On a problem of description of random fields

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We will consider the random fields on the lattice $\mathbb{Z}^d$. Let

$$W = \{J \subset \mathbb{Z}^d: |J| < \infty\}$$

be the set of all finite subsets of the lattice and $X$ be some finite set (the space of spins). Let

$$X^\Lambda = \{(x_t, t \in \Lambda)\}, \Lambda \subset \mathbb{Z}^d, x_t \in X, t \in \Lambda$$

be the set of all configurations on $\Lambda$ and $\mathcal{F}_{\mathbb{Z}^d}$ be the smallest $\sigma$–algebra generated by cylindrical sets in $\mathbb{Z}^d$. 
A probability measure $\mathbb{P}$ on $(X^{\mathbb{Z}^d}, \mathcal{F}^{\mathbb{Z}^d})$ will be called \textit{random field}.

We will use also an equivalent definition of random fields as a system of finite-dimensional probability distributions consistent in Kolmogorov sense:

\[
\mathbb{P} = \{P_{\Lambda}, \Lambda \in W\}, \quad (P_{\Lambda})_I(x) = P_I(x), x \in X^\Lambda, \quad (P_{\Lambda})_I(x) = \sum_{y \in X^\Lambda \setminus I} P_{\Lambda}(x, y)
\]
The random field $\mathbb{P}$ is called Gibbsian if

1. $P_\Lambda(x) > 0$ for all $\Lambda \in \mathcal{W}$ and $x \in X^\Lambda$,

2. The following limits are strictly positive and the convergence is uniform by $\bar{x}$

$$\lim_{\Lambda \uparrow \mathbb{Z}^d \{t\}} \frac{P\{t\cup\Lambda(x,\bar{x}_\Lambda)\}}{P_\Lambda(\bar{x}_\Lambda)}, t \in \mathbb{Z}^d, x \in X, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$$
Theorem (representation)

- If $\mathbb{P}$ is a Gibbs field, then the canonical (one-point) specification allows Gibbsian representation by uniformly converging potential. If random field $\mathbb{P}$ has a version of conditional distribution, which allows Gibbsian representation with uniformly converging potentials, then the field $\mathbb{P}$ is Gibbsian.
If the field $\mathbb{P}$ is Gibbsian, then the above limits define some system of one-point distributions $Q^{(1)}$, which is the unique quasilocal one-point specification (canonical specification).

The set of Gibbs random fields $\mathcal{G}$ is not empty. From its definition it follows that $\mathcal{G}$ contains the set $\mathcal{M}$ of strictly positive Markov random fields.

From the other hand, not all strictly positive random fields are Gibbsian.
Example 1

Let $X = \{0,1\}$. Consider the random field $\mathbb{P}$ with finite-dimensional distribution

$$P_\Lambda(x) = \frac{1}{(|\Lambda| + 1)C_{|\Lambda|}^{|x|}}, \quad \Lambda \in W, x \in X^\Lambda,$$

where $|x| = |\{t \in \Lambda: x_t = 1\}|$.

For all $t \in \mathbb{Z}^d$, $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$ and $\Lambda \in W (\mathbb{Z}^d \setminus \{t\})$ the one-point conditional distributions will have the following form

$$q_t^{\bar{x}_\Lambda}(1) = \frac{P_{t \cup \Lambda}(1\bar{x}_\Lambda)}{P_\Lambda(\bar{x}_\Lambda)} = \frac{|\bar{x}_\Lambda| + 1}{|\Lambda| + 2}.$$
Let $\alpha, p_1, p_2 \in (0,1), p_1 \neq p_2$. Consider the random field $\mathbb{P}$, which is the composition of Bernulian random fields $B^{p_1}$ and $B^{p_2}$ with coefficients $\alpha$ and $\beta = 1 - \alpha$:

$$P_{\Lambda}(x) = \alpha p_1^{|x|}(1 - p_1)^{|\Lambda| - |x|} + \beta p_2^{|x|}(1 - p_2)^{|\Lambda| - |x|},$$

$\Lambda \in W, x \in X^\Lambda$.

For all $t \in \mathbb{Z}^d, \bar{x} \in X^\mathbb{Z}^d \setminus \{t\}$ and $\Lambda \in W(\mathbb{Z}^d \setminus \{t\})$, let

$$q_t(1) = \frac{P_{t \cup \Lambda}(1 \bar{x}_\Lambda)}{P_{\Lambda}(\bar{x}_\Lambda)} = \frac{\alpha p_1 + \beta p_2 \exp\{(|\Lambda|H_{\Lambda}(\bar{x}_\Lambda))\}}{\alpha + \beta \exp\{(|\Lambda|H_{\Lambda}(\bar{x}_\Lambda))\}},$$

where

$$H_{\Lambda}(\bar{x}_\Lambda) = \frac{|\bar{x}_\Lambda|}{|\Lambda|} \ln \frac{p_2}{p_1} + \left(1 - \frac{|\bar{x}_\Lambda|}{|\Lambda|}\right) \ln \frac{1 - p_2}{1 - p_1}.$$
Let $\mathbb{P} = \{P_{\Lambda}, \Lambda \in W\}$ be a positive random field
$P_{\Lambda}(x) > 0, \ x \in X^{\Lambda}, \Lambda \in W$

Let $x \in X^{\Lambda}, y \in X^{V}, \ \Lambda, V \in W, \Lambda \cap V = \emptyset$. We will introduce the conditional probabilities (distributions) on $X^{\Lambda}$ with conditions on $X^{V}$:

$$q_{\Lambda}^{y}(x) = \frac{P_{\Lambda \cup V}(x, y)}{P_{V}(y)}$$
Let $\bar{x} \in X^{\mathbb{Z}^d}, \Lambda_n \uparrow \mathbb{Z}^d \setminus \Lambda$. Consider the following sequence of conditional distributions

$$q_{\Lambda}^{\bar{x}\Lambda_n}(x), x \in X, n = 1,2, \ldots$$

It is known that the given sequence converges always surely by probability $\mathbb{P}$:

$$\lim_{n \to \infty} q_{\Lambda}^{\bar{x}\Lambda_n}(x) = q_{\Lambda}^{\bar{x}}(x), \quad \bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}, x \in X$$

The system $Q = \{q_{\Lambda}^{\bar{x}}, \Lambda \in W, \bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}\}$ is called the conditional distribution of random field $\mathbb{P}$. 
Elements of $Q$ are distributions on $X^\Lambda$ with fixed boundary conditions $\bar{x}$

$$q_\Lambda^\bar{x}(x) \geq 0, \quad \sum_{x \in X^\Lambda} q_\Lambda^\bar{x}(x) = 1$$

and are satisfying the following condition of consistency

$$\frac{q_\Lambda^\bar{x}(x, y)}{(q_\Lambda^\bar{x})_{\Lambda \setminus V}(y)} = q_V^\bar{x} y(x)$$

for all $V \in \Lambda, x \in X^V, y \in X^{\Lambda \setminus V}, \bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}$.

$$(q_\Lambda^\bar{x})_{\Lambda \setminus V}(y) = \sum_{z \in X^V} q_\Lambda^\bar{x}(z, y), \quad y \in X^{\Lambda \setminus V}$$
A consistent system of distributions

\[ Q = \{q^\bar{x}_\Lambda, \Lambda \in W, \bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}\} \]

defined everywhere and parameterized with boundary conditions is called *specification*.

What are the conditions of existence and uniqueness of a random field with conditional distribution, which almost surely coincides with?
Suppose specification $Q$ is such, that for all $x \in X^\Lambda$ 
\[ \sup \left| q^\Lambda(x) - q^\tilde{\Lambda}(x) \right| \xrightarrow{\tilde{\Lambda} \uparrow \mathbb{Z}^d \setminus \Lambda} 0 \]
where sup is taken by all $\tilde{x}, \tilde{x} \in X^\mathbb{Z}^d \setminus \Lambda$ such, that $\tilde{x}^\Lambda = \tilde{x}^\Lambda$. Then there exists a random field whose conditional distribution almost surely coincides with $Q$.

Specifications satisfying the above condition are called quasilocal.
For all pairs of points $t, s \in \mathbb{Z}^d$, let us denote

$$\rho_{s,t} = \sup \frac{1}{2} \sum_{x \in X} |q_t^x(x) - q_t^{\bar{x}}(x)|$$

where sup is taken by all $\bar{x}, \tilde{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$ such, that $\bar{x}_t = \tilde{x}_s, t \neq s$.

Theorem (uniqueness). Let $Q$ be a quasilocal specification and let the following condition hold

$$\sum_{s \in \mathbb{Z}^d \setminus \{t\}} \rho_{s,t} \leq \alpha < 1$$

Then the field with given specification $Q$ is unique.
Let $Q^{(1)} = \{q_t^{\bar{x}}, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}\}$ be a system of one-point distributions ($q_t^{\bar{x}}(x) > 0, \sum_{x \in X} q_t^{\bar{x}}(x) = 1$).

1. What are the conditions of consistency for system $Q^{(1)}$ to be a subsystem of some specification?
2. Does it follow from consistent one-point subsystem properties, the corresponding properties of the whole specification (continuity, positivity, stationary, etc.)?
3. Let $P_1$ be a system of random fields corresponding to specification $Q$, and let $P_2$ be the system of random fields corresponding to consistent one-point subsystem $Q_1$. Will the equation $P_1 = P_2$ hold?
The Solution

Theorem 1  For the system

\[ Q^{(1)} = \{ q_t^\bar{x}, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d\setminus\{t\}} \} \]

to be a subsystem of some specification \( Q \), it is necessary and sufficient that for all \( t, s \in \mathbb{Z}^d \) and \( x, u \in X^t, y, v \in X^s \) the following equation holds

\[
q_t^\bar{x}v(x)q_s^\bar{x}x(y)q_t^\bar{y}u(x)q_s^\bar{u}v(y) = q_s^\bar{u}y(x)q_t^\bar{y}x(u)q_s^\bar{x}v(u)q_t^\bar{x}x(v)
\]

More, the specification \( Q \) will be unique.
Theorem 2  Let $Q^{(1)}$ be consistent and positive and let $Q$ be a specification defined by $Q^{(1)}$. Then
1. Elements of $Q$ are positive;
2. If $Q^{(1)}$ is quasilocal then $Q$ is also quasilocal;
3. System of random fields with the given one-point specification $Q^{(1)}$ will coincide with the system of random fields with the given specification $Q$. 
Theorem 3  Let $Q^{(1)}$ be quasilocal and positive. Then:

1. There exists a random field $\mathbb{P}$ with the given $Q^{(1)}$;

2. $\mathbb{P}$ is unique if Dobrushin’s condition holds for $Q^{(1)}$. 
$q_\bar{x}(x) = q_\bar{x}(u) = \frac{q_{t_1} x_{u_{t_2} \ldots u_{t_n}}(x_{t_1}) q_{t_2} x_{u_{t_3} \ldots u_{t_n}}(x_{t_2}) \ldots q_{t_n} x_{u_{t_1} \ldots u_{t_{n-1}}}(x_{t_n})}{q_{t_1} u_{t_1} q_{t_2} u_{t_2} \ldots q_{t_n} u_{t_n}} q_\Lambda(u)$,

$x = (x_{t_1}, x_{t_2}, \ldots, x_{t_n})$,

$u = (u_{t_1}, u_{t_2}, \ldots, u_{t_n})$,

$\Lambda = \{t_1, t_2, \ldots, t_n\}$, $\bar{x} \in X^{\mathbb{Z}^d \setminus \Lambda}$. 
\[ q_t^{\bar{x}v}(x)q_s^{\bar{x}x}(y)q_t^{\bar{x}y}(u)q_s^{\bar{x}u}(v) \]

\[ = q_s^{\bar{x}u}(y)q_t^{\bar{x}y}(x)q_s^{\bar{x}x}(v)q_t^{\bar{x}v}(u) \]
Let \((\Omega, \mathcal{F}, P)\) be a probabilistic space. 

\[ A, B, C, D \in \mathcal{F}, \]
\[ P(A), P(B), P(C), P(D) > 0 \]

The following equality is correct

\[
P(A/B)P(B/C)P(C/D)P(D/A)
= P(A/D)P(D/C)P(C/B)P(B/A).
\]
Physical Interpretation

Let $\Delta^{\bar{x}}_t (x, u)$ be the energy needed for the particle in point $t$ to change his state from $x$ to $u$, under the boundary conditions $\bar{x}$.

The following equalities are natural:

1. $\Delta^{\bar{x}}_t (x, u) = \Delta^{\bar{x}}_t (x, z) + \Delta^{\bar{x}}_t (z, u)$ (condition of balance), $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}}$

2. $\Delta^{\bar{x}y}_t (x, u) + \Delta^{\bar{x}u}_s (y, v) = \Delta^{\bar{x}x}_s (y, v) + \Delta^{\bar{x}v}_t (x, u)$, $\bar{x} \in X^{\mathbb{Z}^d \setminus \{t, s\}}$
\{H_{t}^{\bar{x}}(x) : x \in X, t \in \mathbb{Z}^d, \bar{x} \in X^{\mathbb{Z}^d \setminus \{t\}} \}

H_{t}^{\bar{x}y}(x) - H_{t}^{\bar{x}y}(u) + H_{s}^{\bar{x}u}(y) - H_{s}^{\bar{x}u}(v)
= H_{t}^{\bar{x}v}(x) - H_{t}^{\bar{x}v}(u) + H_{s}^{\bar{x}x}(y) - H_{s}^{\bar{x}x}(v)
\[ q = \left\{ q_t^x(x) = \frac{\exp\{-\Delta_t^x(x, \alpha)\}}{\sum_z \exp\{-\Delta_t^x(z, \alpha)\}} \right\}, \]

\[ x, \alpha \in X, t \in \mathbb{Z}^d, x^t, u^t \in X^{\mathbb{Z}^d \setminus \{t\}} \]

\[ \Delta = \left\{ \Delta_t^x(x, u) = H_t^x(x) - H_t^x(u), \right\} \]

\[ x, u \in X, t \in \mathbb{Z}^d, x^t, u^t \in X^{\mathbb{Z}^d \setminus \{t\}} \]

\[ q_t^x(x) = \frac{\exp\{-H_t^x(x)\}}{\sum_z \exp\{-H_t^x(z)\}} \]
Algebraic Interpretation (two-dimensional case)

\[ q_{t,s}(i,j) = q^j_t(i) \sum_{\alpha=1}^{N} q_{t,s}(\alpha,j) \]

\[ q_{t,s}(i,j) = q^i_s(j) \sum_{\beta=1}^{N} q_{t,s}(i,\beta) \]

Denote

\[ q_{t,s}(i,j) = z_{ij}, \quad q^j_t(i) = A_{ij}, \quad q^i_s(j) = B_{ji}, \]

\[ A_{ij}, B_{ji} > 0, \sum_j A_{ij} = 1, \sum_j B_{ji} = 1 \]
We will have the following system of equations

\[
\begin{align*}
  z_{ij} &= A_{ij} \sum_{\alpha=1}^{N} z_{\alpha j}, \\
  z_{ij} &= B_{ji} \sum_{\beta=1}^{N} z_{i\beta}, \\
  z_{ij} &> 0, \quad \sum_{i,j} z_{ij} = 1.
\end{align*}
\]

This system is equivalent to the following one

\[
\begin{align*}
  z_{ij} &= \frac{A_{ij}}{A_{kj}} z_{kj}, \\
  z_{ij} &= \frac{B_{ji}}{B_{ki}} z_{ki}, \\
  z_{ij} &> 0, \quad \sum_{i,j} z_{ij} = 1.
\end{align*}
\]
Let $X = \{0,1\}$. The system will take the following form

\[
\begin{align*}
\begin{cases}
  z_{00} &= \frac{A_{00}}{A_{10}} z_{10}, \\
  z_{10} &= \frac{B_{10}}{B_{11}} z_{11}, \\
  z_{11} &= \frac{A_{11}}{A_{01}} z_{01}, \\
  z_{01} &= \frac{B_{01}}{B_{00}} z_{00}
\end{cases}
\end{align*}
\]

\[
\det\begin{pmatrix}
  A_{10} & 0 & -A_{00} & 0 \\
  0 & 0 & B_{11} & -B_{10} \\
  0 & -A_{11} & 0 & A_{01} \\
  -B_{01} & B_{00} & 0 & 0
\end{pmatrix} = A_{00}B_{10}A_{11}B_{01} - B_{00}A_{01}B_{11}A_{10} = 0
\]
Given a fixed $\Lambda \in \mathcal{W}$ and a vector $\bar{z} \in X^{Z^d \setminus \Lambda}$, we have that

$$q^\bar{z}_\Lambda(x, u) = q^\bar{z}u(x) \sum_{z \in X^I} q^\bar{z}_\Lambda(z, u),$$

for all $I \subset \Lambda$.

If we consider all $q^\bar{z}u$ functions are known, we can restore the function $q^\bar{z}_\Lambda$ by solving the above system of linear equations.
Algebraic Interpretation

It is easy to see that the mentioned system can be reduced to the following functional equation

\[ a_{\bar{x}} = A(\bar{x}, \bar{y}) a_{\bar{y}}, \quad \bar{x}, \bar{y} \in X^\Lambda \quad (*) \]

where

\[ a_{\bar{x}} = q_{\Lambda}^\bar{z}(\bar{x}), \quad A(\bar{x}, \bar{y}) = \frac{q_\Lambda^\bar{z}u(x)}{q_\Lambda^{\bar{z}u}(y)}, \]

\[ \bar{x} = \{x, u\}, \bar{y} = \{y, u\}, \quad x, y \in X^I, u \in X^\Lambda \setminus I \]
Let function $A^* (\bar{x}, \bar{y})$ be defined on pairs of vectors $\bar{x}, \bar{y} \in X^\Lambda$, which are different only in one point.

$$A^* (\bar{x}, \bar{y}) = \frac{q_t^{\bar{z}u}(x)}{q_t^{\bar{z}u}(y)},$$

$\bar{x} = \{x, u\}, \bar{y} = \{y, u\}, \quad x, y \in X^t, u \in X^\Lambda \setminus \{t\}$
There exists a solution of the equation (*) if and only if for all $\bar{x}, \bar{y}, \bar{z} \in X^\Lambda$ the following equality is fulfilled

$$A (\bar{x}, \bar{y}) = A (\bar{x}, \bar{z}) A (\bar{z}, \bar{y})$$

(**)

And the general solution has the form

$$a_{\bar{x}} = A (\bar{x}, \bar{z}_0) C (\bar{z}_0), \quad \bar{x} \in X^\Lambda, \bar{z}_0 \in X^\Lambda.$$
In case if the following additional condition takes place

\[ \sum_{\bar{x} \in X^\Lambda} a_{\bar{x}} = 1 \]

the solution is unique and is written

\[ a_{\bar{x}} = \frac{A(\bar{x}, \bar{z}_0)}{\sum_{\bar{y}} A(\bar{y}, \bar{z}_0)}, \quad \bar{x} \in X^\Lambda, \bar{z}_0 \in X^\Lambda. \]
Particularly, if

\[ A(\bar{x}, \bar{z}_0) = \frac{e^{U(\bar{x})}}{e^{U(\bar{z}_0)}} \]

we obtain the Gibbsian form for the solution of the equation (*) \[5\]

\[ a_{\bar{x}} = \frac{e^{U(\bar{x})}}{\sum_{\bar{y}} e^{U(\bar{y})}}, \quad \bar{x} \in X^A. \]
Function properties

It follows from (**) that the function $A(\bar{x}, \bar{y})$ should possess the following properties:

1. $A(\bar{x}, \bar{x}) = 1$,
2. $A(\bar{x}, \bar{y}) A(\bar{y}, \bar{x}) = 1$, and generally,

$$A(\bar{x}, \bar{y}_1) A(\bar{y}_1, \bar{y}_2) A(\bar{y}_{n-1}, \bar{y}_n) A(\bar{y}_n, \bar{x}) = 1$$

if a sequence of vectors $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n \in X^\Lambda$ is given.
The extension of function $A^*(\bar{x}, \bar{y})$ on the space $X^\Lambda \times X^\Lambda$ exists if and only if for all $t, s \in \mathbb{Z}^d$ the following condition is true

$$A^* \left( \bar{x}, D_u^{(t)} \bar{x} \right) A^* \left( D_u^{(t)} \bar{x}, D_v^{(s)} D_u^{(t)} \bar{x} \right) = A^* \left( \bar{x}, D_v^{(s)} \bar{x} \right) A^* \left( D_v^{(s)} \bar{x}, D_u^{(t)} D_v^{(s)} \bar{x} \right)$$

where

$$u, v \in X, \quad D_{a}^{(p)} \bar{x} = \{a, \bar{x}_{\Lambda\setminus p}\}, \quad p \in \Lambda, a \in X.$$
The Extension Formula

The extension will be written by the following formula

\[ A(\bar{x}, \bar{y}) = A^*(\bar{x}, \bar{x}_1)A^*(\bar{x}_1, \bar{x}_2) \ldots A^*(\bar{x}_n, \bar{y}) \]

where \( \bar{x}_i, i \in 1, n \) are such that the pairs of vectors \((\bar{x}, \bar{x}_1), (\bar{x}_1, \bar{x}_2), \ldots, (\bar{x}_n, \bar{y})\) are different in only one point.

More, the extended function will remain the same regardless of the choice of sequence \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n \).
Finally, the solution of (*) is

\[ a_{\bar{x}} = \frac{A(\bar{x}, \bar{z}_0)}{\sum_{\bar{y}} A(\bar{y}, \bar{z}_0)} = \frac{A^*(\bar{x}, \bar{x}_1)A^*(\bar{x}_1, \bar{x}_2)\ldots A^*(\bar{x}_n, \bar{z}_0)}{\sum_{\bar{y}} A^*(\bar{y}, \bar{y}_1)A^*(\bar{y}_1, \bar{y}_2)\ldots A^*(\bar{y}_m, \bar{z}_0)} \]
References


