On a problem of description of random fields

Nahapetian B. S. (Institute of Mathematics of NAS RA)
Horomyan G. T. (Russian-Armenian Slavonic University)

Random Fields

We will consider the random fields on the lattice $\mathbb{Z}^d$. Let

$$W = \{J \subset \mathbb{Z}^d : |J| < \infty\}$$

be the set of all finite subsets of the lattice and X be some finite set (the space of spins).

Let

$$X^\Lambda = \{(x_t, t \in \Lambda)\}, \Lambda \subset \mathbb{Z}^d, x_t \in X, t \in \Lambda$$

be the set of all configurations on Λ and $\mathfrak{I}^{\mathbb{Z}^d}$ be the smallest $\sigma - algebra$ generated by cylindrical sets in $\mathbb{Z}^d$.

A probability measure $\mathbb{P}$ on $(X^{\mathbb{Z}^d}, \mathfrak{I}^{\mathbb{Z}^d})$ will be called *random field.*

We will use also an equivalent definition of random fields as a system of finite-dimensional probability distributions consistent in Kolmogorov sense:

$$\mathbb{P} = \{P_\Lambda, \Lambda \in W\}, \qquad (P_\Lambda)_I(x) = P_I(x), x \in X^\Lambda,$$

$$(P_\Lambda)_I(x) = \sum_{y \in X^{\Lambda \setminus I}} P_\Lambda(x, y)$$

Gibbs Random Fields[1]

The random field $\mathbb{P}$ is called Gibbsian if

1. $P_\Lambda(x) > 0$ for all $\Lambda \in W$ and $x \in X^\Lambda$,
2. The following limits are strictly positive and the convergence is uniform by $\bar{x}$

$$\lim_{\Lambda \uparrow \mathbb{Z}^d \{t\}} \frac{P_{\{t\} \cup \Lambda}(x, \bar{x}_\Lambda)}{P_\Lambda(\bar{x}_\Lambda)}, t \in \mathbb{Z}^d, x \in X, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$$

Theorem (representation)

- *If $\mathbb{P}$ is a Gibbs field, then the canonical (one-point) specification allows Gibbsian representation by uniformly converging potential. If random field $\mathbb{P}$ has a version of conditional distribution, which allows Gibbsian representation with uniformly converging potentials, then the field $\mathbb{P}$ is Gibbsian.*

If the field $\mathbb{P}$ is Gibbsian, then the above limits define some system of one-point distributions $Q^{(1)}$, which is the unique quasilocal one-point specification (canonical specification).

The set of Gibbs random fields $\mathcal{G}$ is not empty. From its definition it follows that $\mathcal{G}$ contains the set $\mathcal{M}$ of strictly positive Markov random fields.

From the other hand, not all strictly positive random fields are Gibbsian.

Example 1

Let $X = \{0,1\}$. Consider the random field $\mathbb{P}$ with finite-dimensional distribution

$$P_\Lambda(x) = \frac{1}{(|\Lambda| + 1)C_{|\Lambda|}^{|x|}}, \qquad \Lambda \in W, x \in X^\Lambda,$$

where $|x| = |\{t \in \Lambda : x_t = 1\}|$.

For all $t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$ and $\Lambda \in W(\mathbb{Z}^d \setminus \{t\})$ the one-point conditional distributions will have the following form

$$q_t^{\bar{x}_\Lambda}(1) = \frac{P_{t \cup \Lambda}(1\bar{x}_\Lambda)}{P_\Lambda(\bar{x}_\Lambda)} = \frac{|\bar{x}_\Lambda| + 1}{|\Lambda| + 2}.$$

Example 2

Let $\alpha, p_1, p_2 \in (0,1), p_1 \neq p_2$. Consider the random field $\mathbb{P}$, which is the composition of Bernulian random fields B^{p_1} and B^{p_2} with coefficients α and $\beta = 1 - \alpha$:

$$P_\Lambda(x) = \alpha p_1^{|x|}(1-p_1)^{|\Lambda|-|x|} + \beta p_2^{|x|}(1-p_2)^{|\Lambda|-|x|},$$
$$\Lambda \in W, x \in X^\Lambda.$$

For all $t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$ and $\Lambda \in W(\mathbb{Z}^d \setminus \{t\})$, let

$$q_t^{\bar{x}_\Lambda}(1) = \frac{P_{t\cup\Lambda}(1\bar{x}_\Lambda)}{P_\Lambda(\bar{x}_\Lambda)} = \frac{\alpha p_1 + \beta p_2 \exp\{|\Lambda| H_\Lambda(\bar{x}_\Lambda)\}}{\alpha + \beta \exp\{|\Lambda| H_\Lambda(\bar{x}_\Lambda)\}},$$

where

$$H_\Lambda(\bar{x}_\Lambda) = \frac{|\bar{x}_\Lambda|}{|\Lambda|} \ln \frac{p_2}{p_1} + \left(1 - \frac{|\bar{x}_\Lambda|}{|\Lambda|}\right) \ln \frac{1-p_2}{1-p_1}.$$

Dobrushin's Description

Let $\mathbb{P} = \{P_\Lambda, \Lambda \in W\}$ be a positive random field

$$P_\Lambda(x) > 0, x \in X^\Lambda, \Lambda \in W$$

Let $x \in X^\Lambda, y \in X^V,\ \Lambda, V \in W, \Lambda \cap V = \emptyset$.
We will introduce the conditional probabilities (distributions) on X^Λ with conditions on X^V:

$$q_\Lambda^y(x) = \frac{P_{\Lambda \cup \mathrm{V}}(x, y)}{P_V(y)}$$

Let $\bar{x} \in X^{\mathbb{Z}^d}, \Lambda_n \uparrow \mathbb{Z}^d \setminus \Lambda$. Consider the following sequence of conditional distributions

$$q_{\Lambda}^{\bar{x}_{\Lambda_n}}(x), x \in X, n = 1,2, \ldots$$

It is known that the given sequence converges always surely by probability $\mathbb{P}$:

$$\lim_{n\to\infty} q_{\Lambda}^{\bar{x}_{\Lambda_n}}(x) = q_{\Lambda}^{\bar{x}}(x), \qquad \bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}, x \in X$$

The system $Q = \{q_{\Lambda}^{\bar{x}}, \Lambda \in W, \bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}\}$ is called the conditional distribution of random field $\mathbb{P}$.

Elements of Q are distributions on X^{Λ} with fixed boundary conditions $\bar{x}$

$$q_{\Lambda}^{\bar{x}}(x) \geq 0, \qquad \sum_{x \in X^{\Lambda}} q_{\Lambda}^{\bar{x}}(x) = 1$$

and are satisfying the following condition of consistency

$$\frac{q_{\Lambda}^{\bar{x}}(x,y)}{\left(q_{\Lambda}^{\bar{x}}\right)_{\Lambda \setminus \mathrm{V}}(y)} = q_{\mathrm{V}}^{\bar{x}y}(x)$$

for all $V \in \Lambda, x \in X^{V}, y \in X^{\Lambda \setminus \mathrm{V}}, \bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}$.

$$\left(q_{\Lambda}^{\bar{x}}\right)_{\Lambda \setminus \mathrm{V}}(y) = \sum_{z \in X^{\mathrm{V}}} q_{\Lambda}^{\bar{x}}(z,y)\,, y \in X^{\Lambda \setminus \mathrm{V}}$$

Specification

A consistent system of distributions

$$Q = \{q_{\Lambda}^{\bar{x}}, \Lambda \in W, \bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}\}$$

defined everywhere and parameterized with boundary conditions is called *specification*.

What are the conditions of existence and uniqueness of a random field with conditional distribution, which almost surely coincides with?

Theorem (existence)

Suppose specification Q is such, that for all $x \in X^{\Lambda}$

$$sup\left|q_{\Lambda}^{\bar{x}}(x) - q_{\Lambda}^{\tilde{x}}(x)\right| \xrightarrow[\widetilde{\Lambda}\uparrow\mathbb{Z}^d\setminus\Lambda]{} 0$$

where sup is taken by all $\bar{x}, \tilde{x} \in X^{\mathbb{Z}^d\setminus\Lambda}$ such, that $\bar{x}_{\widetilde{\Lambda}} = \tilde{x}_{\widetilde{\Lambda}}$. Then there exists a random field whose conditional distribution almost surely coincides with Q.

Specifications satisfying the above condition are called *quasilocal*.

For all pairs of points $t, s \in \mathbb{Z}^d$, let us denote

$$\rho_{s,t} = \sup \frac{1}{2} \sum_{x \in X} |q_t^{\bar{x}}(x) - q_{\mathrm{t}}^{\tilde{x}}(x)|$$

where sup is taken by all $\bar{x}, \tilde{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$ such, that $\bar{x}_t = \tilde{x}_s, t \neq s$.

Theorem (uniqueness). *Let Q be a quasilocal specification and let the following condition hold*

$$\sum_{s \in \mathbb{Z}^d \setminus \{t\}} \rho_{s,t} \leq \alpha < 1$$

Then the field with given specification Q is unique.

Let $Q^{(1)} = \{q_{\mathrm{t}}^{\bar{x}}, \mathrm{t} \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus \{\mathrm{t}\}}\}$ be a system of one-point distributions ($q_{\mathrm{t}}^{\bar{x}}(x) > 0, \sum_{x \in X} q_{\mathrm{t}}^{\bar{x}}(x) = 1$).

1. What are the conditions of consistency for system $Q^{(1)}$ to be a subsystem of some specification?
2. Does it follow from consistent one-point subsystem properties, the corresponding properties of the whole specification (continuity, positivity, stationary, etc.)?
3. Let $\mathcal{P}_1$ be a system of random fields corresponding to specification Q, and let $\mathcal{P}_2$ be the system of random fields corresponding to consistent one-point subsystem Q_1. Will the equation $\mathcal{P}_1 = \mathcal{P}_2$ hold?

The Solution

Theorem 1 *For the system*

$$Q^{(1)} = \{q_t^{\bar{x}}, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}\}$$

to be a subsystem of some specification Q, it is necessary and sufficient that for all $t, s \in \mathbb{Z}^d$ and $x, u \in X^t$, $y, v \in X^s$ the following equation holds

$$q_t^{\bar{x}v}(x) q_s^{\bar{x}x}(y) q_t^{\bar{x}y}(u) q_s^{\bar{x}u}(v) = q_s^{\bar{x}u}(y) q_t^{\bar{x}y}(x) q_s^{\bar{x}x}(v) q_t^{\bar{x}v}(u)$$

More, the specification Q will be unique.

Theorem 2 *Let $Q^{(1)}$ be consistent and positive and let Q be a specification defined by $Q^{(1)}$. Then*

1. *Elements of Q are positive;*

2. *If $Q^{(1)}$ is quasilocal then Q is also quasilocal;*

3. *System of random fields with the given one-point specification $Q^{(1)}$ will coincide with the system of random fields with the given specification Q.*

Theorem 3 Let $Q^{(1)}$ be quasilocal and positive. Then:

1. There exists a random field $\mathbb{P}$ with the given $Q^{(1)}$;

2. $\mathbb{P}$ is unique if Dobrushin's condition holds for $Q^{(1)}$.

Specification Restoration Formula

$$q_\Lambda^{\bar{x}}(x) =$$

$$= \frac{q_{t_1}^{\bar{x}u_{\{t_2,\dots,t_n\}}}(x_{t_1})\, q_{t_2}^{\bar{x}x_{t_1}u_{\{t_3,\dots,t_n\}}}(x_{t_2}) \dots q_{t_n}^{\bar{x}x_{\{t_1,t_2,\dots,t_{n-1}\}}}(x_{t_n})}{q_{t_1}^{\bar{x}u_{\{t_2,\dots,t_n\}}}(u_{t_1})\, q_{t_2}^{\bar{x}x_{t_1}u_{\{t_3,\dots,t_n\}}}(u_{t_2}) \dots q_{t_n}^{\bar{x}x_{\{t_1,t_2,\dots,t_{n-1}\}}}(u_{t_n})}\, q_\Lambda^{\bar{x}}(u),$$

$$x = \left(x_{t_1}, x_{t_2}, \dots, x_{t_n}\right),$$

$$u = \left(u_{t_1}, u_{t_2}, \dots, u_{t_n}\right),$$

$$\Lambda = \{t_1, t_2, \dots, t_n\}, \qquad \bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}.$$

$$q_t^{\bar{x}v}(x)q_s^{\bar{x}x}(y)q_t^{\bar{x}y}(u)q_s^{\bar{x}u}(v)$$

$$= q_s^{\bar{x}u}(y)q_t^{\bar{x}y}(x)q_s^{\bar{x}x}(v)q_t^{\bar{x}v}(u)$$

Probabilistic Interpretation

Let $(\Omega, \mathcal{F}, P)$ be a probabilistic space.

$$A, B, C, D \in \mathcal{F},$$
$$P(A), P(B), P(C), P(D) > 0$$

The following equality is correct

$$P(A/B)P(B/C)P(C/D)P(D/A) = P(A/D)P(D/C)P(C/B)P(B/A).$$

Physical Interpretation

Let $\Delta_t^{\bar{x}}(x,u)$ be the energy needed for the particle in point t to change his state from x to u, under the boundary conditions $\bar{x}$.

The following equalities are natural:

1. $\Delta_t^{\bar{x}}(x,u) = \Delta_t^{\bar{x}}(x,z) + \Delta_t^{\bar{x}}(z,u)$ (condition of balance), $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$

2. $\Delta_t^{\bar{x}y}(x,u) + \Delta_s^{\bar{x}u}(y,v) = \Delta_s^{\bar{x}x}(y,v) + \Delta_t^{\bar{x}v}(x,u)$, $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t,s\}}$

$$\{H_t^{\bar{x}}(x) : x \in X, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}\}$$

$$H_t^{\bar{x}y}(x) - H_t^{\bar{x}y}(u) + H_s^{\bar{x}u}(y) - H_s^{\bar{x}u}(v)$$
$$= H_t^{\bar{x}v}(x) - H_t^{\bar{x}v}(u) + H_s^{\bar{x}x}(y) - H_s^{\bar{x}x}(v)$$

$$q = \left\{ q_t^{\bar{x}}(x) = \frac{\exp\left\{-\Delta_t^{\bar{x}}(x,\alpha)\right\}}{\sum_z \exp\left\{-\Delta_t^{\bar{x}}(z,\alpha)\right\}}, \right.$$
$$\left. x, \alpha \in X, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}} \right\}$$

$$\Delta = \left\{ \Delta_t^{\bar{x}}(x,u) = H_t^{\bar{x}}(x) - H_t^{\bar{x}}(u), \right.$$
$$\left. x, u \in X, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}} \right\}$$

$$q_t^{\bar{x}}(x) = \frac{\exp\left\{-H_t^{\bar{x}}(x)\right\}}{\sum_z \exp\left\{-H_t^{\bar{x}}(z)\right\}}$$

Algebraic Interpretation (two-dimensional case)

$$q_{t,s}(i,j) = q_t^j(i) \sum_{\alpha=1}^{N} q_{t,s}(\alpha, j)$$

$$q_{t,s}(i,j) = q_s^i(j) \sum_{\beta=1}^{N} q_{t,s}(i, \beta)$$

Denote

$$q_{t,s}(i,j) = z_{ij}, \qquad q_t^j(i) = A_{ij}, \qquad q_s^i(j) = B_{ji},$$

$$A_{ij}, B_{ji} > 0, \sum_j A_{ij} = 1, \sum_j B_{ji} = 1$$

We will have the following system of equations

$$\begin{cases} z_{ij} = A_{ij} \sum_{\alpha=1}^{N} z_{\alpha j}\,, \\ z_{ij} = B_{ji} \sum_{\beta=1}^{N} z_{i\beta}\,, \\ z_{ij} > 0,\ \sum_{i,j} z_{ij} = 1. \end{cases}$$

This system is equivalent to the following one

$$\begin{cases} z_{ij} = \dfrac{A_{ij}}{A_{kj}} z_{kj}, \\ z_{ij} = \dfrac{B_{ji}}{B_{ki}} z_{ki}, \\ z_{ij} > 0,\ \sum_{i,j} z_{ij} = 1. \end{cases}$$

Let $X = \{0,1\}$. The system will take the following form

$$\begin{cases} z_{00} = \dfrac{A_{00}}{A_{10}} z_{10}, \\ z_{10} = \dfrac{B_{10}}{B_{11}} z_{11}, \\ z_{11} = \dfrac{A_{11}}{A_{01}} z_{01}, \\ z_{01} = \dfrac{B_{01}}{B_{00}} z_{00} \end{cases}$$

$$\det\begin{pmatrix} A_{10} & 0 & -A_{00} & 0 \\ 0 & 0 & B_{11} & -B_{10} \\ 0 & -A_{11} & 0 & A_{01} \\ -B_{01} & B_{00} & 0 & 0 \end{pmatrix} = A_{00}B_{10}A_{11}B_{01} - B_{00}A_{01}B_{11}A_{10} = 0$$

Algebraic Interpretation

Given a fixed $\Lambda \in W$ and a vector $\bar{z} \in X^{Z^d \setminus \Lambda}$, we have that

$$q_\Lambda^{\bar{z}}(x,u) = q_{\mathrm{I}}^{\bar{z}u}(x) \sum_{z \in X^I} q_\Lambda^{\bar{z}}(z,u)\,,$$

for all $I \subset \Lambda$.

If we consider all $q_{\mathrm{I}}^{\bar{z}u}$ functions are known, we can restore the function $q_\Lambda^{\bar{z}}$ by solving the above system of linear equations.

Algebraic Interpretation

It is easy to see that the mentioned system can be reduced to the following functional equation

$$a_{\bar{x}} = A(\bar{x}, \bar{y}) a_{\bar{y}}, \qquad \bar{x}, \bar{y} \in X^{\Lambda} \qquad (*)$$

where

$$a_{\bar{x}} = q_{\Lambda}^{\bar{z}}(\bar{x}), \qquad A(\bar{x}, \bar{y}) = \frac{q_{\mathrm{I}}^{\bar{z}u}(x)}{q_{\mathrm{I}}^{\bar{z}u}(y)},$$

$$\bar{x} = \{x, u\}, \bar{y} = \{y, u\}, \qquad x, y \in X^{I}, u \in X^{\Lambda \setminus \mathrm{I}}$$

Notation

Let function $A^*(\bar{x}, \bar{y})$ be defined on pairs of vectors $\bar{x}, \bar{y} \in X^{\Lambda}$, which are different only in one point.

$$A^*(\bar{x}, \bar{y}) = \frac{q_{\mathrm{t}}^{\bar{z}u}(x)}{q_{\mathrm{t}}^{\bar{z}u}(y)},$$

$$\bar{x} = \{x, u\}, \bar{y} = \{y, u\}, \qquad x, y \in X^t, u \in X^{\Lambda \setminus \{\mathrm{t}\}}$$

Assertion

There exists a solution of the equation (*) if and only if for all $\bar{x}, \bar{y}, \bar{z} \in X^{\Lambda}$ *the following equality is fulfilled*

$$A(\bar{x}, \bar{y}) = A(\bar{x}, \bar{z})\, A(\bar{z}, \bar{y}) \qquad (**)$$

And the general solution has the form

$$a_{\bar{x}} = A(\bar{x}, \bar{z}_0)\, C(\bar{z}_0), \quad \bar{x} \in X^{\Lambda}, \bar{z}_0 \in X^{\Lambda}.$$

Algebraic Interpretation

In case if the following additional condition takes place

$$\sum_{\bar{x} \in X^{\Lambda}} a_{\bar{x}} = 1$$

the solution is unique and is written

$$a_{\bar{x}} = \frac{A\left(\bar{x}, \bar{z}_0\right)}{\sum_{\bar{y}} A\left(\bar{y}, \bar{z}_0\right)}, \quad \bar{x} \in X^{\Lambda}, \bar{z}_0 \in X^{\Lambda}.$$

The Gibbsian Form

Particularly, if

$$A\left(\bar{x}, \bar{z}_0\right) = \frac{e^{U(\bar{x})}}{e^{U(\bar{z}_0)}}$$

we obtain the Gibbsian form for the solution of the equation (*) [5]

$$a_{\bar{x}} = \frac{e^{U(\bar{x})}}{\sum_{\bar{y}} e^{U(\bar{y})}}, \quad \bar{x} \in X^{\Lambda}.$$

Function properties

It follows from (**) that the function $A(\bar{x}, \bar{y})$ should possess the following properties:

1. $A(\bar{x}, \bar{x}) = 1,$
2. $A(\bar{x}, \bar{y})\, A(\bar{y}, \bar{x}) = 1,$ and generally,

$$A(\bar{x}, \bar{y}_1)\, A(\bar{y}_1, \bar{y}_2)\, A(\bar{y}_{n-1}, \bar{y}_n)\, A(\bar{y}_n, \bar{x}) = 1$$

if a sequence of vectors $\bar{y}_1, \bar{y}_2, , \bar{y}_n \in X^{\Lambda}$ is given.

Theorem

The extension of function $A^*(\bar{x}, \bar{y})$ *on the space* $X^\Lambda \times X^\Lambda$ *exists if and only if for all* $t, s \in Z^d$ *the following condition is true*

$$A^*\left(\bar{x}, D_u^{(t)}\bar{x}\right) A^*\left(D_u^{(t)}\bar{x}, D_v^{(s)} D_u^{(t)}\bar{x}\right) = \\ A^*\left(\bar{x}, D_v^{(s)}\bar{x}\right) A^*\left(D_v^{(s)}\bar{x}, D_u^{(t)} D_v^{(s)}\bar{x}\right)$$

where

$$u, v \in X, \quad D_a^{(p)}\bar{x} = \left\{a, \bar{x}_{\Lambda\setminus p}\right\}, \quad p \in \Lambda, a \in X.$$

The Extension Formula

The extension will be written by the following formula

$$A(\bar{x}, \bar{y}) = A^*(\bar{x}, \bar{x}_1)A^*(\bar{x}_1, \bar{x}_2) \dots A^*(\bar{x}_n, \bar{y})$$

where $\bar{x}_i, i \in \overline{1, n}$ are such that the pairs of vectors $(\bar{x}, \bar{x}_1), (\bar{x}_1, \bar{x}_2), \dots, (\bar{x}_n, \bar{y})$ are different in only one point.

More, the extended function will remain the same regardless of the choice of sequence $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$.

The solution

Finally, the solution of (*) is

$$a_{\bar{x}} = \frac{A(\bar{x}, \bar{z}_0)}{\sum_{\bar{y}} A(\bar{y}, \bar{z}_0)} = \frac{A^*(\bar{x}, \bar{x}_1) A^*(\bar{x}_1, \bar{x}_2) \ldots A^*(\bar{x}_n, \bar{z}_0)}{\sum_{\bar{y}} A^*(\bar{y}, \bar{y}_1) A^*(\bar{y}_1, \bar{y}_2) \ldots A^*(\bar{y}_m, \bar{z}_0)}$$

References

[1] S.Dachian, B.S. Nahapetian, "On Gibbsiannes of Random Fields", Markov Processes and Related Fields, 15(1), 81-105(2009).

[2] R. L. Dobrushin, "The description of a random field by means of conditional probabilities and conditions of its regularity", (Russian) Teor. Verojatnost. i Primenen 13, 201-229 (1968).

[3] S. Yu. Dashyan, B. S. Nahapetian, "Description of random fields by means of one-point conditional distributions and some applications", Markov Process. Related Fields, 7(2), 193-214 (2001).

[4] S. Yu. Dashyan, B. S. Nahapetian, "Description of specifications by means of probability distributions in small volumes under condition of very weak positivity", J. Statist. Phys., 117 (1-2), 281-300 (2004).

[5] R. L. Dobrushin, "Gibbsian random fields for lattice systems with pairwise interactions", (Russian) Funktsional. Analiz i Ego Prilozhen. 2 No. 4, 31-43 (1968).