RANDOM WALK ON $\mathbb{Z}$ WITH ONE-POINT INHOMOGENEITY

A. Pellegrinotti *

1 Generalities and main result

The model that I want to consider here is a particular case of the non-homogeneous case introduced by Minlos and Zhizhina in 1994 [1]. They consider a discrete time random walk on $\mathbb{Z}^d$ with transitions probabilities

$$\mathcal{P}(y \mapsto x) = P_0(x-y) + V(x-y, y).$$

Where $P_0(u), u \in \mathbb{Z}^d$ is a symmetric probability and $V(\cdot,\cdot)$ is such that

$$P_0(u) + V(u,y) \in [0,1) \quad \forall \ y, u \in \mathbb{Z}^d$$

and

$$\sum_u V(u,y) = 0 \quad \forall \ y \in \mathbb{Z}^d$$

They also assume that $P_0(\cdot)$ and $V(\cdot,\cdot)$ are local, i.e. exist positive numbers R_0, R such that

$$P_0(u) = 0, \ if \ |u| > R_0 \ \ and \ \ V(u,\cdot) = 0 \ if \ |u| > R.$$

An important assumption is that there are no traps i.e. there is not set $Y \subset \mathbb{Z}^d$ such that a walk starting at Y cannot leave the set.

For $d = 1$, if we denote by X_t the position of the random walk at time t, under the assumptions above is proved in [1] the following asymptotics:
if $\max\{|x|, |y|\} < \epsilon\sqrt{t \log t}$ ($\epsilon > 0$,small) then

$$\mathcal{P}(X_t = x | X_0 = y) = \frac{1}{\sqrt{2\pi t}\sigma}\left[e^{-\frac{(x-y)^2}{2t\sigma^2}} + k\ sign(x) e^{-\frac{(|x|+|y|)^2}{2t\sigma^2}} + \mathcal{O}\left(\frac{1}{|x|+1}\right)\right] + o(1/\sqrt{t}),$$

where $\sigma^2 = \sum_u u^2 P_0(u)$ and k is a constant depending on the parameters of the model. The methods of the proof is purely analityc and as the authors state in the paper it would be desiderable to find a more probabilistic approach to the problem.

*Dipartimento di Matematica, Università di Roma Tre, Largo S. Leonardo Murialdo 1, 00146 Rome, Italy. Partially supported by INdAM (G.N.F.M.), M.I.U.R. and University research founds.

As a first step in this direction we consider a simplified version of the model, i.e. we consider a transition probabilities of the form

$$\mathcal{P}(y \mapsto x) = P_0(x-y) + \delta_{0,y} c(x-y),$$

where $P_0(\cdot)$ and $c(\cdot)$ verify the same hypothesis as in the general case. The only difference is that we assume exponential decay i.e. $\exists\ B>0$ such that

$$\sum_u (P_0(u) + |c(u)|) e^{B|u|} < \infty. \tag{1.1}$$

Considering the functions

$$\tilde{p}_0(\lambda) = \sum e^{i(\lambda,u)} P_0(u)$$
$$\tilde{c}(\lambda) = \sum e^{i(\lambda,u)} c(u)$$

by assumption (1.1) we have that they can be extended to a strip in the complex plane: $\lambda \mapsto \lambda + i\mu$ with $|\mu| < B$.

We also assume that $|\tilde{p}_0(\lambda)| < 1$ for $\lambda \neq 0$ i.e. the corresponding random walk is irreducible. We take for simplicity $y = 0$. Then we have the following local limit asymptotics.

Theorem *If $x = o(t^{\frac{3}{4}})$, $x \neq 0$, then the following asymptotics holds as $t \to \infty$, for some constant $\kappa > 0$,*

$$P(X_t = x | X_0 = 0) = \frac{1}{\sqrt{2\pi t}\sigma} \left[e^{-\frac{x^2}{2t\sigma^2}} + \frac{sign(x)\ b}{\sigma^2 I} e^{-\frac{x^2}{2t\sigma^2}} + e^{-\kappa|x|} \frac{\Phi(x)}{I} \right] + o(1/\sqrt{t})$$

where $b = \sum_u uc(u)$, the function $\Phi(x)$ is uniformly bounded and independent of t, and

$$I = \frac{1}{2\pi} \int_T \frac{1 - \tilde{p}_0(\mu) - \tilde{c}(\mu)}{1 - \tilde{p}_0(\mu)} d\mu.$$

.

The proof of this Theorem is in [3].

In order to sketch the proof of this result we introduce quantities that can be introduced also in a more general setup.

2 Preliminary constructions

In this section we describe our basic formulas for the proof of Theorem . We consider the general case of dimension $d \geq 1$, as the basic constructions do not depend on d. We only assume that the decay is fast enough for the Fourier transforms to exist.

Consider an inhomogeneous random walk on $\mathbb{Z}^d$ with probabilities $P(x \to y)$, let

$$P^{(t)}(x|y) = P(X_t = x | X_0 = y), \qquad P^{(0)}(x|y) = \delta_{x,y},$$

and denote by $f_t(x|y)$, for $t = 1, 2, \ldots$, the probability that the random walk goes from y to x without going through the origin

$$f_t(x|y) = \sum_{y_1 \neq 0} P(y \to y_1) \sum_{y_2 \neq 0} P(y_1 \to y_2) \ldots \sum_{y_{t-1} \neq 0} P(y_{t-2} \to y_{t-1}) P(y_{t-1} \to x). \qquad (2.1)$$

There is (as is well know) a relation between $P^{(t)}(x|y)$ and $f_t(x|y)$ i.e.

$$P^{(t)}(x|y) = f_t(x|y) + \sum_{k=1}^{t-1} P^{(k)}(0|y) f_{t-k}(x|0), \qquad t > 1$$

and

$$f_1(x|y) = P(y \to x) = P^{(1)}(x|y).$$

For $y = 0$ we set for brevity $P^{(t)}(x|0) = P^{(t)}(x), f_t(x|0) = f_t(x)$. We introduce the generating functions

$$H(z;x) = \sum_{t=0}^{\infty} P^{(t)}(x) z^t, \qquad F(z;x) = \sum_{t=1}^{\infty} f_t(x) z^t. \qquad (2.2)$$

Now it is easy to show that

$$H(z;x) = \delta_{x,0} + H(z;0) F(z;x). \qquad (2.3)$$

Inverting the relation (2.2), using (2.3), we have ($t > 0$)

$$P^{(t)}(x) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z^{t+1}} H(z;x) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z^{t+1}} H(z;0) F(z;x),$$

where γ is an anticlockwise loop which contains the origin and does not contain the singularities of $H(z;x)$.

In order to make the asymptotics analysis we take the Fourier transforms:

$$\tilde{H}(z;\lambda) = \sum_{x \in \mathbb{Z}^d} H(z;x) e^{i(\lambda,x)}, \qquad \tilde{F}(z;\lambda) = \sum_{x \in \mathbb{Z}^d} F(z;x) e^{i(\lambda,x)},$$

and using the previous relations we get

$$\tilde{H}(z;\lambda) = 1 + H(z;0) \tilde{F}(z;\lambda)$$

and from this we have

$$\tilde{F}(z;\lambda) = \frac{\tilde{H}(z;\lambda) - 1}{H(z;0)}, \qquad \tilde{H}(z;\lambda) = 1 + \frac{\tilde{F}(z;\lambda)}{1 - F(z;0)}.$$

It is possible to express these quantities in terms of two integrals:

$$J_0(z) = \int_{T^d} \frac{dm(\mu)}{1 - z\tilde{p}_0(\mu)}, \qquad J(z) = \int_{T^d} \frac{\tilde{c}(\mu)\, dm(\mu)}{1 - z\tilde{p}_0(\mu)}, \qquad (2.4)$$

where $dm(\lambda) = \frac{d\lambda}{(2\pi)^d}$ denotes the normalized Haar measure on T^d.The expressions (2.4) define analytic functions for $z \in \mathbb{C} \setminus \mathcal{C}$, where $\mathcal{C} = \{z = \frac{1}{\tilde{p}_0(\mu)} : \mu \in T^d\}$ is the cut on the real axis. Setting $q = \min \tilde{p}_0(\mu)$, then $\mathcal{C} = [1, 1/q]$, if $q \geq 0$ ($[1, +\infty)$ if $q = 0$), and $\mathcal{C} = [1, \infty) \cup (-\infty, 1/q]$ if $q < 0$.

In what follows the superscript (0) denotes quantities related to the homogeneous random walk P_0, such as $f_t^{(0)}$, $H^{(0)}(z;x)$, $F^{(0)}(z;x)$.

Lemma 2.1 *For the random walk with transition probability*

$$\mathcal{P}(y \mapsto x) = P_0(x-y) + \delta_{0,y} c(x-y)$$

we have, for $z \in \mathbb{C} \setminus \mathcal{C}$,

$$\tilde{H}(z;\lambda) = \tilde{H}^{(0)}(z;\lambda) + \tilde{H}^{(1)}(z;\lambda), \qquad \tilde{F}(z;\lambda) = \tilde{F}^{(0)}(z;\lambda) + \tilde{F}^{(1)}(z;\lambda)$$

where, setting $I(z) = 1 - zJ(z)$,

$$\tilde{H}^{(0)}(z;\lambda) = \frac{1}{1 - z\tilde{p}_0(\lambda)}, \qquad \tilde{H}^{(1)}(z;\lambda) = \frac{z\,\tilde{c}(\lambda)}{1 - z\tilde{p}_0(\lambda)} \frac{J_0(z)}{I(z)}$$

$$\tilde{F}^{(0)}(z;\lambda) = \frac{z\,\tilde{p}_0(\lambda)}{(1 - z\tilde{p}_0(\lambda))J_0(z)} \qquad \tilde{F}^{(1)}(z;\lambda) = \frac{z}{1 - z\tilde{p}_0(\lambda)} \left[\tilde{c}(\lambda) - z\tilde{p}_0(\lambda)\frac{J(z)}{J_0(z)}\right].$$

The proof of Lemma 2.1 is a direct computation.

Now using the Cauchy integral formula we get:

$$P^{(t)}(x) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z^{t+1}} H(z;x) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z^{t+1}} \int_{T^d} e^{-i(\lambda,x)} \tilde{H}(z;\lambda) dm(\lambda). \qquad (2.5)$$

We have that the last integral in (2.5) is written in the following way:

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z^{t+1}} \int_{T^d} e^{-i(\lambda,x)} \tilde{H}^{(0)}(z;\lambda) dm(\lambda) + \frac{1}{2\pi i}\int_\gamma \frac{dz}{z^{t+1}} \int_{T^d} e^{-i(\lambda,x)} \tilde{H}^{(1)}(z;\lambda) dm(\lambda).$$

To prove our theorem we need to study the quantity:

$$R_t(x) \equiv \frac{1}{2\pi i}\int_\gamma \frac{dz}{z^{t+1}} \int_{T^d} e^{-i(\lambda,x)} \tilde{H}^{(1)}(z;\lambda) dm(\lambda),$$

where γ is an anticlockwise loop which contains the origin and does not contain the singularities of $\tilde{H}^{(1)}(z;\lambda)$.

Remark

All the mathematical objects introduced until now do not depend on the dimension. From now on we will restrict to the case $d = 1$ and to fix the notation we assume $q > 0$.

3 Main estimates

The main object of our analysis is the quantity:

$$R_t(x) \equiv \frac{1}{2\pi i}\int_\gamma \frac{dz}{z^{t+1}}\int_T e^{-i\lambda x}\tilde{H}^{(1)}(z;\lambda)dm(\lambda), \tag{3.1}$$

where T is the 1 dimensional torus. We take the loop γ in (3.1) as made of two parts, a circle of radius $1+\delta$, and a small loop γ_1 which closes at $1+\delta$ and goes around the point $z=1$ clockwise. The main contribution to $R_t(x)$ is coming from the integral on γ_1. We perform the change of variables $\beta = \frac{z-1}{z}$ and we get

$$R_t(x) = \frac{1}{2\pi i}\int_{\gamma_*} d\beta(1-\beta)^t D(\beta)\int_T \frac{\tilde{c}(\lambda)e^{-i\lambda x}}{1-\tilde{p}_0(\lambda)-\beta}dm(\lambda) + \mathcal{O}(e^{-\delta t}), \tag{3.2}$$

where the loop γ_* is the image of γ_1, and goes around the origin clockwise, and

$$D(\beta) = \frac{\int_T \frac{dm(\mu)}{1-\tilde{p}_0(\mu)-\beta}}{1-\int_T \frac{\tilde{c}(\mu)dm(\mu)}{1-\tilde{p}_0(\mu)-\beta}}. \tag{3.3}$$

The expressions (3.2), (3.3) define analytic functions outside the cut $\tilde{\mathcal{C}} = [0, 1-q]$, where $q = \min \tilde{p}_0(\lambda)$.

There is a neighborhood $\mathcal{U}$ of the origin such that for $\beta \in \mathcal{U} \setminus \tilde{\mathcal{C}}$ we have,

$$\int_T \frac{dm(\mu)}{1-\tilde{p}_0(\mu)-\beta} = h_0(\beta)(-\beta)^{-\frac{1}{2}} + H_0(\beta), \tag{3.4}$$

$$\int_T \frac{\tilde{c}(\lambda)\, dm(\lambda)}{1-\tilde{p}_0(\lambda)-\beta} = \int_T \frac{c_1(\lambda)\, dm(\lambda)}{1-\tilde{p}_0(\lambda)-\beta} = h_1(\beta)(-\beta)^{\frac{1}{2}} + H_1(\beta). \tag{3.5}$$

Here h_0, H_0, h_1, H_1 are analytic functions in $\mathcal{U}$ and the phase of $(-\beta)^{\frac{1}{2}}$ in the cut plane is determined by the condition that $(-\beta)^{\frac{1}{2}}$ is real and positive on the negative real axis. In (3.5) we take into account that $\tilde{c}(\lambda)$ is the Fourier transform of a real function $c(u)$, so that $\tilde{c}(\lambda) = c_1(\lambda) + ic_2(\lambda)$ where $c_1(\lambda)$ is even, and c_2 is odd. As $\tilde{c}(0) = 0$, for small λ we have $c_1(\lambda) = \mathcal{O}(\lambda^2)$, which gives the representation (3.5). To get (3.4) and (3.5) we use the results in [2].

Combining (3.4) and (3.5) we get

$$D(\beta) = h(\beta)(-\beta)^{-\frac{1}{2}} + H(\beta), \qquad \beta \in \mathcal{U} \setminus \tilde{\mathcal{C}},$$

where h, H are analytic in $\mathcal{U}$.

We first compute the integral

$$R_t^{(1)}(x) = \frac{1}{2\pi i}\int_{\gamma_*} (1-\beta)^t H(-\beta)d\beta \int_T \frac{\tilde{c}(\lambda)e^{-i\lambda x}}{1-\tilde{p}_0(\lambda)-\beta}dm(\lambda).$$

Because the function inside the integral is integrable w.r.t. the Lebesgue measure $d\beta dm(\lambda)$ we can change the order of integration and write

$$\int_{\gamma_*} (1-\beta)^t \frac{H(-\beta)}{1-\tilde{p}_0(\lambda)-\beta} d\beta = 2\pi i (\tilde{p}_0(\lambda))^t H(-1+\tilde{p}_0(\lambda))\ \chi(|\lambda|<\lambda_*),$$

where λ_* is related (see [3]) to the solution of the equation $1-\tilde{p}_0(\lambda)-\beta=0$.

Then we have

$$R_t^{(1)}(x) = \int_{-\lambda_*}^{\lambda_*} \tilde{c}(\lambda) e^{-i\lambda x} (\tilde{p}_0(\lambda))^t H(-1+\tilde{p}_0(\lambda)) dm(\lambda).$$

Using a result that we will report briefly in the appendix we get:

$$\left| R_t^{(1)}(x) \right| \le C_1 \frac{|x|+1}{t^{\frac{3}{2}}} e^{-\frac{x^2}{2\sigma^2 t}}.$$

The other term is

$$R_t^{(2)}(x) = \frac{1}{2\pi i} \int_{\gamma_*} d\beta\ h(-\beta)(-\beta)^{-\frac{1}{2}} (1-\beta)^t \int_T \frac{\tilde{c}(\lambda) e^{-i\lambda x}}{1-\tilde{p}_0(\lambda)-\beta} dm(\lambda).$$

In order to evaluate this term we need to take the contribution given by the cut. This gives as result that

$$R_t^{(2)}(x) = \frac{1}{\pi} \int_0^{\delta_*} \frac{dy}{\sqrt{y}} h(-y)(1-y)^t PP \int_T \frac{\tilde{c}(\lambda) e^{-i\lambda x}}{1-\tilde{p}_0(\lambda)-y} dm(\lambda).$$

For what concerns the PP we have

$$PP \int_T \frac{\tilde{c}(\lambda) e^{-i\lambda x}}{1-\tilde{p}_0(\lambda)-y} dm(\lambda) = e^{-\kappa|x|} \Psi(y;x) + \frac{\mathcal{J}(2y)}{\sqrt{2y}} G(\lambda^\sharp(y);x)$$

where $\lambda^\sharp(y) = \lambda(\sqrt{2y})$ is one of the solutions of the equation $1-\tilde{p}_0(\lambda)=\beta$ and $\mathcal{J}$ is the Jacobian related to the Morse's lemma. Moreover

$$\Psi(y;x) = \int_T \frac{e^{-i\lambda x} \tilde{c}(\lambda - i\ sign(x)\ \kappa)}{1-\tilde{p}_0(\lambda - i\ sign(x)\ \kappa) - y} dm(\lambda),$$

$$G(\lambda;x) = c_1(\lambda)\sin(\lambda|x|) + sign(x) c_2(\lambda)\cos(\lambda x). \tag{3.6}$$

where $\tilde{c}(\lambda) = c_1(\lambda) + i c_2(\lambda)$.

Now the contribution of the first term is

$$R_t^{(2;1)}(x) = \frac{e^{-\kappa|x|}}{\pi} \int_0^{\delta_*} \frac{h(-y)}{\sqrt{y}} (1-y)^t \Psi(y;x) dy.$$

Now performing the change of variables $y = \frac{u^2}{2t}$ we get

$$R_t^{(2;1)}(x) = \frac{e^{-\kappa|x|}}{\pi} \int_0^{\sqrt{2t\delta_*}} h(-\frac{u^2}{2t})(1-\frac{u^2}{2t})^t \Psi(\frac{u^2}{2t};x) \sqrt{\frac{2}{t}} du.$$

Considering the behaviour for large t we have

$$R_t^{(2;1)}(x) \sim \frac{e^{-\kappa|x|}}{\pi}\sqrt{\frac{2}{t}}h(0)\Psi(0;x)\int_0^\infty e^{-\frac{u^2}{2}}du.$$

By direct calculation we have

$$h(0) = \frac{h_0(0)}{1-H_1(0)} = \frac{1}{\sqrt{2}\sigma I}$$

and so we get

$$R_t^{(2;1)}(x) \sim \frac{e^{-\kappa|x|}}{\sqrt{2\pi t}}\frac{1}{\sigma I}\Psi(0;x)$$

The contribution of the second term is:

$$R_t^{(2;2)}(x) = \frac{1}{\sqrt{2\pi}}\int_0^{\delta_*}\mathcal{J}(2y)h(-y)(1-y)^t\frac{G(\lambda^\sharp(y);x)}{y}dy.$$

Now $G(\cdot,\cdot)$ is the sum of 2 terms (see (3.6)) . We take into account the first, which we denote as

$$S_t^{(1)}(x) = \frac{1}{\sqrt{2\pi}}\int_0^{\delta_*}\mathcal{J}(2y)h(-y)(1-y)^t\frac{c_1(\lambda^\sharp(y))\sin(\lambda^\sharp(y)|x|)}{y}dy.$$

Performing the change of variables $y=\frac{u^2(\lambda)}{2}$, where $\frac{u^2(\lambda)}{2}$ is the new variable given by the Morse's Lemma, and observing that

$$\mathcal{J}(2y)dy = u(\lambda)d\lambda,$$

we get

$$S_t^{(1)}(x) = \frac{\sqrt{2}}{\pi}\int_0^{\lambda_*}\tilde{p}_0(\lambda)^t\lambda g(\lambda)\sin(\lambda|x|)h(\tilde{p}_0(\lambda)-1)d\lambda.$$

Using a result that we will report briefly in the appendix we get:

$$\left|S_t^{(1)}(x)\right| \le C_2\frac{|x|+1}{t^{\frac{3}{2}}}e^{-\frac{x^2}{2\sigma^2 t}}.$$

For the contribution of the second term in (3.6), by the same change of variables $y\to\lambda$, setting $g_1(\lambda)=\frac{c_2(\lambda)}{u(\lambda)}$ we get

$$S_t^{(2)}(x) = sign(x)\frac{\sqrt{2}}{\pi}\int_0^{\lambda_*}(\tilde{p}_0(\lambda))^t g_1(\lambda)\cos(\lambda x)h(\tilde{p}_0(\lambda)-1)d\lambda.$$

As $c_2(\lambda)=b\lambda+\mathcal{O}(\lambda^3)$, where $b=\sum_u uc(u)$, we get $g_1(0)=\frac{b}{\sigma}$. Using a result that we will discuss in the next section, taking into account the expressions of $h(0)$, we see that the asymptotics is

$$S_t^{(2)}(x) = \frac{sign(x)}{\sqrt{2\pi\sigma^2 t}}\frac{b}{\sigma^2 I}e^{-\frac{x^2}{2\sigma^2 t}}+o(t^{-\frac{1}{2}}).$$

Setting $\Phi(x) = \Psi(0; x)$, we get the proof of the Theorem.

The case $x = 0$ is studied in the following

Corollary 1 *For $x = 0$, as $t \to \infty$, we have*

$$P(X_t = 0|X_0 = 0) \sim \frac{1}{\sqrt{2\pi t}\, \sigma I}.$$

Moreover the characteristic function $\phi(\lambda)$ of the first time of return to the origin behaves for small λ as

$$\phi(\lambda) = F(e^{i\lambda}; 0) = 1 - \sqrt{2}\, \sigma\, I\, (1 - e^{i\lambda})^{\frac{1}{2}} + \mathcal{O}(1 - e^{i\lambda}). \tag{3.7}$$

(Here $F(z; x)$ denotes the generating function defined by (2.2).)

Corollary 1 shows that the quantity I, which in terms of the probabilities can be written as

$$I = 1 - \sum_{k=0}^{\infty} (c * P_0^{(k)})(0),$$

controls the returns to the origin of the random walk.

If we consider the total time T_n of the first n returns to the origin, which is a sum of independent variables with characteristic function given by the function ϕ in (3.7), we have the following result.

Corollary 2 *The characteristic function of the random variable T_n/n^2 behaves, for $n \to \infty$ as*

$$\left(1 - \sqrt{2}\sigma\, I\, (1 - e^{i\frac{\lambda}{n^2}})^{\frac{1}{2}} + \mathcal{O}(1 - e^{i\frac{\lambda}{n^2}})\right)^n \to e^{-\sqrt{-2i\lambda}\, \sigma I(1)}.$$

Therefore the asymptotic distribution of T_n/n^2 for our inhomogeneous walk is the same as that of a homogeneous random walk with dispersion $\sigma^2 I^2$.

4 Appendix

The main tool to get the result is the study of the integral of the following kind:

$$I(\beta|f) = \int_{-\pi}^{\pi} \frac{f(\lambda)\, dm(\lambda)}{1 - \tilde{p}_0(\lambda) - \beta}$$

where $\beta \in \mathbb{C}$. In particular the integral as function of β is analityc outside the cut $\tilde{C} = [0, 1 - q]$, $q = \min \tilde{p}_0(\lambda)$. We are interested in the behaviour of $I(\beta|f)$ for small β and near the cut.

We assume (as in our model) that $f(\lambda)$ and $\tilde{p}_0(\lambda)$ can be extended by analiticity to a complex neighborhood of the torus.

In order to study $I(\beta|f)$ we must introduce the following region: let $\tau > 0$

$$R(\tau) = \{\lambda = \lambda_1 + i\lambda_2 \mid \lambda_1 \in (-\pi, \pi],\ |\lambda_2| < \tau\}$$

Then we have the following result:
Lemma 1 *There is a value* $\tau_* \in (0, B)$ *such that for* $|\beta| < \frac{\sigma^2\tau_*^2}{4}$ *the equation for the complex variable* $\lambda = \lambda_1 + i\lambda_2$

$$1 - \tilde{p}_0(\lambda) = \beta$$

has only two solutions $\lambda = \pm\lambda(\sqrt{2\beta})$ *in the region* $R(\tau_*)$.

This result means that the singular point of $I(\beta|f)|$ are only 2, if $|\beta| < \frac{\sigma^2\tau_*^2}{4}$. Now we assume τ_* fixed and δ is a positive number such that $\delta < \frac{\sigma^2\tau_*^2}{4}$.

Now we define the following integral:

$$P(\beta, \mu|f) = \int_{-\pi}^{\pi} \frac{f(\lambda + i\mu)\, dm(\lambda)}{1 - \tilde{p}_0(\lambda + i\mu) - \beta},$$

where $\mu \in \mathbb{R}$ and $|\mu| < \tau_*$ and $|\beta| < \delta$.

Obiously

$$P(\beta, 0|f) = I(\beta|f)$$

and when $\beta = y$ is real and positive we define as usual the principal part as

$$PP \int_{-\pi}^{\pi} \frac{f(\lambda)\, dm(\lambda)}{1 - \tilde{p}_0(\lambda) - y} = \lim_{\epsilon \to 0} \int_{|1-\tilde{p}_0(\lambda)-y|>\epsilon} \frac{f(\lambda)\, dm(\lambda)}{1 - \tilde{p}_0(\lambda) - y}. \tag{4.1}$$

The existence of the limit (4.1) is proved in the following lemma:
Lemma 2 *Let* f *be as above,* $0 < \kappa < \tau_*$ *and* $0 < y < \delta$. *The following assertions hold.*

i) If f *is even then the following type of Sokhotski relations for the complex torus hold*

$$\lim_{\epsilon \to 0} I(y \pm i\epsilon|f) = P(y, \pm\kappa|f) \pm i\frac{\mathcal{J}(2y)}{\sqrt{2y}} f(\lambda(\sqrt{2y})).$$

$$PP \int_{-\pi}^{\pi} \frac{f(\lambda)\, dm(\lambda)}{1 - \tilde{p}_0(\lambda) - y} = P(y, \pm\kappa|f).$$

If f *is odd, then*

$$P(y, \pm\kappa|f) \pm i\frac{\mathcal{J}(2y)}{\sqrt{2y}} f(\lambda(\sqrt{2y})) = 0.$$

$J(\cdot)$ is the Jacobian related with the application of Morse Lemma.

In the paper [3] it is proved that in a neighborhood of the origin $\mathcal{U}$ the representation

$$I(\beta|f) = h(\beta)(-\beta)^{-\frac{1}{2}} + H(\beta), \qquad \beta \in \mathcal{U} \setminus \tilde{C}$$

holds, where h, H are analytic in $\mathcal{U}$. The use of Sokhotski's formula identifies the functions h, H in terms of the even part of f. In fact we have the following
Remark *If* $f(\lambda)$ *is neither even or odd it can be decomposed as* $f(\lambda) = f^{(e)}(\lambda) + f^{(o)}(\lambda)$, *where* $f^{(e)}$ *is even and* $f^{(o)}$ *is odd. Then, for* $\beta = y \in (0, \delta)$ *we have*

$$\lim_{\epsilon \to 0} I(y \pm i\epsilon|f) = PP \int_{-\pi}^{\pi} \frac{f^{(e)}(\lambda)\, dm(\lambda)}{1 - \tilde{p}_0(\lambda) - y} \pm i\frac{\mathcal{J}(2y)}{\sqrt{2y}} f^{(e)}(\lambda(\sqrt{2y})).$$

$$PP\int_{-\pi}^{\pi}\frac{f(\lambda)\,dm(\lambda)}{1-\tilde{p}_0(\lambda)-y}=P(y,\pm\kappa|f)\pm i\frac{\mathcal{J}(2y)}{\sqrt{2y}}f^{(o)}(\lambda(\sqrt{2y})).$$

Now we consider the case $f(\lambda)=g(\lambda)e^{-i\lambda x}$ where $x\in\mathbb{Z},\ x\neq 0$ and $g(\cdot)$ is analytic in a complex neighborhood of T. We define

$$Q_g(y,\mu;x)=\int_{-\pi}^{\pi}\frac{g(\lambda+i\mu)e^{-i\lambda x}}{1-\tilde{p}_0(\lambda+i\mu)-y}dm(\lambda).$$

For $\mu\neq 0$ and small, Q_g is uniformly bounded in x, and the following relations hold.

i) If g is even, then

$$PP\int_{-\pi}^{\pi}\frac{g(\lambda)e^{-i\lambda x}\,dm(\lambda)}{1-\tilde{p}_0(\lambda)-y}=e^{-\kappa|x|}Q_g(y,-\ sign(x)\kappa;x)+$$

$$+\frac{\mathcal{J}(2y)}{\sqrt{2y}}g(\lambda(\sqrt{2y}))\sin(\lambda(\sqrt{2y})|x|).$$

ii) If g is odd, then

$$PP\int_{-\pi}^{\pi}\frac{g(\lambda)e^{-i\lambda x}\,dm(\lambda)}{1-\tilde{p}_0(\lambda)-y}=e^{-\kappa|x|}Q_g(y,-\ sign(x)\kappa;x)-$$

$$-\ isign(x)\ \frac{\mathcal{J}(2y)}{\sqrt{2y}}g(\lambda(\sqrt{2y}))\cos(\lambda(\sqrt{2y})|x|).$$

Moreover $Q_g(y,-\ sign(x)\kappa;x)$ is even in x for even g and is odd for odd g.

References

[1] Minlos, R.A.; Zhizhina, E.A. *A local limit theorem for nonhomogeneous random walk on a lattice.* (Russian) Teor. Veroyatnost. i Primenen. 39 (1994), no. 3, 513–529; translation in Theory Probab. Appl. 39 (1994), no. 3, 490–503 (1995).

[2] Boldrighini, C.; Minlos, R.A.; Pellegrinotti, A. *Random walks in quenched i.i.d. space-time random environment are always a.s. diffusive.* Probability Theory Rel. Fields, 129, n. 1, pp. 133-156 (2004)

[3] Boldrighini, C.; Pellegrinotti, A. *Random walk on $\mathbb{Z}$ with one-point inhomogeneity.* Markov processes and related fields, 18, 421-440 (2012)