Matroids and Resummation

Matroids and Resummation

- What is a matroid?
- What is it good for?

- A matroid is a collection of subsets with nice properties.
- These subsets can index interesting sums.

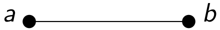
Outline

Matroids and Resummation

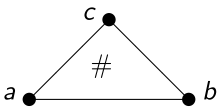
- Abstract simplicial complexes
- Matroids
- Resummation
- Uniform matroids and inclusion-exclusion
- Graphic matroids and tree bounds

What is a simplex?

A simplex consists of all subsets of a set Y.
The dimension of the simplex is $\#Y - 1$.
Example of a 1 dimensional simplex:
face set $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.
facet set $\{\{a, b\}\}$

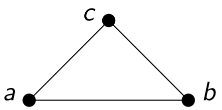


Example of a 2 dimensional simplex:
face set $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$.
facet set $\{\{a, b, c\}\}$

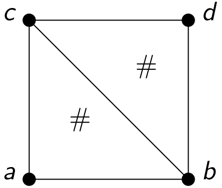


What is a complex?

A complex on *ground set E* is a family of subsets that is downward closed.
Example of a 1 dimensional complex:
face set $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}\}$.
facet set $\{\{a,b\}, \{b,c\}, \{a,c\}\}$.



Example of a 2 dimensional complex:
face set
$\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{b,c\}, \{a,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{b,c,d\}\}$.
facet set $\{\{a,b,c\}, \{b,c,d\}\}$.



Abstract complexes

If Y is a set, then the *power set*

$$P[Y] = \{X \mid X \subseteq Y\}.$$

A *downward closed* collection $\mathcal{L}$ is a collection $\mathcal{L} \subseteq P[E]$ such that

- $Y \in \mathcal{L}$, $X \subseteq Y$ implies $X \in \mathcal{L}$.

A *complex* Δ is a *ground set* E together with a collection $\mathrm{face}(\Delta) \subseteq P[E]$. The only axiom is

- $\mathrm{face}(\Delta)$ is downward closed.

A set $X \subseteq E$ is a *face* of Δ if $X \in \mathrm{face}(\Delta)$.
A set $X \subseteq E$ is a *facet* of Δ if it is a maximal face of Δ.
The *dimension* of a face X is

$$\dim(X) = \#X - 1.$$

For each ground set E there is a *void complex* with no faces.
If the complex Δ is not void, then $\emptyset \in \mathrm{face}(\Delta)$.
If Δ is not void, then

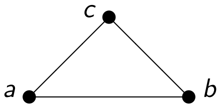
$$\dim(\Delta) = \max\{\dim(X) \mid X \in \mathrm{facet}(\Delta)\}.$$

The Euler characteristic

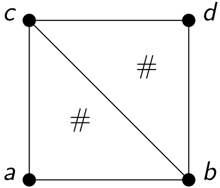
Define the *reduced Euler characteristic* of Δ by

$$\chi(\Delta) = \sum_{Y \in \mathrm{face}(\Delta)} (-1)^{\dim(Y)}.$$

Previous example of a 1 dimensional complex:
face set $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}\}$.
$\chi(\Delta) = -1 + 3 - 3 = -1$.

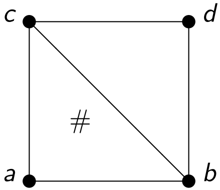


Previous example of a 2 dimensional complex:
face set
$\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{b,c\}, \{a,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{b,c,d\}\}$.
$\chi(\Delta) = -1 + 4 - 5 + 2 = 0$.

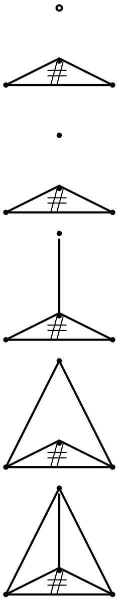


Pure complexes

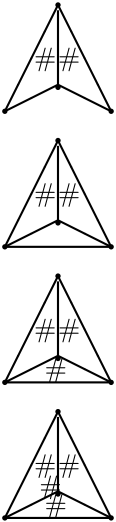
A complex is *pure* if every facet has the same dimension.
Example of a 2 dimensional complex that is not pure:
face set
$\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{b,c\}, \{a,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}\}$.
facet set $\{\{a,b,c\}, \{b,d\}, \{c,d\}\}$.
$\chi(\Delta) = -1 + 4 - 5 + 1 = -1$.



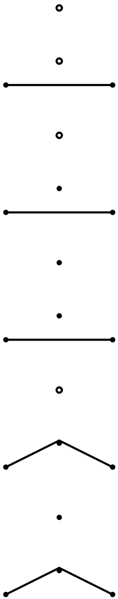
Complexes: 4 point ground set, dim$= 2$



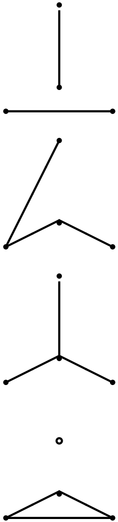
Complexes: 4 point ground set, dim= 2 (more)



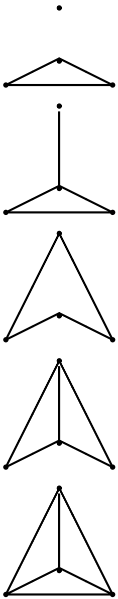
Complexes: 4 point ground set, dim= 1



Complexes: 4 point ground set, dim= 1 (more)



Complexes: 4 point ground set, dim= 1 (yet more)



What is a matroid?

The complex Δ is *pure* if it is not void and every facet has the same dimension.
If Δ is a complex with ground set E, and $K \subseteq E$, then the *restriction* $\Delta|K$ is the complex with ground set K and

$$\mathrm{face}(\Delta|K) = \{X \subseteq K \mid X \in \mathrm{face}(\Delta)\}$$

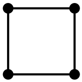
A *matroid* is a complex $\Delta = (E, \mathrm{face}(\Delta))$ that satisfies the following additional axiom.

- For every $K \subseteq E$, the restricted complex $\Delta|K$ is pure.

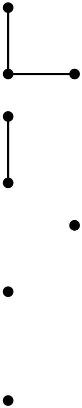
For matroids the terminology is different. Suppose M is a matroid. Then

- A face is an *independent set*. $\mathcal{I}(M)$ is the collection of independent sets.
- A facet is a *basis*. $\mathcal{B}(M)$ is the collection of bases.

Restrictions of a 1-dimensional matroid



Restrictions from 4 points to 3,2,2,1 point subsets:



Geometric representation of matroids

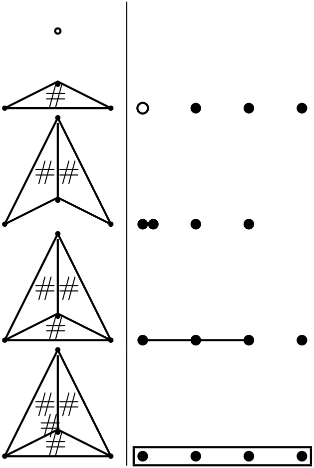
For $X \subseteq E$ define $\dim(X) = \dim(M|X)$. Every subset has a dimension.
If X is independent, then $M|X = P[X]$ and $\dim(X) = \#X - 1$.
Every matroid has a geometric representation. This involves:

- Dependent points of dimension -1.
- Multiple points of dimension 0.
- Lines of dimension 1.
- Planes of dimension 2.
- Etc.

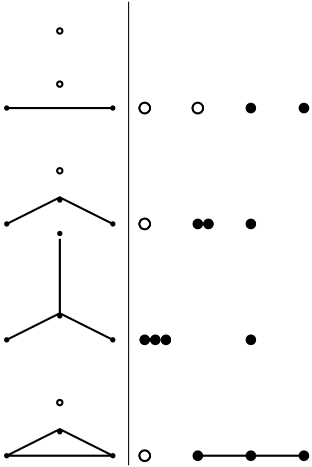
A set X is dependent if it has:

- 1 or more dependent points.
- 2 or more points in a multiple point class.
- 3 or more points in a line.
- 4 or more points in a plane.
- Etc.

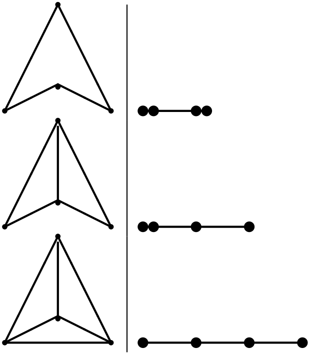
Matroids: 4 point ground set, dim= 2



Matroids: 4 point ground set, dim= 1

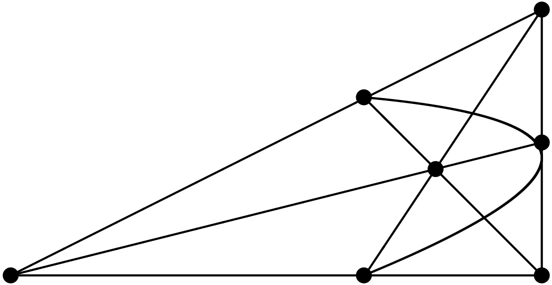


Matroids: 4 point ground set, dim= 1 (more)



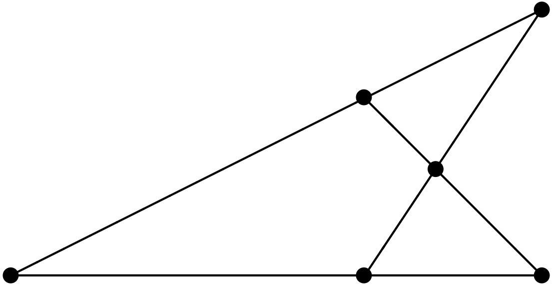
The 7-point projective plane

7 points, 7 lines in plane
two-dimensional: $\binom{7}{3} - 7 = 35 - 7 = 28$ bases each with 3 points



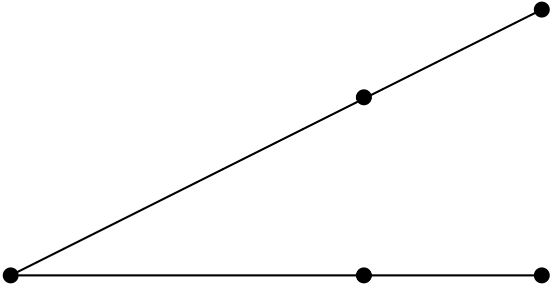
The 6-point complete quadrilateral

6 points, 4 lines in plane
two-dimensional: $\binom{6}{3} - 4 = 20 - 4 = 16$ bases each with 3 points



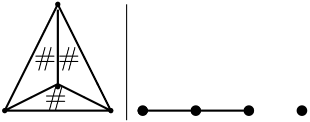
The 5-point intersecting lines

5 points, 2 lines in plane
two-dimensional: $\binom{5}{3} - 2 = 10 - 2 = 8$ bases each with 3 points

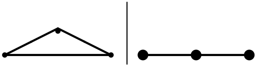


4 and 3 point restrictions

4 points, 1 line in plane.
Two-dimensional: $\binom{4}{3} - 1 = 4 - 1 = 3$ bases (3 points).



3 points, 1 line.
One-dimensional: $\binom{3}{2} = 3$ bases (2 points)



Independent sets, bases, spanning sets

For each matroid M there are various collections of sets

- $X \in \mathcal{I}(M)$ is an *independent set* if X is a subset of a basis.
- $X \in \mathcal{B}(M)$ is a *basis* if it is a maximal independent set or a minimal spanning set.
- $X \in \mathcal{S}(M)$ is a *spanning set* if it is a superset of a basis.

Matroid duality

For each matroid on E there is a *dual matroid* M^*.

- $X \in \mathcal{I}(M)$ is equivalent to $E \setminus X \in \mathcal{S}(M^*)$.
- $X \in \mathcal{B}(M)$ is equivalent to $E \setminus X \in \mathcal{B}(M^*)$.
- $X \in \mathcal{S}(M)$ is equivalent to $E \setminus X \in \mathcal{I}(M^*)$.

For an example, take the matroid M with ground set $E = \{1, 2, 3, 4\}$. It may be characterized by:

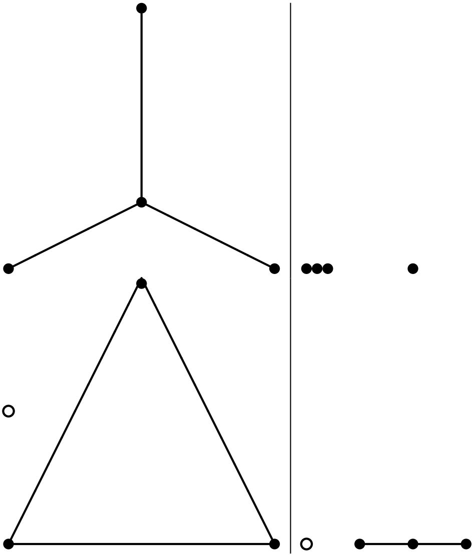
- $\mathcal{B}(M) = \{\{1, 3\}, \{2, 3\}, \{3, 4\}\}$.

$\chi(M) = -1 + 4 - 3 = 0$.
The dual matroid has ground set $E = \{1, 2, 3, 4\}$.

- $\mathcal{B}(M^*) = \{\{2, 4\}, \{4, 1\}, \{1, 2\}\}$.

$\chi(M^*) = -1 + 3 - 3 = -1$.

Dual matroids



Resummation and topology

Let M be a matroid. Then there exists a function $R : \mathcal{B}(M) \to \mathcal{I}(M)$ with

$$\sum_{Y \in \mathcal{I}(M)} \prod_{\ell \in Y} t_\ell = \sum_{X \in \mathcal{B}(M)} \left(\prod_{\ell \in R(X)} t_\ell \prod_{\ell \in X \setminus R(X)} (1 + t_\ell) \right).$$

Recall the *reduced Euler characteristic* of M:

$$\chi(M) = \sum_{Y \in \mathcal{I}(M)} (-1)^{\dim(Y)}.$$

Take all $t_\ell = -1$. Then

$$\chi(M) = (-1)^d \# \{X \in \mathcal{B}(M) \mid R(X) = X\},$$

It follows that

$$|\chi(M)| \leq \# \mathcal{B}(M).$$

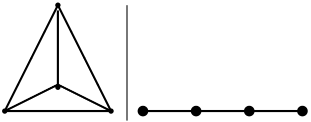
The reduced Euler characteristic is bounded by the number of facets. For a matroid, the topology is dominated by top dimension.

The uniform matroid $U_{k,n}$

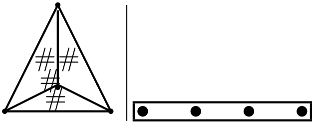
The *uniform matroid* is $M = U_{k,n}$. It has ground set E with $\#E = n$.

- Independent sets: $\mathcal{I}(U_{k,n}) = \{X \mid \#X \leq k\}$
- Basis sets: $\mathcal{B}(U_{k,n}) = \{X \mid \#X = k\}$
- Spanning sets: $\mathcal{S}(U_{k,n}) = \{X \mid \#X \geq k\}$

$U_{2,4}$



$U_{3,4}$



Euler characteristic of $U_{k,n}$

$$\chi(U_{k,n}) = \sum_{j=0}^{k} (-1)^{j-1} \binom{n}{j}.$$

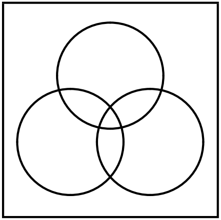
$$|\chi(U_{k,n})| \leq \binom{n}{k}.$$

Illustrates: For a matroid, the topology is dominated by top dimension.

Inclusion-exclusion

Take each $c_\ell = 0$ or 1.
Inclusion-exclusion with three sets:

$$(1 - c_1)(1 - c_2)(1 - c_3) = 1 - c_1 - c_2 - c_3 + c_2 c_3 + c_1 c_3 + c_1 c_2 - c_1 c_2 c_3$$



Inclusion-exclusion with n sets.

$$\prod_{\ell \in E} (1 - c_\ell) = \sum_{Y \subseteq E} (-1)^{\# Y} \prod_{\ell \in Y} c_\ell .$$

Resummation for spanning sets

There exists a function $S : \mathcal{B}(M) \to \mathcal{S}(M)$ with

$$\sum_{Y \in \mathcal{S}(M)} \prod_{\ell \in Y} t_\ell = \sum_{X \in \mathcal{B}(M)} \left(\prod_{\ell \in X} t_\ell \prod_{\ell \in S(X) \setminus X} (1 + t_\ell) \right) .$$

A huge sum of products is written as a smaller sum of products.
Rewrite with $c_\ell = -t_\ell$.

$$\sum_{Y \in \mathcal{S}(M)} (-1)^{\# Y} \prod_{\ell \in Y} c_\ell = (-1)^k \sum_{X \in \mathcal{B}(M)} \left(\prod_{\ell \in X} c_\ell \prod_{\ell \in S(X) \setminus X} (1 - c_\ell) \right) .$$

The tail for $U_{k,n}$

For each $\ell \in E$ let c_ℓ be 0 or 1. Take $t_\ell = -c_\ell$ in the distributive law. Then

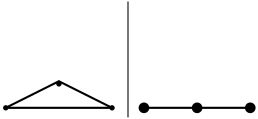
$$\sum_{\#Y \geq k} (-1)^{\#Y} \prod_{\ell \in Y} c_\ell = (-1)^k \sum_{\#X = k} \left(\prod_{\ell \in X} c_\ell \prod_{\ell \in S(X) \setminus X} (1 - c_\ell) \right).$$

The right hand side has known sign $(-1)^k$. This is the Bonferonni result for the tail in the inclusion-exclusion formula. This also gives an estimate for the tail of an alternating sum.

$$|\sum_{\#Y \geq k} (-1)^{\#Y} \prod_{\ell \in Y} c_\ell| \leq \sum_{\#X = k} \prod_{\ell \in X} c_\ell.$$

The tail for $U_{2,3}$

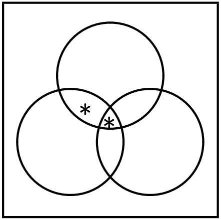
$U_{2,3}$ has spanning sets $\{2,3\}, \{1,3\}, \{1,3\}, \{1,2,3\}$.
$U_{2,3}$ has basis sets $\{2,3\}, \{1,3\}, \{1,3\}$.



Take each $c_\ell = 0$ or 1.
The tail is a sum over spanning sets; write as sum over basis sets:

$$c_2c_3 + c_1c_3 + c_1c_2 - c_1c_2c_3 = c_2c_3 + c_1c_3 + c_1c_2(1 - c_3)$$

All terms are positive.



The graphic matroid $M(K_n)$

Let V be a set with $\#V = n$ points (particles).
Let let E be the set of all two-element sets of V, so $\#E = \binom{n}{2}$. The *complete graph* K_n has vertex set V and edge set $E(K_n) = E$.
A *simple graph* G on *vertex set* V is specified by its *edge set* $E(G) \subseteq E$.
Define the matroid $M = M(K_n)$ with ground set E and

- $X \in \mathcal{I}(M)$ iff $X = E(G)$ for G a forest graph (no cycles) on V.
- $X \in \mathcal{B}(M)$ iff $X = E(G)$ for G a tree graph on V.
- $X \in \mathcal{S}(M)$ iff $X = E(G)$ for G a connected graph on V.

We have $\dim(M) = n - 1$ and $\#\mathcal{B}(M) = n^{n-2}$. Note that $\#\mathcal{S}(M) \geq 2^{\binom{n}{2}-(n-1)}$.
If E has n points, the reduced Euler characteristic is

$$\chi(M) = (-1)^{n-1}(n-1)!.$$

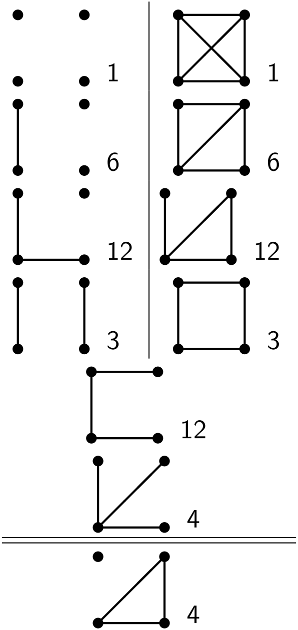
The graphic matroid $M(K_4)$

The vertex set has $\#V = 4$; the edge set has $\#E = 6$.
An edge in the graph is a point in the matroid.
The number of $X \subseteq E$ with $\#X = 0, 1, 2, 3, 4, 5, 6$ is $1, 6, 15, 20, 15, 6, 1$.

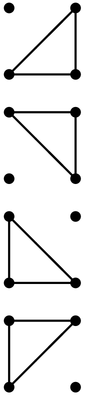
- Trees: $\#\mathcal{B}(M) = 4^2 = 16$ (of 20).
- Forests: $\#\mathcal{I}(M) = 1 + 6 + 15 + 16 = 38$.
- Connected graphs: $\#\mathcal{S}(M) = 1 + 6 + 15 + 16 = 38$.

Each tree has 3 edges; that is, each basis has three points.

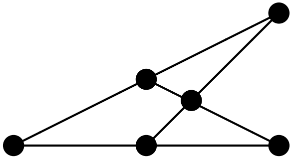
Independent sets, spanning sets, bases in $M(K_4)$



Three-element non-bases in $M(K_4)$



Replace edges by points, triangles by lines:



The graphic matroid $M(K_5)$

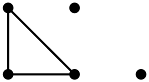
The vertex set has $\#V = 5$; the edge set has $\#E = 10$.
An edge in the graph is a point in the matroid.

- Four-edge trees are four-point basis sets.
- The number of independent four-point sets is $\#\mathcal{B}(M) = 5^3 = 125$.

- Geometric description: 3 edge triangles are 3 point lines; 6 edge complete subgraphs are 6 point planes.
- There are 10 lines and 5 planes.
- A four-point set is dependent if it has 3 collinear points, or 4 non-collinear points in a plane.
- The number of dependent four-point sets is $10 \cdot 7 + 5 \cdot 3 = 85$.

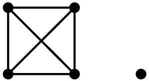
Lines and planes in $M(K_5)$

10 of these:

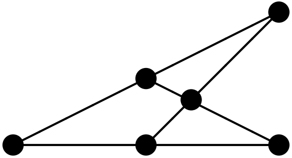


Replace edges by points, triangles by lines:

5 of these:

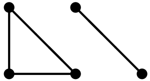


Replace edges by points, triangles by lines:



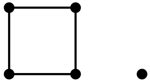
Four-element non-bases in $M(K_5)$

70 with three of four points on a line:

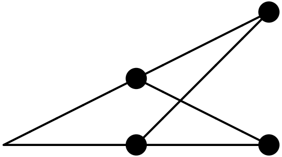


Replace edges by points, triangles by lines:

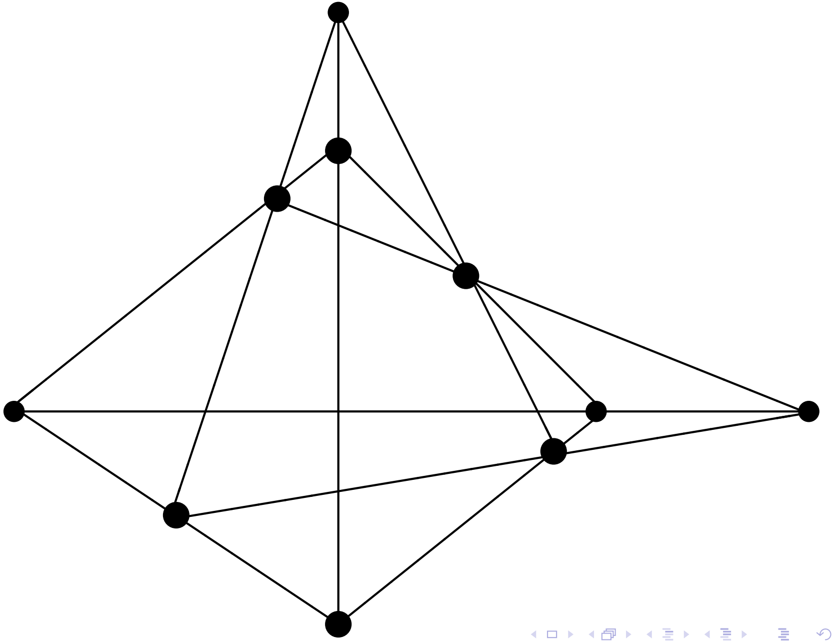
15 with four of four points in a plane:



Replace edges by points, triangles by lines:



Geometry of $M(K_5)$



Penrose tree bound for connected graph sums of $M(K_n)$

Connected graph sums are used to calculate pressure, density, etc. Here $E = E(K_n)$ is the set of pairs of particles, ℓ denotes a pair of particles, $0 \leq V_\ell \leq +\infty$ is their potential energy, β is the inverse temperature, $\exp(-\beta V_\ell)$ is a Boltzmann factor. A key quantity is

$$t_\ell = \exp(-\beta V_\ell) - 1.$$

Thus $-1 \leq t_\ell \leq 0$.

$$\sum_{G \in \mathrm{conn}[V]} \prod_{\ell \in E(G)} t_\ell = \sum_{T \in \mathrm{tree}[V]} \left(\prod_{\ell \in E(T)} t_\ell \prod_{\ell \in S(E(T)) \setminus E(T)} (1 + t_\ell) \right).$$

$$\Big| \sum_{G \in \mathrm{conn}[V]} \prod_{\ell \in E(G)} t_\ell \Big| \leq \sum_{T \in \mathrm{tree}[V]} \prod_{\ell \in E(T)} |t_\ell|.$$

The connected graph sum is bounded by the tree sum.

The fundamental theorem of calculus for a matroid

- William G. Faris, The fundamental theorem of calculus for a matroid, J. Math. Phys. 53, 063305 (2012).
- A more sophisticated resummation: involves iterated fundamental theorem of calculus for many variables.
- Generalizes work of Brydges-Kennedy-Abdesselam-Rivasseau.