Construction of point processes for classical and quantum gases

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General setting

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Infinitely divisible point processes with Lévy measure L

Definition

A point process P is infinitely divisible if for every $n \in \mathbb{N}$ there exists a point process P_n such that

$$P = \underbrace{P_n * \ldots * P_n}_{n-times}$$

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- **Classical result:** For every infinitely divisible point process P there exists a measure L on $\mathcal{M}^{\cdot\cdot}(X)$ (Lévy measure or cluster measure) such that $\int\limits_{\mathcal{M}^{\cdot\cdot}(X)} (1 - e^{-\mu(f)}) L(\mathrm{d}\,\mu) < \infty$ for $f \in F_{bc,+}(X)$ and

$$\mathcal{L}_P(f) = \int\limits_{\mathcal{M}^{\cdot\cdot}(X)} e^{-\mu(f)}\, P(\mathrm{d}\,\mu) = \exp(-\int\limits_{\mathcal{M}^{\cdot\cdot}(X)} (1 - e^{-\mu(f)})\, L(\mathrm{d}\,\mu))$$

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- We call $\Im_L = P$ the random KMM-process for L.

Infinitely divisible point processes with Lévy measure L

- L measure on $\mathcal{M}^{\cdot\cdot}(X)$ (cluster measure) such that the first moment of L is locally finite. That is

$$\int_{\mathcal{M}^{\cdot\cdot}(X)} \mu(f)\, L(\mathrm{d}\,\mu) < \infty \text{ for all } f \in F_{bc,+}(X).$$

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- **Classical result:** Let P_L Poisson point process in $\mathcal{M}^{\cdot\cdot}(X)$ then

$$\Im_L = P_L \circ \xi^{-1}$$

where

$$\xi : \begin{cases} \mathcal{M}^{\cdot\cdot}(\mathcal{M}^{\cdot\cdot}(X)) \to \mathcal{M}^{\cdot\cdot}(X) \\ \mu = \delta_{\nu_1} + \delta_{\nu_2} + \ldots \mapsto \sum_j \nu_j \end{cases}$$

Characterization of infinitely divisible P.P. via an integral equation

Theorem (Mecke, 1969)

Let P be a point process then $P = \Im_L \Leftrightarrow$

$$C_P(h) = \iint h(x,\mu)\,\mu(\mathrm{d}\,x)P(\mathrm{d}\,\mu) = \int\int\int h(x,\nu+\mu)\,C_L(\mathrm{d}\,x\,\mathrm{d}\,\nu)P(\mathrm{d}\,\mu)$$

for all $h \geq 0$ *measurable on* $X \times \mathcal{M}^{\cdot\cdot}(X)$.
For short

$$C_P = C_L \star P.$$

Point processes with a signed Lévy pseudo measure

- Let now L^+ and L^- be two Lévy measures on $\mathcal{M}_f^{\cdot\cdot}(X)$, that is

$$\int_{\mathcal{M}_f^{\cdot\cdot}(X)} (1 - e^{-\mu(f)}) L^{\pm}(\mathrm{d}\,\mu) < \infty \text{ for all } f \in F_{bc,+}(X).$$

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- **Main question:** When does exist a point process P in X such that

$$\mathcal{L}_P(f) = \exp\left(- \int_{\mathcal{M}_f^{\cdot\cdot}(X)} (1-e^{-\mu(f)})\,(L^+ - L^-)(\mathrm{d}\,\mu)\right), \quad f \in F_{bc,+}(X).$$

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- Equivalently:

$$\Im_{L^+} = \Im_{L^-} * P.$$

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- $L = L^+ - L^-$ is called **signed Lévy pseudo measure**.

Point processes with a signed Lévy pseudo measure

- Let L^+ and L^- be of the form

$$\int_{\mathcal{M}_f^{\cdot\cdot}(X)} \varphi(\mu)\, L^{\pm}(\mathrm{d}\,\mu) = \sum_{n=1}^{\infty} \frac{1}{n} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \Theta_n^{\pm}(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_n),$$

where Θ_n^+ and Θ_n^- are Radon (locally finite) measures on X^n.

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- Introduce the signed Radon measures $\Theta_n = \Theta_n^+ - \Theta_n^-$.

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Theorem

Let L^+ and L^- be measures on $\mathcal{M}_f^{\cdot\cdot}(X)$ of the above form such that

$$\int_{\mathcal{M}_f^{\cdot\cdot}(X)} (1 - e^{-\mu(f)}) L^{\pm}(\mathrm{d}\,\mu) < \infty \text{ for all } f \in F_{bc,+}(X),$$

$$\varrho_k\Big(\bigotimes_{j=1}^{k} f_j\Big) = \sum_{\sigma \in S_k} \prod_{\omega \in \sigma} \Theta_{|\omega|}\Big(\bigotimes_{j \in \omega} f_j\Big) \geq 0 \text{ for all } f_1, \ldots, f_k \in F_{bc,+}(X).$$

Then there exists a point process $\Im_L$ such that $\Im_{L^+} = \Im_{L^-} * \Im_L$.

Idea of the proof

- Let $L = L^+ - L^-$. Take $\Lambda \subset X$ bounded then

$$L_\Lambda(\varphi) = L(1_{\mathcal{M}_f^{\cdot\cdot}(\Lambda)}\,\varphi), \quad \varphi \geq 0 \text{ measurable},$$

  is the finite signed Lévy measure of

$$Q_\Lambda(\varphi) = \frac{1}{\Xi(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, \varrho_n(\mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n).$$

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- That means

$$\mathcal{L}_{Q_\Lambda}(f) = \exp\left(- \int_{\mathcal{M}_f^{\cdot\cdot}(\Lambda)} (1 - e^{-\mu(f)})\, L(\mathrm{d}\,\mu) \right).$$

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$$\mathcal{L}_{Q_\Lambda}(f) = \exp\left(- \int\limits_{\mathcal{M}_f^{\cdot\cdot}(\Lambda)} (1 - e^{-\mu(f)})\, L(\mathrm{d}\,\mu)\right).$$

- Conclude with Lévy's continuity theorem.

Characterization of point processes with a signed Lévy pseudo measure via an integral equation

Theorem

Let $|L| = L^+ + L^-$ be of first order and P a point process, then $P = \Im_L$ if and only if

$$(C_{L^-} \star P)(h) + C_P(h) = (C_{L^+} \star P)(h)$$

for all $h \geq 0$ measurable on $X \times \mathcal{M}^{\cdot\cdot}(X)$. Recall

$$(C_{L^\pm} \star P)(h) = \int \int \int h(x, \nu + \mu)\, C_{L^\pm}(\mathrm{d}\, x\, \mathrm{d}\, \nu) P(\mathrm{d}\, \mu)$$

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Corollary

For a particular class of testfunctions h we have

$$C_{Q_\Lambda}(h) \to C_{\Im_L}(h), \qquad \text{as } \Lambda \uparrow X.$$

Classical Gibbs processes

Let $X = \mathbb{R}^d$ and $\phi : X \times X \to \mathbb{R} \cup \{\infty\}$ be a stable and regular pair potential, that is

$$E_\phi(\delta_{x_1} + \ldots + \delta_{x_n}) := \sum_{1 \le i < j \le n} \phi(x_i, x_j) \ge -B\, n, \text{ for some } B \in [0, \infty),$$

$$C_\phi := \sup_{x \in X} \int \left|1 - e^{-\phi(x,y)}\right| \mathrm{d}\, y < \infty.$$

Let us denote by U_ϕ the Ursell function corresponding to ϕ, that is

$$U_\phi(x_1, \ldots, x_n) = \sum_{G \in \mathcal{C}_n} \prod_{\{i,j\} \in G} (e^{-\phi(x_i, x_j)} - 1),$$

where $\mathcal{C}_n$ denotes the set of all undirected connected graphs with n vertices.

Classical Gibbs processes

$$L_\phi(\varphi) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, U_\phi(x_1, \ldots, x_n)\, \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n.$$

Locally finiteness of first moment of $|L_\phi| = L_\phi^+ + L_\phi^-$. Let $f \in F_{bc,+}(X)$

$$\begin{aligned}
&\int \mu(f)\, |L_\phi|(\mathrm{d}\, \mu) \\
&= \int \mathrm{d}\, x\, f(x) \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!} \int_{X^{n-1}} \mathrm{d}\, x_2 \ldots \mathrm{d}\, x_n \,|U_\phi(x, x_2, \ldots, x_n)|
\end{aligned}$$

Ruelle's algebraic approach yields

$$Q_\Lambda(\varphi) = \frac{1}{\Xi(\Lambda)} \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, e^{-E_\phi(\delta_{x_1} + \ldots + \delta_{x_n})}\, \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n.$$

Classical Gibbs processes

Theorem

For sufficiently small $z \in (0, \infty)$ *and a stable and regular pair potential* ϕ, $\Im_{L_\phi}$ *exists and is a Gibbs process.*

Classical Gibbs processes

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For sufficiently small $z \in (0, \infty)$ and a stable and regular pair potential ϕ, $\Im_{L_\phi}$ exists and is a Gibbs process.

P is a *Gibbs point process* to the pair potential ϕ and activity $z \in (0, \infty)$ if it is a solution to the *Campbell equilibrium equation*

$$\iint h(x, \mu)\, \mu(\mathrm{d}\, x) P(\mathrm{d}\, \mu) = \iint h(x, \mu + \delta_x)\, e^{-E_\phi(x,\mu)}\, z\lambda(\mathrm{d}\, x)\, P(\mathrm{d}\, \mu),$$

where $h \geq 0$ on $X \times \mathcal{M}^{\cdot\cdot}(X)$ and the *conditional energy* is given by

$$E_\phi(x, \mu) = \begin{cases} \int \phi(x, y)\, \mu(\mathrm{d}\, y), & \int |\phi(x, y)|\, \mu(\mathrm{d}\, y) < \infty \\ \infty, & \text{else.} \end{cases}$$

Determinantal processes

Let $k : X \times X \to \mathbb{R}$ be a bounded and non negative definite kernel, that is for any $z_1, \dots, z_n \in \mathbb{R}$ and $x_1, \dots, x_n \in X$

$$\sum_{i,j=1}^{n} z_i k(x_i, x_j) z_j \geq 0.$$

Additionally assume

$$\gamma_k = \sup_{y \in X} \int |k(x,y)| \, \mathrm{d}\, x < 1.$$

$$L_k(\varphi) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n}) \, k(x_1, x_2) \ldots k(x_n, x_1) \, \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n.$$

Determinantal processes

Locally finiteness of first moment of $|L_k|$. Let $f \in F_{bc,+}(X)$

$$\int \mu(f)\,|L_k|(\mathrm{d}\,\mu)$$

$$= \sum_{n=1}^{\infty} \int_{X^n} f(x_1)\,|k(x_1,x_2)||k(x_2,x_3)|\dots|k(x_n,x_1)|\,\mathrm{d}\,x_1\dots\mathrm{d}\,x_n$$

$$\leq \|k\|_\infty\,\lambda(f) \sum_{n=1}^{\infty} \gamma_k^{n-1}$$

Furthermore we have

$$Q_\Lambda(\varphi)$$

$$= \frac{1}{\Xi(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \varphi(\delta_{x_1} + \dots + \delta_{x_n})\,\det(\{k(x_i,x_j)\}_{i,j})\,\mathrm{d}\,x_1\dots\mathrm{d}\,x_n.$$

Determinantal processes

Theorem

Let $k : X \times X \to \mathbb{R}$ be a bounded and non negative definite kernel such that

$$\gamma_k = \sup_{y \in X} \int |k(x,y)| \, \mathrm{d}\, x < 1.$$

Then the point process $\Im_{L_k}$ corresponding to the signed Lévy pseudo measure

$$L_k(\varphi) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_{X^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n}) \, k(x_1, x_2) \ldots k(x_n, x_1) \, \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n.$$

is a determinantal process with correlation kernel

$$K(x,y) = \sum_{m=1}^{\infty} (-1)^{m-1} k^{(m)}(x,y),$$

where $k^{(1)}(x,y) = k(x,y)$ and $k^{(m)}(x,y) = \int k^{(m-1)}(x,z) k(z,y) \, \mathrm{d}\, z$, $m \geq 2$.

Gibbs - determinantal processes

Is there a Gibbsian modification of determinantal processes? That is, does the thermodynamic limit of the local processes

$$Q_\Lambda(\varphi) = \frac{1}{\Xi(\Lambda)} \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\int_{\Lambda^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n})\, e^{-E_\phi(\delta_{x_1}+\ldots+\delta_{x_n})} \det(\{k(x_i, x_j)\}_{i,j})\, \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n$$

as $\Lambda \uparrow X$ exist?

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as $\Lambda \uparrow X$ exist?

This will be the subject of the next talk!

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