Limiting behaviour of two-dimensional self-avoiding polygons

Ostap Hryniv
Dept. of Mathematical Sciences
Durham University

Ostap.Hryniv@durham.ac.uk

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Outline

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Self-avoiding polygons

A SAP is a closed loop \( \omega = (\omega_0, \omega_1, \ldots, \omega_n) \), with \( \omega_j \in \mathbb{Z}^2, j \geq 0 \),

\[
\omega_0 = \omega_n, \quad |\omega_k - \omega_{k-1}| = 1 \quad \forall k = 1, 2, \ldots, n,
\]

\( \omega_j \neq \omega_k \quad \forall 0 \leq j < k \leq n \)

and \( |\omega| = n \) is the length of \( \omega \). [cf. V.Betz's talk!]

Probability distribution:

\[
P_\beta(\omega) \propto \exp\{ -\beta|\omega| \}, \quad \beta > 0.
\]

AIM: statistical properties of 2D SAPs enclosing large area \( Q \).
SAP: centering

Consider $\mathcal{Z}^\circ \equiv \{\omega : \text{centered at } 0\}$, i.e., $Q_1 = Q_3$, $Q_2 = Q_4$, where

$$P_\beta(\omega) = \frac{1}{Z_\beta} \exp\{-\beta|\omega|\}, \quad Z_\beta = \sum_{\omega \in \mathcal{Z}^\circ} \exp\{-\beta|\omega|\}$$

is well defined for each $\beta > \bar{\beta}_{cr}$. 
Model

Consider:

\[ \mathcal{Z}_Q^\odot \equiv \{ \omega : \text{centered at 0, enclosed area } Q \} \]

i.e., \( Q_1 = Q_3 \), \( Q_2 = Q_4 \), \( Q_1 + Q_2 + Q_3 + Q_4 = Q \).

**AIM:** for fixed \( \beta > \bar{\beta}_{cr} \), in the limit \( Q \to \infty \),
evaluate

\[ Z_\beta^Q = \sum_{\omega \in \mathcal{Z}_Q^\odot} \exp\{ -\beta |\omega| \} , \]

and describe

\[ P_\beta( \cdot | \mathcal{Z}_Q^\odot ) . \]
Remarks

- A similar question can be asked about other 2D polygons, e.g., phase boundaries of finite-range models [Pirogov-Sinai?]

- we develop the geometric part of the sharp large deviation theory for some of these models, e.g., low-temperature 2D Ising

- some results can also be extended to higher temperatures
Ising model: large deviations

Gibbs probability distribution in $\Omega_V \equiv \{-1, +1\}^V$, $V \in \mathbb{Z}^2$:

$$\mathbb{P}^\beta_V(\sigma) = \frac{1}{Z(V, \beta)} \exp\left\{ \beta \sum_{\langle x, y \rangle \in V} \sigma_x \sigma_y \right\}, \quad \beta > 0, \sigma \in \Omega_V.$$

Q.: distribution of $S_V = \sum_{x \in V} \sigma_x$ under $\mathbb{P}^\beta_V(\cdot)$ as $V \uparrow \mathbb{Z}^2$?

typical $\sigma$ under $\mathbb{P}^\beta_V(\cdot \mid S_V)$?

more precisely: fix $a_V$ s.t. $a_V \simeq c|V|$, $|c| < 1$, $a_V \equiv |V|$ (mod 2)

asymptotics of $\mathbb{P}^\beta_V(S_V = a_V)$ as $V \uparrow \mathbb{Z}^2$?

typical $\sigma$ under $\mathbb{P}^\beta_V(\cdot \mid S_V = a_V)$?
2D Ising model: phase transition

\[ \exists \beta_{cr} \in (0, \infty) \text{ s.t.:} \]

- \( \beta < \beta_{cr} : \) unique limit \( \mu \) of \( \mathbb{P}_V^\beta(\cdot) \) as \( V \uparrow \mathbb{Z}^2 \)
- \( \beta > \beta_{cr} : \) limit depends on b.c.; each such \( \mu \) is a convex combination of \( \mu^+ \) and \( \mu^- \), where

\[ \mu^\pm = \lim_{V \uparrow \mathbb{Z}^2} \mathbb{P}_V^{\beta, \pm}(\cdot), \]

\[ \mathbb{P}_V^{\beta, \pm}(\sigma) = \frac{\exp\{\pm \beta \sum_{x \in \partial V} \sigma_x\}}{Z(V, \beta, \pm)} \mathbb{P}_V^\beta(\sigma); \]

in particular, \( m_\beta \equiv \langle \sigma_0 \rangle_{\mu^+} > 0 \text{ iff } \beta > \beta_{cr}. \]
**DKS theory** (Dobrushin-Kotecký-Shlosman ['92])

\[ V = V_N \equiv T_N, \bar{\sigma} \text{ — periodic b.c., } 0 < \beta < \infty \text{ — “large”} \]

- provides the asymptotics of \( \mathbb{P}_N^{\beta, \text{per}} (S_N = a_N) \) as \( N \to \infty \)
  
  \[ \mathbb{P}_N^{\beta, \text{per}} \left( \left| S_N = a_N \right. \right) \to 1. \]

- describes “typical configurations” corresponding to such event: as \( N \to \infty \),

  \[ \mathbb{P}_N^{\beta, \text{per}} \left( \left| S_N = a_N \right. \right) \to 1. \]
Theorem [DKS’92]

Let $\rho_N \equiv \frac{a_N}{N^2} \rightarrow \rho \in (\rho_0, m_\beta)$ as $N \rightarrow \infty$ with $\rho_0 > 1/2$. Then $\exists$ positive constants $\beta_0(\rho_0)$, $K$, and $\kappa$, such that $\forall \beta \geq \beta_0$ the following holds with probability $\rightarrow 1$ (as $N \rightarrow \infty$):

- $\exists$! macroscopic contour $\Gamma_0$ (of correct sign)
- its area satisfies

$$\left| \text{Int}(\Gamma_0) - \lambda_N N^2 \right| \leq K N^{6/5} (\log N)^\kappa$$

with $\lambda_N = 1/2 - a_N/2m_\beta N^2$;

- $\exists x = x(\sigma)$ s.t.

$$\text{dist}_H(\Gamma_0 + x(\sigma), N \gamma_{\beta, \rho_N}) \leq K N^{3/4} (\log N)^{3/2},$$

$\gamma_{\beta, \rho_N}$ being the Wulff shape of “phase volume” $\rho_N$;

- all other contours have size $\leq K \log N$. 

Wulff theory

- surface tension

\[
\tau_\beta(\cdot) = \lim_{N \to \infty} \lim_{M \to \infty} \frac{\cos \varphi}{\beta N} \log \frac{Z(V_{NM}, \beta, +)}{Z(V_{NM}, \beta, \bar{\sigma} \varphi)}
\]

- Wulff functional (surface energy)

\[
\gamma \mapsto \mathcal{W}_\beta(\gamma) = \int_\gamma \tau_\beta(\vec{n}_s) \, ds
\]

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\gamma \mapsto \mathcal{W}_\beta(\gamma) = \int_\gamma \tau_\beta(\vec{n}_s) \, ds
\]
• Wulff variational problem

\[ \mathcal{W}_\beta(\gamma) \rightarrow \inf : \quad \text{Vol}(\gamma) \geq 1 \quad (\star) \]

• Wulff shape \( \gamma = \gamma_1 \) — the (unique) solution to (\( \star \))

• Wulff construction [Wulff 1901] :

Wulff shape = boundary of \( \mathcal{W}_{\lambda_0} \) s.t. \( \text{Vol}(\mathcal{W}_{\lambda_0}) = 1 \) and

\[ \mathcal{W}_\lambda \equiv \bigcap_{\bar{n} \in \mathbb{S}^1} \left\{ x \in \mathbb{R}^2 : (x, \bar{n}) \leq \lambda \tau_\beta(\bar{n}) \right\} \]
Corollary
Under the conditions of the theorem above, \( \exists \alpha \in (0, 1) \) s.t.

\[
\mathbb{P}_{N}^{\beta, \text{per}} (S_N = a_N) = \exp \left\{ -\beta N \mathcal{W}_\beta (\gamma_{\beta, \rho_N}) + o(N^\alpha) \right\}
\]

as \( N \to \infty \); here \( \mathcal{W}_\beta (\gamma_{\beta, \rho_N}) \) is the surface energy of the Wulff shape \( \gamma_{\beta, \rho_N} \).

generalized to all \( \beta > \beta_{\text{cr}} \) in Ioffe-Schonmann ['98]
Self-avoiding polygons in $\mathbb{Z}^2$, “centered at the origin”:

- $Q_1$
- $Q_2$
- $Q_3$
- $Q_4$

Condition on the “parts on the diagonals”, then re-sum
Step I: a single-quarter problem $Z_{Q}^{a,b}$

- vertical strip + (free) diagonal ends
- renewal type analysis on micro scale
- cluster expansions + renormalisation
- key property: to massgap positivity [Chayes-Chayes ’86, Ioffe ’98]

$\Rightarrow$ Ornstein-Zernike asymptotics [LLT]
SAP: Step 1

Key result:

\[
Z_Q^{a,b} = \sum_\omega \exp\{-\beta |\omega|\} \approx c \frac{\sqrt{D}}{Q} \exp\{-W(a, b, Q)\},
\]

as \( Q \to \infty \), where \( D = \det \text{Hess}_{a,b} W(\cdot, \cdot, Q) \).

[LLT; Ornstein-Zernike behaviour; cf. A.Pellegrinotti's talk!]

Also:

surface energy \( W(\cdot, \cdot, Q) \) is symmetric, smooth, strictly convex.
Step II: macroscopic variational problem

Minimise

\[ W(a, b, Q_1) + W(b, c, Q_2) + W(c, d, Q_3) + W(d, a, Q_4) \]

subject to

\[ Q_1 = Q_3, \quad Q_2 = Q_4, \quad Q_1 + Q_2 + Q_3 + Q_4 = Q. \]

symm. + convexity \[\implies a = b = c = d \equiv \bar{\rho} \]

smoothness \[\implies Q_1 = Q_2 = Q_3 = Q_4 \equiv Q/4 \]

\[\implies\] optimal shape – \textbf{Wulff droplet}, area \( Q \), “diag. diam.” \( 2\bar{\rho} \)
Asymptotics of $Z_Q^\beta$

III step: sum over all contours in $Z_Q^\circ$

$$P_\beta(Z_Q^\circ) \approx c \ Q^{-5/2} \exp\{-\sqrt{QW_1}\}.$$  

IV step: local entropy factor $\approx \bar{\rho}^2 \approx Q$.

**Thm 1:** $\forall \beta > \beta_{cr}$ fixed, as $Q \to \infty$:

$$Z_Q^\beta = \sum_{\omega \in Z_Q^\circ} e^{-\beta|\omega|} \approx c \ Q^{-3/2} \exp\{-\sqrt{QW_1}\}.$$
Consider $X_Q(s), s \in [0, \bar{l}], \text{ — } Q^{-1/4}$-scaled normal fluctuations of $\omega \in \mathcal{Z}_Q^\odot$ around the Wulff shape $\mathcal{W}_Q$ at the origin.
SAP: fluctuations

Let
\[ d\xi(s) = \frac{1}{\sqrt{\beta(\tau_\beta(\vec{n}_s) + \tau''_\beta(\vec{n}_s))}} dw(s) \]
\[ \bar{l} \text{ – length of } \partial \mathcal{W}_1 \]
\[ q_i = \int_{\gamma_i} \xi(s) \, ds \]
\[ x(s) \text{ – periodic Gibbs field on } T_{[0,\bar{l}]} \text{ (i.e. periodic extension of } \xi(s)) \]
on the event
\[ \circ \equiv \{ q_1 = q_3, q_2 = q_4, q_1 + q_2 + q_3 + q_4 = 0 \} \]

**Thm 2**: \( \forall \text{ fixed } \beta > \beta_{cr} \), as \( Q \to \infty \), the distribution of \( X_Q(s) \) converges to that of \( x(s) \).
possible extensions to other 2D models:

- low-temperature Ising model [in preparation]
- finite range (Pirogov-Sinai?) models
- Ising at all subcritical temperatures [with R.Kotecký, D.Ioffe; in progress]
- moderate deviations for the Ising model [with R.Kotecký, D.Ioffe; in progress]
- ...
Conclusions

main results:

• sharp asymptotics of the partition function
• functional CLT for normal fluctuations from the Wulff shape
• results valid for all $\beta > \beta_{cr}$

main tools:

• variational calculus [Wulff construction]
• analytic perturbations theory [cluster expansions]
• local limit theorems
• renormalization [mesoscopic scale]
• . . .