The Martingale Method in the Theory of Random Fields

Nahapetian B.S.

(Institute of Mathematics, National Academy of Science of Armenia),

Khachatryan L.A.

(Russian-Armenian (Slavonic) University)

S A M M P

Yerevan, 2012

One-dimensional case

Well-known martingale method has wide range of applications in the theory of stochastic processes, particulary in problems of convergence of sequences of random variables and in limit theorems.

Definition. A stochastic sequence of random variables $\xi_1, \xi_2, ..., \xi_k, ...$ is called ***martingale*** if for any $k \geq 0$

$$E\left|\xi_k\right| < \infty \qquad \text{and} \qquad E\left(\xi_{k+1}\Big/\sigma\left(\xi_s, 0 \leq s \leq k\right)\right) = \xi_k \text{ a.s.}$$

We see that the notion of martingale is essentially used the complete ordering property of real line. The absence of this property in multidimensional structures doesn't allow directly extend the martingale method on this case. Nevertheless several important results in this topic were obtained. These results laid down the foundations of the theory of the multidimensional martingales.

The main goal of our talk is to demonstrate how the martingale method can be applied in the theory of Gibbs random fields.

The definition of martingale in multidimensional case as a rule uses monotone sequences of subsets of some set. The choice of such sequences of subsets determines various classes of multidimensional martingales.

We will consider a sequences of finite subsets of integer lattice $\mathbb{Z}^d$. Let

$$W = \left\{ V \subset \mathbb{Z}^d, |V| < \infty \right\}$$

$\{\Im_V, V \in W\}$ — partially ordered set of σ-algebras, i.e. $\Im_V \in \Im$, $V \in W$, $\tilde{V} \subset V \Rightarrow \Im_{\tilde{V}} \subset \Im_V$, $\Im_\emptyset = \{\emptyset, \Omega\}$.

<u>Definition.</u> A stochastic family of random variables $(S_V, \Im_V)$ is called a ***martingale*** if for any $\tilde{V}, V \in W$, $\tilde{V} \subset V$ we have

$$E\,|S_V| < \infty \qquad \text{and} \qquad E\left(S_V \middle/ \Im_{\tilde{V}}\right) = S_{\tilde{V}} \qquad \text{a.s.}$$

We will consider the martingales, which definition is based on the notion of the martingale-difference random field, introduced by Nahapetian and Petrosian in 1992.

Definition. A random field ξ_t, $t \in \mathbb{Z}^d$ is called a ***martingale-difference random field*** if for any $t \in \mathbb{Z}^d$

$$E\left|\xi_t\right| < \infty \qquad \text{and} \qquad E\left(\xi_t \big/ \sigma\left(\xi_s, s \in \mathbb{Z}^d \setminus \{t\}\right)\right) = 0 \text{ a.s.}$$

We will consider only random fields with finite phase spaces.

Denote

$$S_V = \sum_{t \in V} \xi_t \qquad \text{and} \qquad \Im_V = \sigma\left(\xi_t, t \in V\right), V \in W.$$

If ξ_t, $t \in \mathbb{Z}^d$ is a martingale-difference random field then the family $\left(S_V, \Im_V\right), V \in W$, forms a martingale with respect to any sequence $V_n \in W, n = 1, 2, \ldots$ of increasing finite subsets of $\mathbb{Z}^d$.

Conversely, if for given random field ξ_t, $t \in \mathbb{Z}^d$ a stochastic family $\left(S_V, \Im_V\right), V \in W$ is a martingale with respect to any sequence of increasing finite subsets, then this field is a martingale-difference random field.

Constructions of martingale-difference random fields

Construction 1

Suppose that $X \in \mathbb{R}^1$ is a set symmetric with respect to zero and let $B(X)$ be the σ-algebra of its Borelian subsets. Consider on $B(X)$ a symmetric measure (i.e. $\mu(A) = \mu(-A)$, $A \in B(X)$). Let ξ_t, $t \in \mathbb{Z}^d$ be a random field with phase space X such that its finite-dimensional distributions are absolutely continuous with respect to the product-measure μ^V, $V \in W$ with even densities $p_V(x_t, t \in V)$, $V \in W$ i.e.

$$p_V(\theta_t x_t, t \in V) = p_V(x_t, t \in V)$$

for any $\theta_t \in \{1, -1\}$. Such a random field ξ_t, $t \in \mathbb{Z}^d$ represents a martingale-difference random field.

Construction 2

Consider a finite subset X of a real line $\mathbb{R}^1$ such that $X = \bigcup\limits_{k=1}^{n} X_k$, $X_i \cap X_j = \emptyset$, $i \neq j$ and

$$\sum_{x \in X_k} x = 0, \qquad k = \overline{1,n}.$$

Let there exists a random field ξ_t, $t \in \mathbb{Z}^d$ taking values in X such that its conditional distribution

$$q_t^{\bar{x}}(x) = P\left(\xi_t = x \big/ \xi_s = \bar{x}_s, s \in \mathbb{Z}^d \setminus \{t\}\right)$$

$x \in X$, $\bar{x} = \left(\bar{x}_s, s \in \mathbb{Z}^d \setminus \{t\}\right) \in X^{\mathbb{Z}^d \setminus \{t\}}$, $t \in \mathbb{Z}^d$ has the constant value $q_{t,k}^{\bar{x}}$ when $x \in X_k$, $k = \overline{1,n}$, i.e.

$$q_t^{\bar{x}}(x) = q_{t,k}^{\bar{x}}, \qquad x \in X_k, \ k = \overline{1,n}.$$

Then the random field ξ_t, $t \in \mathbb{Z}^d$ is a martingale-difference random field.

Construction 3

Let $T = \left\{T_j\right\}$ be a partition of the lattice $\mathbb{Z}^d$ ($\mathbb{Z}^d = \bigcup\limits_j T_j$, $T_j \cap T_k = \emptyset$, $j \neq k$) and suppose that a random field ξ_t, $t \in \mathbb{Z}^d$ has the following property: for any j the random field ξ_t, $t \in T_j$ represents a martingale-difference, i.e. for any $t \in T_j$

$$E\left(\xi_t \middle/ \sigma\left(\xi_s, s \in T_j \backslash \{t\}\right)\right) = 0 \text{ a.s.}$$

If in addition the random variables ξ_s and ξ_r are independent when $s \in T_j$, $r \in T_k$, $j \neq k$, then the random field ξ_t, $t \in \mathbb{Z}^d$ is a martingale-difference random field.

Gibbs martingale-difference random fields

Let ξ_t, $t \in \mathbb{Z}^d$ be a Gibbs random field, corresponding to the potential Φ , which components take values in symmetric with respect to zero set X. If the potential Φ is even, i.e.

$$\Phi_V\left(\theta_t x_t, t \in V\right) = \Phi_V\left(x_t, t \in V\right), \qquad V \in W$$

for each $\theta_t \in \{1, -1\}$, then such a Gibbs random field is a martingale-difference random field.

Examples of even potentials

$$\Phi_V\left(x_t, t \in V\right) = \prod_{t \in V} |x_t| \cdot |V|^{\gamma}$$

and

$$\Phi_V\left(x_t, t \in V\right) = \exp\left\{ - \sup_{t \in V} |x_t| \cdot |V|^{\gamma} \right\}$$

where $\gamma \in \mathbb{R}^1$, $V \in W$, $x_t \in X$, $t \in V$.

Nahapetian, 1995

Theorem (CPT) . Let $\xi_t,\ t \in \mathbb{Z}^d$ be a homogeneous ergodic martingale-difference random field such that $0 < \sigma^2 = E\xi_0^2 < \infty$. Then

$$\lim_{n\to\infty} P\left(\frac{S_{V_n}}{\sigma \cdot n^{d/2}} < x\right) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-u^2/2} du, \qquad x \in \mathbb{R}^1,$$

where V_n is a d-dimensional cube with side length $n, n = 1, 2, \ldots.$

For Gibbs random fields with symmetric with respect to zero phase spaces there is a corollary from CLT for martingale-difference random fields.

Corollary 1. Let Φ be an even translation-invariant potential such that the corresponding Gibbs random field ξ_t, $t \in \mathbb{Z}^d$ is ergodic and $E\xi_0^2 > 0$. Then for this Gibbs random field CLT is valid.

Remark. Note that if Φ is an even translation-invariant potential such that the corresponding Gibbs random field ξ_t, $t \in \mathbb{Z}^d$ is unique, than this field is ergodic and we can use Corollary 1.

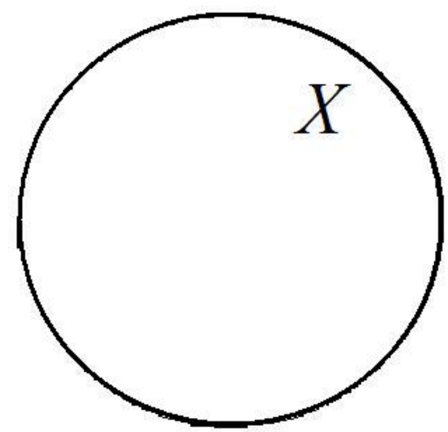
Corollary 2. Let Φ be an even translation-invariant potential such that the corresponding Gibbs random field $\xi_t,\ t \in \mathbb{Z}^d$ is unique. Then for this random field local limit theorem is valid.

1. A new method of constructing martingale-difference random fields

Randomization

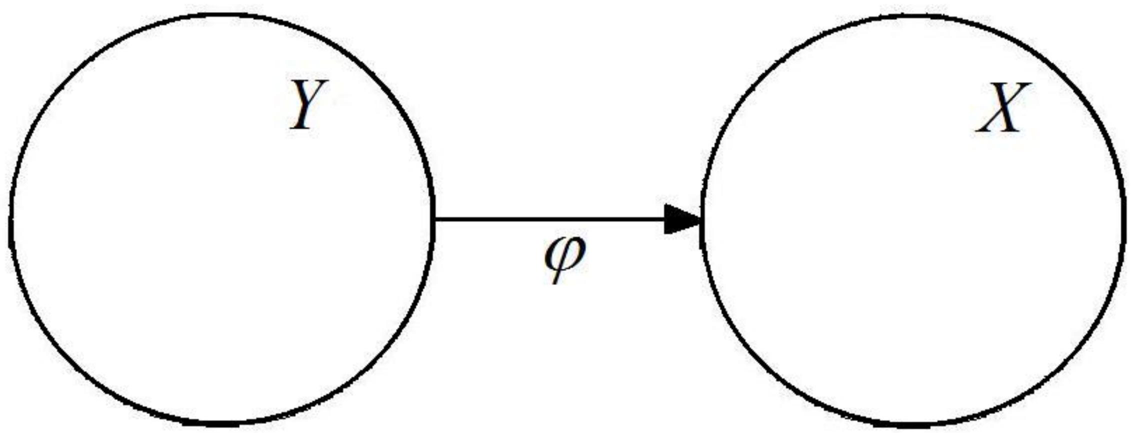
X — finite set

p — probability distributions on X, i.e. $p(x) \geq 0$, $x \in X$ and $\sum_{x \in X} p(x) = 1$



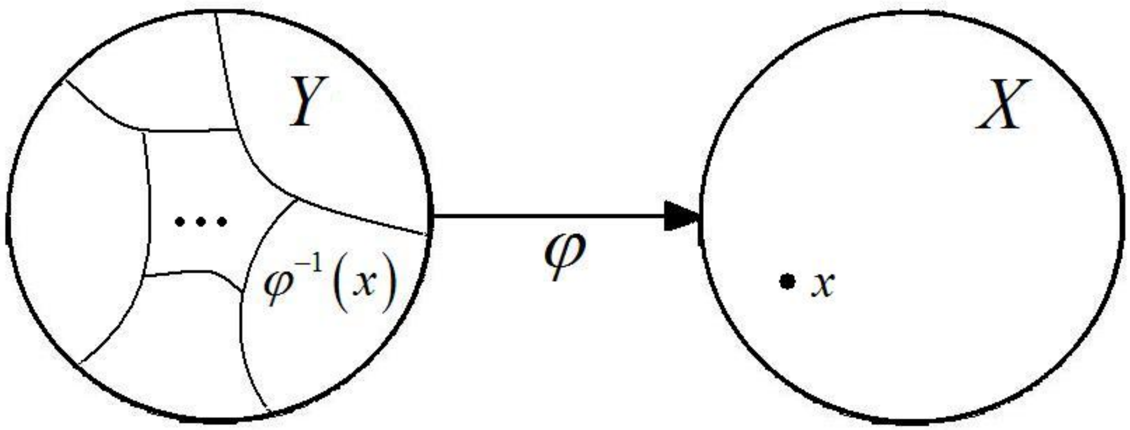
Y — finite set

$\varphi : Y \to X$ — surjective map, i.e. each $x \in X$ has no less than one pre-image $y \in Y$



$$Y = \bigcup_{x \in X} \varphi^{-1}(x),$$

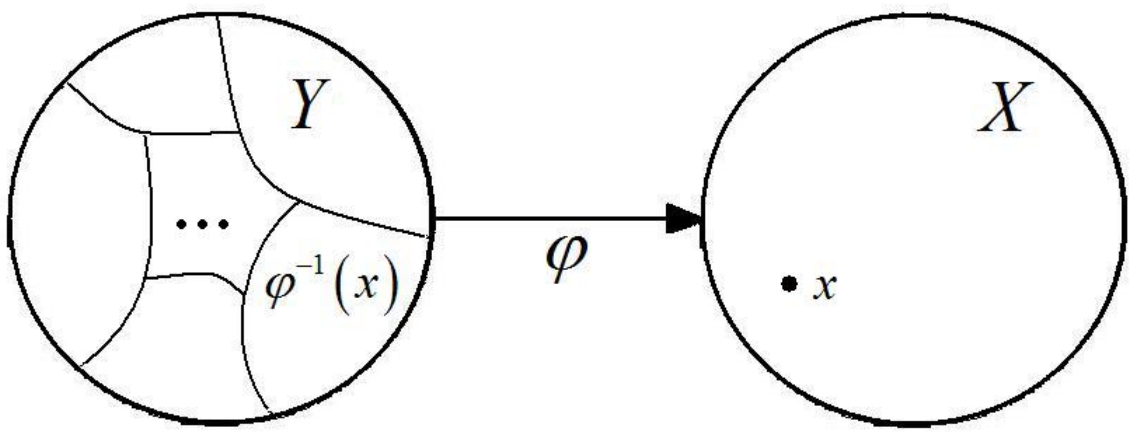
where $\varphi^{-1}(x) = \{y \in Y : \varphi(y) = x\}$ — set of pre-images of $x, x \in X$



$R = \{R^x, x \in X\}$ — family of probability distribution on Y, each of which is concentrated on $\varphi^{-1}(x)$, i.e.

$$R^x(y) \geq 0, \; y \in Y, \qquad \sum_{y \in Y} R^x(y) = 1,$$

$$R^x(y) = 0 \qquad \text{if } y \notin \varphi^{-1}(x)$$



Then we have probability distribution on Y

$$\widehat{p}(y) = \sum_{x \in X} R^x(y)\, p(x), \qquad y \in Y.$$

Indeed,

$$\widehat{p}(y) \geq 0, \qquad y \in Y$$

and

$$\sum_{y \in Y} \widehat{p}(y) = \sum_{x \in X} \sum_{y \in \varphi^{-1}(x)} R^x(y)\, p(x) =$$

$$\sum_{x \in X} p(x) \sum_{y \in \varphi^{-1}(x)} R^x(y) = \sum_{x \in X} p(x) = 1$$

Theorem 1. Suppose we are given the random field ξ_t, $t \in \mathbb{Z}^d$ with phase space X, finite set Y, surjective map φ and the family $R = \{R_t^x, x \in X\}$ of probability distribution on Y , each of which is concentrated on $\varphi^{-1}(x)$. Then there exist a random field η_t, $t \in \mathbb{Z}^d$ with phase space Y such that

$$\xi_t = \varphi(\eta_t), \qquad t \in \mathbb{Z}^d$$

and for any $V \in W$

$$P(\eta_t = y_t, t \in V) = \prod_{t \in V} R_t^{x_t}(y_t) \cdot P(\xi_t = x_t, t \in V),$$

$y_t \in Y$, $x_t \in X$, $t \in V$.

Definition. A random field η_t, $t \in \mathbb{Z}^d$ for which

$$\xi_t = \varphi(\eta_t), \qquad t \in \mathbb{Z}^d,$$

we will call ***associated*** with random field ξ_t, $t \in \mathbb{Z}^d$ (by the means of the map φ).

Remarks

1)Associated random field is not necessary unique

2) By imposing certain conditions on Y, φ and $R_t = \{R_t^x, x \in X\}$, $t \in \mathbb{Z}^d$ we will obtain different associated random fields with certain properties.

Associated martingale-difference random fields

Theorem 2. Let ξ_t, $t \in \mathbb{Z}^d$ be a random field with phase space X, let η_t, $t \in \mathbb{Z}^d$ be a random field with phase space Y associated with ξ_t, $t \in \mathbb{Z}^d$ by the means of the map φ, and let $R_t = \{R_t^x, x \in X\}$, $t \in \mathbb{Z}^d$ be the corresponding to η_t, $t \in \mathbb{Z}^d$ families of probability distributions. If for any $x \in X$ and $t \in \mathbb{Z}^d$

$$\sum_{y \in \varphi^{-1}(x)} y \cdot R_t^x(y) = 0,$$

then the random field η_t, $t \in \mathbb{Z}^d$ is a martingale-difference random field.

If families of probability distributions $R_t = \{R_t^x, x \in X\}$, $t \in \mathbb{Z}^d$ are such that

$$R_t^x\left(y\right) = \begin{cases} \frac{1}{\left|\varphi^{-1}(x)\right|}, & y \in \varphi^{-1}\left(x\right) \\ \\ 0, & y \notin \varphi^{-1}\left(x\right) \end{cases}$$

$x \in X$, $y \in Y$, $t \in \mathbb{Z}^d$, then it is sufficient to claim that a set Y and a map φ be such that

$$\sum_{y \in \varphi^{-1}(x)} y = 0,$$

for any $x \in X$.

Randomization of Gibbs random fields

Theorem 3. Let ξ_t, $t \in \mathbb{Z}^d$ be a Gibbs random field. Then a random field η_t, $t \in \mathbb{Z}^d$ associated with ξ_t, $t \in \mathbb{Z}^d$ is also a Gibbs random field.

Theorem 4. Let ξ_t, $t \in \mathbb{Z}^d$ be a Gibbs random field with phase space X and potential $\Phi(x_t x_V)$, $x_t \in X$, $x_V \in X^V$, $t \in \mathbb{Z}^d$, $V \in W\left(\mathbb{Z}^d \backslash t\right)$. Let there exists a partition of X

$$X = \bigcup_{k=1}^{n} X_k, \qquad X_i \cap X_j = \emptyset, \qquad i \neq j,$$

such that

$$\sum_{x \in X_k} x = 0, \qquad k = \overline{1,n}$$

and the potential $\Phi(x_t x_V)$ has the constant value on the elements of partition of X, i.e.

$$\Phi(x_t x_V) = \Phi_k(x_V),$$

$x_t \in X_k$, $k = \overline{1,n}$. Then the Gibbs random field ξ_t, $t \in \mathbb{Z}^d$ is a martingale-difference random field.

Note that symmetric with respect to zero phase space and even potential satisfy conditions of Theorem 4.

Corollary 3 (CLT). Let ξ_t, $t \in \mathbb{Z}^d$ be a translation-invariant ergodic Gibbs random field with phase space X and potential Φ which are satisfy the conditions of Theorem 4. If also $E\xi_0^2 > 0$ then for this random field **CLT is valid**.

Corollary 4 (LLT). Let ξ_t, $t \in \mathbb{Z}^d$ be a translation-invariant ergodic Gibbs random field with phase space X and potential Φ which are satisfy the conditions of Theorem 4. If also $E\xi_0^2 > 0$ then for this random field **local limit theorem is valid**.

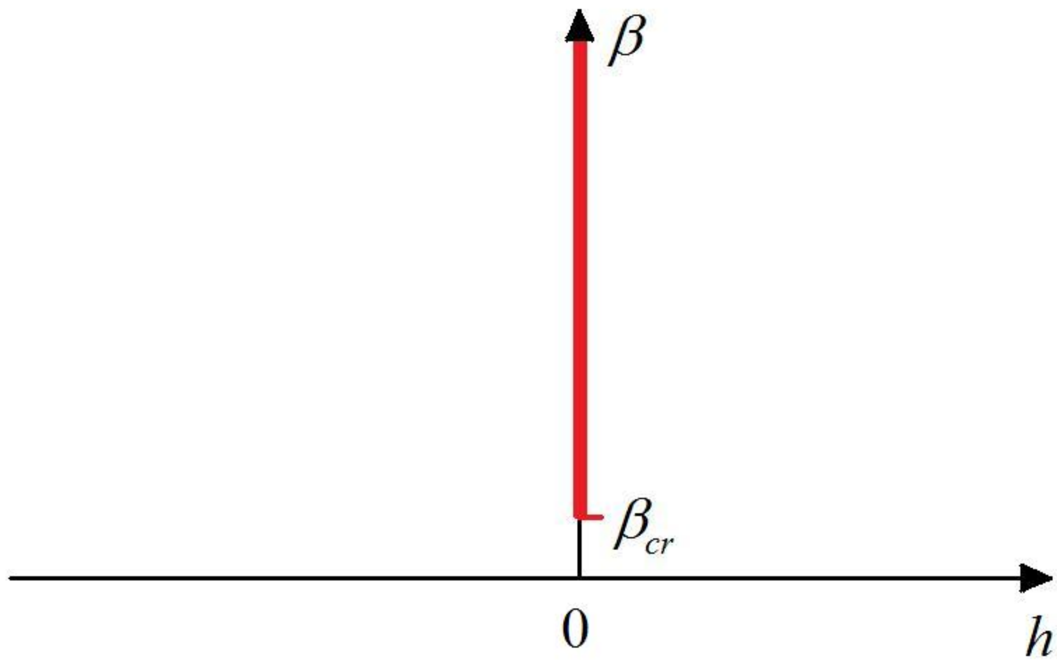
3. Example of applying the randomization method to some Gibbs random field

Ising model

$$\Phi_{\{t,s\}}(x_t, x_s) = \begin{cases} x_t \cdot x_s, & |t-s| = 1, \\ 0, & |t-s| \neq 1, \end{cases}$$

where $x_t, x_s \in X = \{-1, 1\}$ and $|t-s| = \max_{1 \leq i \leq d} \left| t^{(i)} - s^{(i)} \right|$.

Phase diagram for the Ising model



Nahapetian, 1997

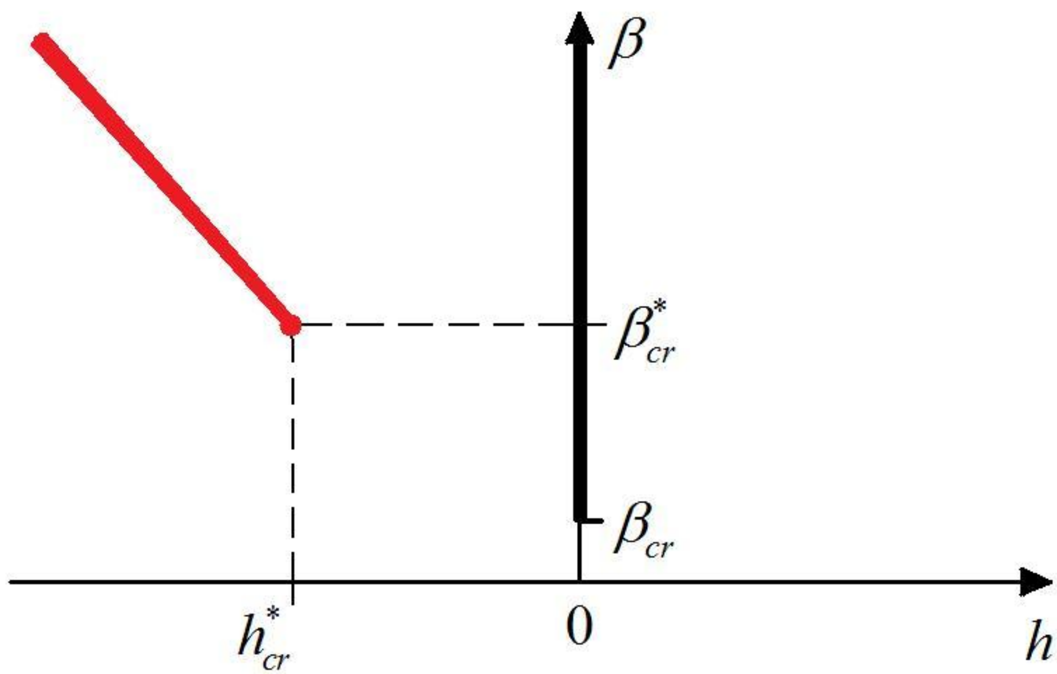
Model with even potential

$$\tilde{\Phi}_{\{t,s\}}(y_t, y_s) = \begin{cases} |y_t| \cdot |y_s|, & |t-s| = 1, \\ 0, & |t-s| \neq 1, \end{cases} \qquad t, s \in \mathbb{Z}^d,$$

where $y_t, y_s \in Y = \{-1, 0, 1\}$.

$$x = 2|y| - 1$$

Phase diagram for the model with even potential



Coordinates of critical point for model with even potential

$$\beta^*_{cr} = 4\beta_{cr} \qquad \text{and} \qquad h^*_{cr} = -\beta^*_{cr} d + \ln 2.$$

The connection formulas of total spins probability distribution

Denote for $V \in W$

S_V^{Is} — total spin of the Ising model

S_V^{ev} — total spin of the model with even potential

$$P\left(S_V^{ev} = k\right) = \sum_{j=0}^{(|V|-k)/2} \binom{k+2j}{2j} 2^{k+2j} P\left(S_V^{Is} = k+2j\right),$$

$k \geq 0$

$$P\left(S_V^{Is}=k\right)=2^{-k}\Bigg[P\left(S_V^{ev}=k\right)-$$

$$-\sum_{j=1}^{(|V|-k)/2}\left(\begin{array}{c}k+2j\\2j\end{array}\right)P\left(S_V^{ev}=k+2j\right)+$$

$$+\sum_{j=1}^{(|V|-k)/2}\left(\begin{array}{c}k+2j\\2j\end{array}\right)2^{2j}\times$$

$$\times\sum_{s=1}^{(|V|-k)/2-j}\left(\begin{array}{c}k+2\left(j+s\right)\\2s\end{array}\right)2^{2s}P\left(S_V^{ev}=k+2\left(j+s\right)\right)-$$

$$-\sum_{j=1}^{(|V|-k)/2}\left(\begin{array}{c}k+2j\\2j\end{array}\right)2^{2j}\sum_{s=1}^{(|V|-k)/2-j}\left(\begin{array}{c}k+2\left(j+s\right)\\2s\end{array}\right)2^{2s}\times$$

$$\times\sum_{l=1}^{(|V|-k)/2-j-s}\left(\begin{array}{c}k+2\left(j+s+l\right)\\2l\end{array}\right)2^{2l}P\left(S_V^{ev}=k+2\left(j+s+l\right)\right)+...\Bigg]$$

ξ_t, $t \in \mathbb{Z}^d$ — the homogenous random field, $X = \{0, 1\}$

η_t, $t \in \mathbb{Z}^d$ — associated random field, $Y = \{-1, 0, 1\}$ such that

$$\xi_t = |\eta_t|\,, \qquad t \in \mathbb{Z}^d$$

and for any $t \in \mathbb{Z}^d$

$$P(\eta_t = \pm 1) = \frac{1}{2} P(\xi_t = 1)\,, \qquad P(\eta_t = 0) = P(\xi_t = 0)\,.$$

The random field η_t, $t \in \mathbb{Z}^d$ is a martingale-difference because

$$\sum_{y \in \varphi^{-1}(x)} y = 0, \qquad \text{for any } x \in X.$$

Connection formula for finite-dimensional probability distributions

$$P(\eta_t = y_t, t \in V) = 2^{-|V|} P(\xi_t = x_t, t \in V)\,,$$

$y_t \in Y$, $x_t \in X$, $t \in V$, $V \in W$.

The connection formulas of total spines probability distributions

$$P\left(S_V^{\eta} = 2s\right) = \sum_{j=s}^{|V|/2} 2^{-2j} \binom{2j}{j-s} P\left(S_V^{\xi} = 2j\right),$$

$$s = 0, 1, ..., |V|/2,$$

$$P\left(S_V^{\eta} = 2s-1\right) = \sum_{j=s}^{|V|/2} 2^{-2j+1} \binom{2j-1}{j-s} P\left(S_V^{\xi} = 2j-1\right),$$

$$s = 1, 2, ..., |V|/2,$$

$$P\left(S_V^{\eta} = -s\right) = P\left(S_V^{\eta} = s\right),$$

$$s = 1, 2, ..., |V|,$$

for any $V \in W$.

$$P\left(S_V^{\xi}=j\right)=2^j\sum_{k=0}^{\frac{|V|-j}{2}}(-1)^k\frac{j+2k}{j+k}\left(\begin{array}{c} j+k \\ k \end{array}\right)P\left(S_V^{\eta}=j+2k\right),$$

$$j=0,1,...,|V|,$$

for any $V\in W$.

Characteristic function of the total spin of associated r.f. by means of total spin probabilities of given r.f.

$$f_{S_V^\eta}(t) = \sum_{j=0}^{|V|} (\cos t)^j \, P\left(S_V^\xi = j\right),$$

$V \in W$.

CLT for associated random field

Theorem 5. Let η_t, $t \in \mathbb{Z}^d$ be a martingale-difference random field associated with random field ξ_t, $t \in \mathbb{Z}^d$. Then

$$f_{\frac{S_V^\eta}{\sqrt{DS_V^\eta}}}(t) \to e^{-t^2/2}, \qquad \text{as } V \uparrow \mathbb{Z}^d.$$

Proof.

$$f_{\frac{S_V^\eta}{\sqrt{DS_V^\eta}}}(t) = \sum_{j=0}^{|V|} \left(\cos \frac{t}{\sqrt{p|V|}}\right)^j P\left(S_V^\xi = j\right) =$$

$$= \sum_{j=0}^{|V|} \left(1 - j \cdot \frac{t^2}{2p|V|} + j \cdot o\left(|V|^{-2}\right) + o\left(|V|^{-4}\right)\right) P\left(S_V^\xi = j\right) =$$

$$= 1 - \frac{t^2}{2p|V|} ES_V^\xi + o\left(|V|^{-2}\right) ES_V^\xi + o\left(|V|^{-4}\right) =$$

$$= 1 - \frac{t^2}{2p|V|} p\,|V| + o\left(|V|^{-2}\right) p\,|V| + o\left(|V|^{-4}\right) =$$

$$= 1 - \frac{t^2}{2} + o\left(|V|^{-1}\right) =$$

$$= e^{-t^2/2} + o\,(1)\,.$$

Characteristic function of total spin of given r.f. by means of total spin probabilities of associated r.f.

$$f_{S_V^\xi}(t) = \sum_{j=-|V|}^{|V|} \cos\left(j \cdot \operatorname{arccos} e^{it}\right) P\left(S_V^\eta = j\right)$$

$V \in W$.

CLT for the Ising model outside the critical point

Theorem 6. Let ξ_t, $t \in \mathbb{Z}^d$ be a Gibbs random field corresponding to Ising model and let η_t, $t \in \mathbb{Z}^d$ be associated with ξ_t, $t \in \mathbb{Z}^d$ martingale-difference random field. If exists the limit $\lim\limits_{V \uparrow \mathbb{Z}^d} \frac{DS_V^\xi}{|V|} = \sigma^2$ and $0 < \sigma^2 < \infty$ then

$$f_{\frac{S_n^\xi - MS_n^\xi}{\sqrt{DS_n^\xi}}}(t) \to e^{-t^2/2} \qquad \text{as } V \uparrow \mathbb{Z}^d.$$

Proof

$$f_{\frac{S_n^\xi}{\sqrt{DS_n^\xi}}}(t) = f_{\frac{S_n^\xi}{\sigma\sqrt{|V|}}}(t) = \sum_{j=-|V|}^{|V|} \cos\left(j \cdot \arccos e^{\frac{it}{\sigma\sqrt{|V|}}}\right) P\left(S_V^\eta = j\right) =$$

$$= \sum_{j=-n}^{n} \left(1 + \left(\frac{it}{\sigma\sqrt{|V|}} - \frac{t^2}{3\sigma^2|V|}\right) j^2 - \frac{t^2}{6\sigma^2|V|} j^4 + o\left(|V|^{-3/2}\right)\right) P\left(S_V^\eta = j\right) =$$

$$= 1 + \left(\frac{it}{\sigma\sqrt{|V|}} - \frac{t^2}{3\sigma^2|V|}\right) E\left(S_V^\eta\right)^2 - \frac{t^2}{6\sigma^2|V|} E\left(S_V^\eta\right)^4 + o\left(|V|^{-3/2}\right) =$$

$$= 1 + \left(\frac{it}{\sigma\sqrt{|V|}} - \frac{t^2}{3\sigma^2|V|}\right) p\,|V| - \frac{t^2}{6\sigma^2|V|}\left(3\sigma^2\,|V| + 3p^2\,|V|^2 - 2p\,|V|\right) +$$

$$+ o\left(|V|^{-3/2}\right) \to e^{it\frac{p}{\sigma}\sqrt{|V|} - \frac{t^2}{2}}$$

Then

$$f_{\frac{S_n^\xi - |V|p}{\sigma\sqrt{|V|}}}(t) = e^{-it\frac{p}{\sigma}\sqrt{|V|}} \cdot f_{\frac{S_n^\xi}{\sigma\sqrt{|V|}}}(t) \to e^{-it\frac{p}{\sigma}\sqrt{|V|}} \cdot e^{it\frac{p}{\sigma}\sqrt{|V|} - \frac{t^2}{2}} = e^{-t^2/2}$$

Thank you for your attention!