Groupoid cocycles and derivations

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One-parameter automorphism groups

In the C*-algebraic formalism, time evolution of a physical system is given by a pair $(\mathcal{A}, \alpha)$, where $\mathcal{A}$ is a C*-algebra called algebra of observables and $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of automorphisms of $\mathcal{A}$. The generator of α is a derivation, usually unbounded, which will be denoted by δ. The pair $(\mathcal{A}, \alpha)$ will be called here a C*-system.

If $\mathcal{A} = \mathcal{B}(\mathcal{H})$ is the algebra of bounded operators on a Hilbert space $\mathcal{H}$, there exists an unbounded self-adjoint operator H on $\mathcal{H}$ such that

$$\alpha_t(A) = e^{itH} A e^{-itH}$$

Then the generator of α is the derivation δ where

$$\delta(A) = i[H, A] = i(HA - AH).$$

Example 0

Suppose that $\mathcal{A} = M_n(\mathbb{C})$ and that α is given by a self-adjoint diagonal operator $H = \mathrm{diag}(h_1, \dots, h_n)$. Then, for $A = (A_{k,l}) \in M_n(\mathbb{C})$, we have:

$$\alpha_t(A)_{k,l} = e^{it(h_k - h_l)} A_{k,l}; \qquad \delta(A)_{k,l} = i(h_k - h_l)A_{k,l}$$

Let us view the matrices as functions on $G = \{1, \dots, n\} \times \{1, \dots, n\}$ and rewrite these formulas after introducing $\gamma = (k, l) \in G$ and $c(\gamma) = h_k - h_l$. This gives

$$\alpha_t(f)(\gamma) = e^{itc(\gamma)} f(\gamma); \qquad \delta(f)(\gamma) = ic(\gamma)f(\gamma)$$

Note that $c : G \to \mathbb{R}$ satisfies the Chasles (or cocycle) relation:

$$c(k, l) + c(l, m) = c(k, m).$$

Groupoids

The previous example fits into the general framework of groupoids and their convolution algebras.

A groupoid is a small category $(G, G^{(0)})$ where each arrow $\gamma \in G$ is invertible. The inverse is denoted γ^{-1}. We denote by $r(\gamma)$ the range of γ and by $s(\gamma)$ its source. A pair of arrows (γ, γ') is composable iff $s(\gamma) = r(\gamma')$; then the composition is denoted by $\gamma\gamma'$.

There are two examples to keep in mind: groups (this is the case when $G^{(0)}$ is reduced to a singleton $\{e\}$) and equivalence relations (this is the case when the map $(r, s) : G \to G^{(0)} \times G^{(0)}$ is injective). In that case, arrows are pairs (x, y); composition law and inverse are respectively

$$(x, y)(y, z) = (x, z); \qquad (x, y)^{-1} = (y, x)$$

Convolution algebra

We assume that G has locally compact Hausdorff topology compatible with its algebraic structure and a Haar system $(\lambda^x)_{x\in G^{(0)}}$ where for all $x \in G^{(0)}$, λ^x is a Radon measure on the fibre $G^x = r^{-1}(x)$ and we have

- (*invariance*) $\gamma\lambda^{s(\gamma)} = \lambda^{r(\gamma)}$;
- (*continuity*) $x \to \int f d\lambda^x$ is continuous for all $f \in C_c(G)$.

We define the C*-algebra $C^*(G)$ as the C*-completion of the $*$-algebra $C_c(G)$ of continuous compactly supported functions on G, where

$$f * g(\gamma) = \int f(\gamma\gamma')g(\gamma'^{-1})d\lambda^{s(\gamma)}(\gamma')$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

Scalar cocycles and derivations

A map $c : G \to \mathbb{R}$ is a cocycle if

$$c(\gamma\gamma') = c(\gamma) + c(\gamma')$$

When $G = X \times X$ (or any equivalence relation), this is above Chasles relation.

We assume that c is continuous and we define $\delta_c : C_c(G) \to C_c(G)$ by

$$\delta_c(f)(\gamma) = ic(\gamma)f(\gamma).$$

It is immediate that δ_c is a derivation:

$$\delta_c(f * g) = \delta_c(f) * g + f * \delta_c(g).$$

Some properties

Define

$$\alpha_t(f)(\gamma) = e^{itc(\gamma)} f(\gamma).$$

It is a one-parameter automorphism group of $C^*(G)$. This shows that δ_c is a pregenerator, in particular it is closable.

Proposition

Let G, c, δ_c be as above. Then,

1. *δ_c is bounded iff c is bounded;*
2. *δ_c is inner if c is a coboundary.*

Recall that a cocycle $c : G \to \mathbb{R}$ is a coboundary if there exists $b : G^{(0)} \to \mathbb{R}$ such that $c = b \circ r - b \circ s$. Here, we are dealing with continuous cocycles and we insist on having b continuous.

Example 1

This example is a groupoid description of the Ising model. As usual, Λ is a lattice in $\mathbb{R}^d$ and each site carries a spin ± 1. The configuration space is $X = \{-1, +1\}^\Lambda$. Let G be the graph of the equivalence relation on X where two configurations are equivalent iff the number of sites where they differ is finite. The C*-algebra $\mathcal{A}$ of observables is an algebra of matrices indexed by G: its elements are functions on G and the operations are the usual matrix multiplication and involution:

$$ab(x,z) = \sum_y a(x,y)b(y,z); \qquad a^*(x,y) = \overline{a(y,x)}$$

The one-parameter automorphism group α is given by

$$\alpha_t(a)(x,y) = e^{itc(x,y)}a(x,y)$$

where the energy cocycle $c : G \to \mathbb{R}$ is defined by

$$c(x,y) = -\sum_{i,j} J(i,j)[x(i)x(j) - y(i)y(j)]$$

where J depends on the nature of the interaction.

Example 2

The Cuntz algebra $\mathcal{O}_d$ is the C*-algebra generated by d isometries $S_1, \dots, S_d$ on a Hilbert space $\mathcal{H}$ whose ranges are mutually orthogonal and span $\mathcal{H}$.

The gauge automorphism group is the one-parameter automorphism group α such that $\alpha_t(S_k) = e^{it} S_k$.

Its groupoid description is $\mathcal{O}_d = C^*(G)$, where

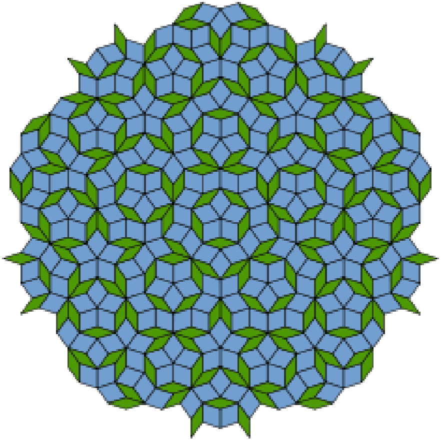
$$G = \{(x, m-n, y) : x, y \in X, m, n \in \mathbb{N}, T^n x = T^m y\}$$

where T is the one-sided shift on $X = \{1, \dots, d\}^{\mathbb{N}}$.

The above gauge automorphism group α is then given by the gauge cocycle $c : G \to \mathbb{R}$ such that $c(x, k, y) = k$.

Example 3

One usually defines a tiling T as a subdivision of the plane $\mathbb{R}^2$ into tiles. Here is a famous example of a tiling (or rather a patch of this tiling):



Example 3 (cont'd)

We denote by $T + x$ the translate of the tiling T by the vector $x \in \mathbb{R}^2$. We define a metric on the set of tilings: T and T' are ϵ close if there exist vectors x and x' of norm $\leq \epsilon$ such that $T + x$ and $T' + x'$ agree on the ball of centre 0 and radius $1/\epsilon$.

The hull of a tiling T_0 is the orbit closure

$$\Omega(T_0) = \overline{T_0 + \mathbb{R}^2}$$

The group $\mathbb{R}^2$ acts continuously on $\Omega(T_0)$ and one can form semi-direct groupoid

$$G(T_0) = \{(T, x, T') \in \Omega(T_0) \times \mathbb{R}^2 \times \Omega(T_0) : T + x = T'\}$$

Example 4 (cont'd)

There is a natural cocycle on $G(T_0)$, namely $c : G(T_0) \to \mathbb{R}^2$ such that $c(T, x, T') = x$. This construction provides a C*-system $(C^*(G(T_0)), \alpha)$.

What is this C*-system good for?

Aperiodic tilings model quasicrystals. The corresponding C*-algebra gives information on the spectrum of its Schrödinger Hamiltonian, in particular on its gaps. A large part of the work is to compute the K-theory of $C^*(G(T_0))$ and its image under a canonical trace.

Non-commutative Dirichlet forms

J.-L. Sauvageot has given in 1989 the following construction in the framework of non-commutative Dirichlet forms:

Let $(T_t)_{t\geq 0}$ be a strongly continuous semi-group of completely positive contractions (also called Markov operators) of a C*-algebra A. We denote by $-\Delta$ its generator (so that $T_t = e^{-t\Delta}$); it is a complete dissipation. We assume that we have a dense sub $*$-algebra $\mathcal{A}$ which is an essential domain of the generator.

The associated Dirichlet form is the sesquilinear map $\mathcal{L} : \mathcal{A} \times \mathcal{A} \rightarrow A$ defined by

$$\mathcal{L}(a, b) = \frac{1}{2}[a^* \Delta(b) + \Delta(a^*)b - \Delta(a^* b)].$$

Then $\mathcal{L}$ is completely positive in the sense that

$$\forall\, n \in \mathbb{N}^*, \forall\, a_1, \ldots, a_n \in \mathcal{A}, \quad [\mathcal{L}(a_i, a_j)] \in M_n(A)^+.$$

C*-correspondence

Before giving Sauvageot's representation theorem, let us recall the definition of a correspondence in the C*-algebraic framework.

Definition

Let A and B be C*-algebras. An (A, B)-*C*-correspondence* is a right B-C*-module $\mathcal{E}$ together with a $*$-homomorphism $\pi : A \to \mathcal{L}_B(\mathcal{E})$.

The inner product $\mathcal{E} \times \mathcal{E} \to B$ will be denoted $< \cdot, \cdot >$.

GNS type construction

The positivity of the non-commutative Dirichlet form gives the following GNS representation:

Theorem (J.-L. Sauvageot, 1989)

Let $(T_t)_{t\geq 0}$ be a semi-group of CP contractions of a C-algebra A as above and let $\mathcal{L}$ be its associated Dirichlet form. Then*

1. *There exists an (A, A)-C*-correspondence $\mathcal{E}$ and a derivation $\delta : \mathcal{A} \to \mathcal{E}$ such that*
   - *for all $a, b \in \mathcal{A}$, $\mathcal{L}(a, b) =< \delta(a), \delta(b) >$;*
   - *the range of δ generates $\mathcal{E}$ as a C*-module.*
2. *If $(\mathcal{E}', \delta')$ is another pair satisfying the same properties, there exists a C*-correspondence isomorphism $u : \mathcal{E} \to \mathcal{E}'$ such that $\delta' = u \circ \delta$.*

Classical example: the heat equation semi-group

Here the C*-algebra is $A = C_0(M)$, where (M, g) is a complete Riemannian manifold, $T_t = \exp(-t\Delta)$ is the heat equation semi-group and Δ is the Laplacian. The Dirichlet form is

$$\mathcal{L}(f,g) = \frac{1}{2}[\overline{f}\Delta(g) + \Delta(\overline{f})g - \Delta(\overline{f}g)].$$

Let us introduce the complex tangent bundle $T_{\mathbb{C}}M$, the corresponding Hilbert C*-module $C_0(M, T_{\mathbb{C}}M)$ over $C_0(M)$, where the inner product is given by $<\xi,\eta>(x) = g_x(\overline{\xi(x)},\eta(x))$, and the gradient

$$\nabla : C_c^\infty(M) \to C_0(M, T_{\mathbb{C}}M).$$

It is a derivation and we have

$$\mathcal{L}(f,g) = <\nabla f, \nabla g> .$$

Thus this is the derivation given by the Sauvageot construction. One retrieves the tangent bundle and the gradient from the laplacian only. This is why Sauvageot calls $\mathcal{E}$ the tangent bimodule in the general case.

Groupoid example: conditionally negative type functions

I want to present another example of Sauvageot's construction. Instead of the commutative C*-algebra $C_0(M)$, we shall consider a groupoid C*-algebra $C^*(G)$. Instead of the Laplacian Δ, we shall consider the operator of pointwise multiplication by a conditionally negative type function ψ on G.

Definition

Let G be a groupoid. A function $\psi : G \to \mathbb{R}$ is said CNT (conditionally of negative type) if

1. $\forall x \in G^{(0)}, \psi(x) = 0$,
2. $\forall \gamma \in G, \psi(\gamma^{-1}) = \psi(\gamma)$,
3. $\forall n \in \mathbb{N}^*$, $\forall \zeta_1, \ldots, \zeta_n \in \mathbb{R}$ such that $\sum_1^n \zeta_i = 0$,

$$\forall x \in G^{(0)}, \quad \forall \gamma_1, \ldots, \gamma_n \in G^x, \quad \sum_{i,j} \psi(\gamma_i^{-1}\gamma_j)\zeta_i\zeta_j \leq 0.$$

CNT functions define dissipations

Proposition

Let (G, λ) be a locally compact groupoid with Haar system and let $\psi : G \to \mathbb{R}$ be a continuous CNT function. Then,

1. *Pointwise multiplication by $e^{-t\psi}$, where $t \geq 0$ defines a completely positive contraction $T_t : C^*(G) \to C^*(G)$ and $(T_t)_{t\geq 0}$ is a strongly continuous semi-group of completely positive contractions of $C^*(G)$.*
2. *Its generator $-\Delta$ has the dense sub-$*$algebra $\mathcal{A} = C_c(G)$ as essential domain and $\Delta f = \psi f$.*

Exercise. Compute the Sauvageot's derivation associated with this semi-group.

CNT functions admit cocycle representations

Proposition (J.-L. Tu, 1999; R 2012)

Let (G, λ) *be a locally compact groupoid with Haar system. Let* $\psi : G \to \mathbb{R}$ *be a continuous function conditionally of negative type. Then*

1. *There exists a pair* (E, c) *consisting of a continuous* G*-Hilbert bundle* E *and a continuous cocycle* $c : G \to r^*E$ *such that*
   - *for all* $\gamma \in G$, $\psi(\gamma) = \|c(\gamma)\|^2$;
   - *for all* $x \in G^{(0)}$, $\{c(\gamma), \gamma \in G^x\}$ *is total in* E_x.
2. *If* (E', c') *is another pair satisfying the same properties, there exists a* G*-equivariant isometric continuous bundle map* $u : E \to E'$ *such that* $c' = u \circ c$.

What is a G-Hilbert bundle?

Definition

- A bundle consists of topological spaces E and X and a continuous, open and surjective map $\pi : E \to X$.
- A bundle $\pi : E \to X$ is a Hilbert bundle if each fiber $E_x = \pi^{-1}(x)$ is a Hilbert space and
  1. addition, scalar multiplication and norm are continuous;
  2. if (u_i) is a net in E such that $\|u_i\| \to 0$ and $\pi(u_i) \to x$, then $u_i \to 0_x$.
- Let G be a topological groupoid with $G^{(0)} = X$. A G-Hilbert bundle is a Hilbert bundle $\pi : E \to X$ endowed with a continuous, linear and isometric action of G.

Linear and isometric means that for all $\gamma \in G$, $L(\gamma) : E_{s(\gamma)} \to E_{r(\gamma)}$ is a linear isometry.

What is a vector-valued cocycle?

Definition

Let $E \to G^{(0)}$ be a G-Hilbert bundle. A cocycle for E is a continuous section $c : G \to r^*E$ (this means that $c(\gamma) \in E_{r(\gamma)}$) such that

$$c(\gamma\gamma') = c(\gamma) + L(\gamma)c(\gamma').$$

The crossed-product C*-correspondence

Given a G-Hilbert bundle E, we let $C_c(G, r^*E)$ be the space of compactly supported continuous sections of r^*E.
We define the left and right actions: for $f, g \in C_c(G)$ and $\xi \in C_c(G, r^*E)$,

$$f\xi(\gamma) = \int f(\gamma')L(\gamma')\xi({\gamma'}^{-1}\gamma)d\lambda^{r(\gamma)}(\gamma')$$

$$\xi g(\gamma) = \int \xi(\gamma\gamma')g({\gamma'}^{-1})d\lambda^{s(\gamma)}(\gamma')$$

and the right inner product: for $\xi, \eta \in C_c(G)$,

$$< \xi, \eta > (\gamma) = \int (\xi({\gamma'}^{-1})|\eta({\gamma'}^{-1}\gamma))_{s(\gamma')}d\lambda^{r(\gamma)}(\gamma').$$

This can be completed into a C*-correspondence $C^*(G, r^*E)$

Construction of the derivation

Let now $c : G \to r^*E$ be a continuous cocycle. Define $\delta_c : C_c(G) \to C_c(G, r^*E)$ by

$$\delta_c(f)(\gamma) = ic(\gamma)f(\gamma).$$

Proposition

1. δ_c *is a derivation:* $\delta_c(f * g) = \delta_c(f)g + f\delta_c(g)$;
2. $\delta_c : C_c(G) \to C_r^*(G, r^*E)$ *is closable.*

Solution of the exercise

Theorem (R 2012)

Let (G, λ) be a locally compact groupoid with Haar system and let $\psi : G \to \mathbb{R}$ be a continuous CNT function. Then the derivation $(\mathcal{E}, \delta)$ associated with the semi-group (T_t) of pointwise multiplication by $e^{-t\psi}$ by Sauvageot is $(C^*(G, r^*E), \delta_c)$, where (E, c) is the cocycle representing ψ.

For the proof, one checks that the Dirichlet form of $(T_t)_{t\geq 0}$ has the desired expression:

$$\mathcal{L}(f, g) =< \delta_c(f), \delta_c(g) >$$

and that the range of δ_c generates $C^*(G, r^*E)$ as a C*-module.

Remarks

- Just as in the scalar case, δ_c is bounded iff c is bounded. It is inner if c is a continuous boundary.
- The theory of unbounded derivations with values in (M, M)-Hilbert modules, where M is a von Neumann algebra (usually equipped with a finite trace) is well developed. This is not the case for unbounded derivations with values in (A, A) C*-bimodules. The above groupoid example may be useful to develop this theory.
- It would be nice to illustrate the vector-valued case by physical examples.

The End

Thank you for your attention!