# Limiting behaviour of two-dimensional self-avoiding polygons

Ostap Hryniv
Dept. of Mathematical Sciences
Durham University

Ostap.Hryniv@durham.ac.uk

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# Outline

- Model
- Motivation
- Results and Ideas
- Conclusions

# Self-avoiding polygons

A SAP is a **closed** loop $\boldsymbol{\omega} = (\omega_0, \omega_1, \ldots, \omega_n)$, with $\omega_j \in \mathbb{Z}^2$, $j \geq 0$,

$$\omega_0 = \omega_n \,, \qquad |\omega_k - \omega_{k-1}| = 1 \quad {}^\forall k = 1, 2, \ldots, n \,,$$
$$\omega_j \neq \omega_k \qquad {}^\forall 0 \leq j < k \leq n$$

and $|\boldsymbol{\omega}| = n$ is the length of $\boldsymbol{\omega}$. [cf. V.Betz's talk!]

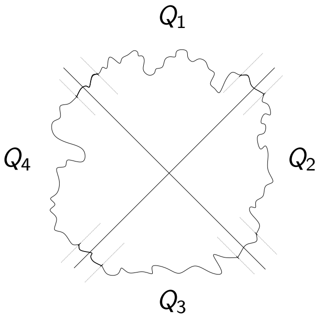
Probability distribution:

$$\mathbf{P}_\beta(\boldsymbol{\omega}) \propto \exp\bigl\{-\beta|\boldsymbol{\omega}|\bigr\} \,, \qquad \beta > 0 \,.$$

AIM: statistical properties of $2D$ SAPs enclosing **large** area $Q$.

# SAP: centering

Consider $\mathcal{Z}^{\odot} \equiv \{\omega : \text{centered at } 0\}$, i.e., $Q_1 = Q_3$, $Q_2 = Q_4$, where




Then

$$\mathbf{P}_{\beta}(\boldsymbol{\omega}) = \frac{1}{Z_{\beta}} \exp\{-\beta|\boldsymbol{\omega}|\}, \qquad Z_{\beta} = \sum_{\boldsymbol{\omega} \in \mathcal{Z}^{\odot}} \exp\{-\beta|\boldsymbol{\omega}|\}$$

is well defined for each $\beta > \bar{\beta}_{\text{cr}}$.

# Model

Consider:

$$\mathcal{Z}_Q^{\odot} \equiv \{\omega : \text{centered at 0, enclosed area } Q\}$$

i.e., $Q_1 = Q_3,\ Q_2 = Q_4,\ Q_1 + Q_2 + Q_3 + Q_4 = Q.$

AIM: for fixed $\beta > \bar{\beta}_{\text{cr}}$, in the limit $Q \to \infty$,

evaluate

$$Z_Q^{\beta} = \sum_{\omega \in \mathcal{Z}_Q^{\odot}} \exp\{-\beta|\omega|\}\,,$$

and describe

$$\mathbf{P}_\beta(\,\cdot\,|\mathcal{Z}_Q^{\odot})\,.$$

# Remarks

- A similar question can be asked about other $2D$ polygons, e.g., phase boundaries of finite-range models [Pirogov-Sinai?]
- we develop the geometric part of the sharp large deviation theory for some of these models, e.g., low-temperature $2D$ Ising
- some results can also be extended to higher temperatures

# Ising model: large deviations

Gibbs probability distribution in $\Omega_V \equiv \{-1,+1\}^V$, $V \Subset \mathbb{Z}^2$:

$$\mathbb{P}_V^\beta(\sigma) = \frac{1}{Z(V,\beta)} \exp\Big\{\beta \sum_{\substack{\langle x,y\rangle \\ x,y\in V}} \sigma_x \sigma_y\Big\}, \qquad \beta > 0\,, \sigma \in \Omega_V\,.$$

Q.: distribution of $S_V = \sum_{x\in V} \sigma_x$ under $\mathbb{P}_V^\beta(\,\cdot\,)$ as $V \uparrow \mathbb{Z}^2$ ?
typical $\sigma$ under $\mathbb{P}_V^\beta(\,\cdot \mid S_V)$ ?

more precisely: fix $a_V$ s.t. $a_V \simeq c|V|$, $|c| < 1$, $a_V \equiv |V| \pmod 2$

asymptotics of $\mathbb{P}_V^\beta\big(S_V = a_V\big)$ as $V \uparrow \mathbb{Z}^2$ ?
typical $\sigma$ under $\mathbb{P}_V^\beta(\,\cdot \mid S_V = a_V)$ ?

# 2D Ising model: phase transition

$^\exists \beta_{\text{cr}} \in (0, \infty)$ s.t.: [cf. L.Khachatrian's talk!]

- $\beta < \beta_{\text{cr}}$ : unique limit $\mu$ of $\mathbb{P}_V^\beta(\,\cdot\,)$ as $V \uparrow \mathbb{Z}^2$
- $\beta > \beta_{\text{cr}}$: limit depends on b.c.; each such $\mu$ is a convex combination of $\mu^+$ and $\mu^-$, where

$$\mu^\pm = \lim_{V \uparrow \mathbb{Z}^2} \mathbb{P}_V^{\beta,\pm}(\,\cdot\,)\,,$$

$$\mathbb{P}_V^{\beta,\pm}(\sigma) = \frac{\exp\bigl\{\pm\beta \sum_{x \in \partial V} \sigma_x\bigr\}}{Z(V, \beta, \pm)}\,\mathbb{P}_V^\beta(\sigma)\,;$$

in particular, $m_\beta \equiv \langle \sigma_0 \rangle_{\mu^+} > 0$ iff $\beta > \beta_{\text{cr}}$.

# DKS theory (Dobrushin-Kotecký-Shlosman ['92])

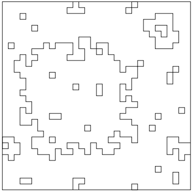
$V = V_N \equiv T_N$, $\bar{\sigma}$ — periodic b.c., $0 < \beta < \infty$ — "large"

- provides the asymptotics of $\mathbb{P}_N^{\beta,\mathrm{per}}\big(S_N = a_N\big)$ as $N \to \infty$

  [cond's on $a_N$]

- describes *"typical configurations"* corresponding to such event: as $N \to \infty$,

$$\mathbb{P}_N^{\beta,\mathrm{per}}\left(\right.$$



$$\left. \,\middle|\, S_N = a_N \right) \to 1.$$

**Theorem [DKS'92]**

Let $\rho_N \equiv \frac{a_N}{N^2} \to \rho \in \left(\rho_0, m_\beta\right)$ as $N \to \infty$ with $\rho_0 > 1/2$.
Then $^\exists$ positive constants $\beta_0(\rho_0)$, $K$, and $\kappa$, such that $^\forall\beta \geq \beta_0$ the following holds with probability $\to 1$ (as $N \to \infty$):

- $\exists!$ macroscopic contour $\Gamma_0$ (of correct sign)
- its area satisfies

$$\left| \left|\mathrm{Int}(\Gamma_0)\right| - \lambda_N N^2 \right| \leq K N^{6/5} (\log N)^\kappa$$

  with $\lambda_N = 1/2 - a_N / 2 m_\beta N^2$;
- $^\exists x = x(\sigma)$ s.t.

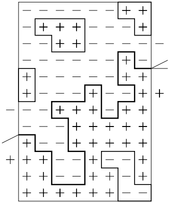
$$\mathrm{dist}_H\left(\Gamma_0 + x(\sigma), N\gamma_{\beta,\rho_N}\right) \leq K N^{3/4} (\log N)^{3/2},$$

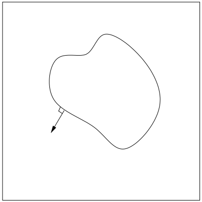
  $\gamma_{\beta,\rho_N}$ being the Wulff shape of "phase volume" $\rho_N$;
- all other contours have size $\leq K \log N$.

# Wulff theory

- surface tension

$$\tau_\beta(\,\cdot\,) = \lim_{N\to\infty}\lim_{M\to\infty}\frac{\cos\varphi}{\beta N}\log\frac{Z(V_{NM},\beta,+)}{Z(V_{NM},\beta,\bar\sigma^\varphi)}$$





- Wulff functional (surface energy)
  $\gamma$ – closed self-avoiding rectifiable curve in $\mathbb{R}^2$

$$\gamma \mapsto \mathcal{W}_\beta(\gamma) = \int_\gamma \tau_\beta(\vec n_s)\,ds$$

- Wulff variational problem

$$\mathcal{W}_\beta(\gamma) \longrightarrow \inf : \qquad \mathrm{Vol}(\gamma) \geq 1 \qquad\qquad (*)$$

- Wulff shape $\gamma = \gamma_1$ — the (unique) solution to $(*)$
- Wulff construction [Wulff 1901] :

Wulff shape = boundary of $W_{\lambda_0}$ s.t. $\mathrm{Vol}(W_{\lambda_0}) = 1$ and

$$W_\lambda \equiv \bigcap_{\vec{n} \in \mathbb{S}^1} \Big\{ x \in \mathbb{R}^2 : (x, \vec{n}) \leq \lambda \tau_\beta(\vec{n}) \Big\}$$

**Corollary**

Under the conditions of the theorem above, $^{\exists}\alpha \in (0,1)$ s.t.

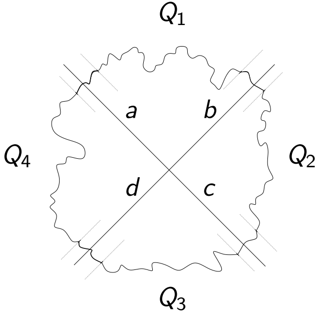
$$\mathbb{P}_N^{\beta,\mathrm{per}}\big(S_N = a_N\big) = \exp\Big\{-\beta N \mathcal{W}_\beta(\gamma_{\beta,\rho_N}) + o(N^\alpha)\Big\}$$

as $N \to \infty$; here $\mathcal{W}_\beta(\gamma_{\beta,\rho_N})$ is the surface energy of the Wulff shape $\gamma_{\beta,\rho_N}$.

generalized to all $\beta > \beta_{\mathrm{cr}}$ in Ioffe-Schonmann ['98]

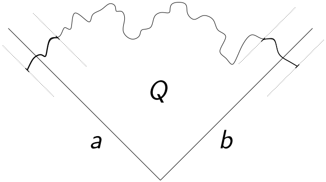
# SAP: ideas

Self-avoiding plygons in $\mathbb{Z}^2$, "centered at the origin":




condition on the "parts on the diagonals", then re-sum

# Step I: a single-quarter problem $\mathcal{Z}_Q^{a,b}$




- vertical strip + (free) diagonal ends
- renewal type analysis on <u>micro</u> scale
- cluster expansions + renormalisation
- key property: to massgap positivity [Chayes-Chayes '86, Ioffe '98]

$\Rightarrow$ Ornstein-Zernike asymptotics [LLT]

# SAP: Step I

**Key result:**

$$Z_Q^{a,b} = \sum_{\omega} \exp\{-\beta|\boldsymbol{\omega}|\} \approx c\frac{\sqrt{D}}{Q}\exp\Big\{-W(a,b,Q)\Big\},$$

as $Q \to \infty$, where $D = \det \mathrm{Hess}_{a,b} W(\cdot,\cdot,Q)$.

[LLT; Ornstein-Zernike behaviour; cf. A.Pellegrinotti's talk!]

**Also**:

surface energy $W(\cdot,\cdot,Q)$ is symmetric, smooth, strictly convex.

# Step II: macroscopic variational problem

Minimise

$$W(a, b, Q_1) + W(b, c, Q_2) + W(c, d, Q_3) + W(d, a, Q_4)$$

subject to

$$Q_1 = Q_3\,, \qquad Q_2 = Q_4\,, \qquad Q_1 + Q_2 + Q_3 + Q_4 = Q\,.$$

symm. + convexity $\Longrightarrow$ $a = b = c = d \equiv \bar{\rho}$

smoothness $\Longrightarrow$ $Q_1 = Q_2 = Q_3 = Q_4 \equiv Q/4$

$\Longrightarrow$ optimal shape – **Wulff droplet**, area $Q$, "diag. diam." $2\bar{\rho}$

# Asymptotics of $Z_Q^\beta$

III step: sum over all contours in $\mathcal{Z}_Q^{\odot}$

$$\mathbf{P}_\beta\big(\mathcal{Z}_Q^{\odot}\big) \approx c\, Q^{-5/2} \exp\big\{-\sqrt{Q}\mathcal{W}_1\big\}\,.$$

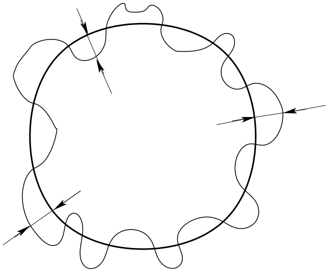
IV step: local entropy factor $\asymp \bar{\rho}^2 \asymp Q$.

**Thm 1**: $\forall \beta > \beta_{\text{cr}}$ fixed, as $Q \to \infty$:

$$Z_Q^\beta = \sum_{\boldsymbol{\omega} \in \mathcal{Z}_Q^{\odot}} e^{-\beta|\boldsymbol{\omega}|} \approx c\, Q^{-3/2} \exp\big\{-\sqrt{Q}\mathcal{W}_1\big\}\,.$$

## SAP: fluctuations

Consider $X_Q(s)$, $s \in [0, \bar{l}]$, — $Q^{-1/4}$-scaled **normal fluctuations** of $\omega \in \mathcal{Z}_Q^{\odot}$ around the Wulff shape $\mathcal{W}_Q$ at the origin



# SAP: fluctuations

Let

$$d\,\xi(s) = \frac{1}{\sqrt{\beta(\tau_\beta(\vec{n}_s) + \tau''_\beta(\vec{n}_s))}}\, dw(s)$$

$\bar{l}$ – length of $\partial \mathcal{W}_1$ $\qquad q_i = \int_{\gamma_i} \xi(s)\, ds$

$x(s)$ – periodic Gibbs field on $\mathbf{T}_{[0,\bar{l}]}$ (i.e. periodic extension of $\xi(s)$) on the event

$$\odot \equiv \left\{ q_1 = q_3, q_2 = q_4, q_1 + q_2 + q_3 + q_4 = 0 \right\}$$

**Thm 2**: $\forall$ fixed $\beta > \beta_{\mathrm{cr}}$, as $Q \to \infty$, the distribution of $X_Q(s)$ converges to that of $x(s)$.

# SAP: remarks

possible extensions to other $2D$ models:

- low-temperature Ising model [in preparation]
- finite range (Pirogov-Sinai?) models
- Ising at all subcritical temperatures [with R.Kotecký, D.Ioffe; in progress]
- moderate deviations for the Ising model [with R.Kotecký, D.Ioffe; in progress]
- ...

# Conclusions

main results:

- sharp asymptotics of the partition function
- functional CLT for normal fluctuations from the Wulff shape
- results valid for all $\beta > \beta_{\text{cr}}$

main tools:

- variational calculus [Wulff construction]
- analytic perturbations theory [cluster expansions]
- local limit theorems
- renormalization [mesoscopic scale]
- ...