mentioned localisation for convenience we consider asymptotics for y varying in an open bounded set $U \subset \mathbb{R}^q$. Moreover, let $V \subset \mathbb{R}^q$ be another open bounded set such that $\overline{U} \subset V$, and assume $\pi_{\mathbb{C}}\mathcal{P}(y) = \emptyset$ for all $y \in \mathbb{R}^q \setminus U$.

Then we define

$$\mathcal{W}^\infty(\mathbb{R}^q, \mathcal{K}_{\mathcal{P}}^{\infty,\gamma}(X^\wedge))$$

to be the space of functions of the form

$$\omega(r) \int e^{iy\eta} \{[\eta]^{(n+1)/2} \langle \hat{\zeta}(y,\eta)_z, (r[\eta])^{-z} \rangle d\eta$$

for any cut-off function $\omega(r)$ and some $\hat{\zeta}(y,\eta) \in C_0^\infty(V_y, \mathcal{A}'(K, \hat{H}^\infty(\mathbb{R}^q_\eta, C^\infty(X))))$, subordinate to $\mathcal{P}$.

Subordinate means that $\hat{\zeta}(y,\eta)$ is carried by $\pi_{\mathbb{C}}\mathcal{P}(y)$ for every y and of order m_j for every j. Recall that $\mathcal{P}(y)=\{(p_j,m_j)\}_{j=1,\dots,J(y)}$.

Subordinate means that $\hat{\zeta}(y, \eta)$ is carried by $\pi_{\mathbb{C}}\mathcal{P}(y)$ for every y and of order m_j for every j. Recall that $\mathcal{P}(y) = \{(p_j, m_j)\}_{j=1,\dots,J(y)}$.

Observe that the above singular fuctions can be written as

$$\omega(r) \int e^{iy\eta} \langle \hat{\zeta}(y, \eta)_z, r^{-z} \rangle d\eta$$

for $\hat{\zeta}(y, \eta) \in C^\infty(V_y, \mathcal{A}'(K, \hat{H}^\infty(\mathbb{R}^q_\eta, C^\infty(X))))$, subordinate to $\mathcal{P}$, or, alternatively, as

$$\omega(r) \langle \hat{\zeta}(y, y')_z, r^{-z} \rangle |_{y'=y}$$

for $\hat{\zeta}(y, y') \in C^\infty(V_y, \mathcal{A}'(K, H^\infty(\mathbb{R}^q_{y'}, C^\infty(X))))$, subordinate to $\mathcal{P}$.

Another equivalent (and most simple) description of smoothing singular functions with variable discrete asymptotics is

$$\{\omega(r)\langle\zeta(y)_z, r^{-z}\rangle : \\ \zeta(y) \in C^\infty(V_y, \mathcal{A}'(K, C^\infty(X))), \zeta \text{ subordinate to } \mathcal{P}\}.$$

Theorem

Let $\mathcal{P}$ be a variable discrete asymptotic type over $\mathbb{R}^q$ and assume that $\pi_{\mathbb{C}}\mathcal{P}(y) \subset K$ for some compact set $K \subset \{(n+1)/2 - \gamma + \vartheta < \mathrm{Re}\, z < (n+1)/2 - \gamma\}$ for all $y \in \mathbb{R}^q$, and let $V \subset \mathbb{R}^q$ be an open bounded set such that $\pi_{\mathbb{C}}\mathcal{P}(y) = \emptyset$ for all $y \in \mathbb{R}^q \setminus U$. Then, modulo $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge)) + \mathcal{W}^\infty(\mathbb{R}^q, \mathcal{K}^{\infty,\gamma}_{\mathcal{P}}(X^\wedge))$ the elements of $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_{\mathcal{P}}(X^\wedge))$ are of the form

$$\omega(r) \int e^{iy\eta} \{[\eta]^{(n+1)/2} \langle \hat{\zeta}(y,\eta)_z, (r[\eta])^{-z} \rangle d\eta$$

for any cut-off function $\omega(r)$ and some $\hat{\zeta}(y,\eta) \in C^\infty(V_y, \mathcal{A}'(K, \hat{H}^s(\mathbb{R}^q_\eta, C^\infty(X))))$, subordinate to $\mathcal{P}$.

Regularity of Solutions

Solutions to elliptic edge-degenerate equations $Au = f$ are regular with variable discrete asymptotics. Here f is assumed to be a weighted edge distirbution with variable branching asymptotics.

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The scheme of the proof is as follows. We construct a parametrix P of the operator A which is sensitive enough to feel the variable asymptotic types. This can be done within a very subtle version of edge pseudo-differential algebra which reflects variable branching asymptotics within the various substructures of smoothing Mellin plus Green operators that are responsible for the asymptotic information.

We multiply the equation $Au = f$ from the left by P which yields $PAu = Pf$. Here $PA = 1 + G$ where G is a Green operator with variable discrete asymptotics. Moreover, Pf is of the desired asymptotic behaviour. Thus the same is true of $u = Pf - Gu$ provided that u is a weighted distribution which is an a priori information, feeded in in the computation of asymptotics.

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These ideas work in principle for many types of singular PDE-problems, e.g., in elasticity, crack theory, mixed and transmission problems, but also in many-particle systems, where for lower particle numbers the asymptotics for Hamiltonians have been computed, see the references below.

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Thank you for your attention!