The role of zero-energy resonances in Quantum Mechanics

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Zero energy resonances play an important role in the study of quantum mechanical systems.

A well known example is the Efimov effect: if particles A,B,C interact via sufficiently regular two body potentials, if the spectrum of each two body sub-system is positive and there is a zero-energy resonance in at least two of the channels, then there are infinitely many eigenvalues which accumulate at zero.
Moreover the eigenvalues satisfy asymptotically the relation $E_n = E_0 e^{-cn}$ where the constant c is a function only of the masses. The eigenfunctions become progressively flatter (it is an infrared phenomenon).

Another interesting effect is this: when a suitably chosen *squeezing* parameter ϵ converges to zero, the presence of a zero energy resonance can change the structure of the limit Schroedinger equation.

For example it provides a boundary condition different from decoupling for the Schroedinger equation on a metric graph if considered as limit of a tubular neighborhood of radius ϵ with Dirichlet boundary conditions.

As another interesting case, consider a particle in R^3 subject to a potential of Rollnik class.
Suppose the hamiltonian is positive and there is a zero energy resonance.
If one scales the potential according to $V^\epsilon(x) = \frac{1}{\epsilon^2} V(\frac{x}{\epsilon})$ it was proved by S.Albeverio and R.Hoegh-Krohn [AKH] that the limit system is described by a *point interaction.*
Point interactions are self-adjont extensions of the symmetric operator $-\Delta$ defined on functions that have support away from the chosen point.
The convergence is in norm resolvent sense.

Notice that the above scaling leaves invariant the stationary Schroedinger equation, maps zero energy resonances to zero energy resonances and leaves invariant the Rollnik norm of the potental.
Under this scaling the L^1 norm of the potential goes to zero, and $\frac{1}{\epsilon} V^\epsilon$ converges in distribution to a delta function.

One can ask if also in the Efimov case one can obtain a contact interaction among three particles as limit of a sequence of potentials V^{ϵ} which satisfy the conditions of Efimov and are scaled as in the case considered in [AHK] (we shall use the name *contact interactions* for the case in which the limit singular set has dimension greater than zero).

This question was considered by Makarov and Melezlik [MM] who hinted to a positive answer.
From the historical point of view, first came the analysis of contact interaction of three particles using the theory of Krein-von Neumann of self-adjoint extension of the Laplacian defined on functions that vanish in a neighborhood of the hyper-planes $\Gamma_{i,j},\ i \neq j$ defined by $x_i = x_j$.
For the case of three particles Minlos and Faddaev [MF] were able to construct, on the Hilbert space of square-integrable permutation invariant and spherically symmetric functions, a two-parameter family $H^{\alpha,\beta}$ of self-adjoint extensions.
The parameter β takes value in $[0, 2\pi)$ and the construction is in analogy to Weyl's limit circle case.

These extensions *are all unbounded below,* with infinitely many negative eigenvalues $E_n^{\alpha,\beta}$.
The eigenvalue satisfy asymptotcally $|E_n^{\alpha,\beta}| = e^{cn}$ where the constant c depends only on the masses of the particlaes. This behavior is exact for $\alpha = 0$; for $\alpha \neq 0$ the behavior survives at infinity because the difference is a bounded operator.
Notice that the point spectra σ_p of $H^{\alpha,\beta}$ are disjoint for different values of β and $\cup_\beta \sigma_p(H^{\alpha,\beta}) = (-\infty, 0)$

One refers to this effect as Thomas effect [Th]. It is called also *fall to the centre* since as the eigenvalues become more negative the corresponding eigenfunctions are more and more concentrated at the barycenter.

For a N-particles system we will call *Thomas effect* the fact that the energy spectrum is unbounded below.

In the Physics literature it goes under the name *instability.*

At the suggestion of Faddaev, who was aware of the effect discovered by Efimov, Yafaev [Ya1], [Ya2] considered the case of three particles with *regular* potentials with a zero energy resonance in at least two channels, and proved that there are infinitely many eigenvalues which accumulate at zero.

Eventually this led to the mathematical proof of the Efimov effect given independently by Sigal [Si] and Tamura [Ta] .

Later Vugaltier and Zihslin [VZ] proved that the Efimov effect *is not present* in the space of functions that are antisymmetric under permutation of the coordinates of two of the particles.
This is due essentially to the fact that the eigenfunctions corresponding to the "tail " of the negative discrete spectrum have corresponding eigenfuctions which are symmetric; requiring antisymmetry forces the kinetic energy to increase and suppresses the Efimov effect.
It is not a priori evident that the same phenomenon occurs in the case of contact interaction among three bodies; however since the increase of the kinetic energy is of the same order as the negative part of the potential one is led to suspect that also in this case the Thomas effect is suppressed by the requirement of antisymmetry.

A slightly more general version of contact interactions is formulated in [DTF] using the theory of quadratic forms.
In this approach, the interaction of a system in R^{3N} is described by the sum of a regular hamiltonian H_0 (typically a free one) and a quadratic (sesquilinear) form $\mathcal{Q}$ defined in an auxiliary space $\mathcal{K}$ (dubbed space of charges in [DFT]) of codimension 3, together with a map $\mathcal{K} \to R^{3N}$ having as kernel the free Green function.
The dynamics is described *formally* by $H_0 + G_0 \mathcal{Q} G_0$.
The space $\mathcal{K}$ is a space of functions on $\cup_{i \neq j} \Gamma_{i,j}$ where $\Gamma_{i,j} = \{X \ : x_i = x_j\}$.
One may visualize the construction by analogy with the usual construction of an elliptic operator in R^3 in presence of a boundary surface Γ.
In that case the space $\mathcal{K}$ is a space of functions on Γ and the bilinear form represents the Weyl-Titchtmarsh operator (Dirichlet to Neumann map).
The main difference is that the co-dimension of the singular set is now three and Sobolev embedding is not sufficient to guarantee that a function in the domain of the laplacian has boundary value in $\mathcal{H}^1$.
This enlarges the set of possible self-adjoint extension; our approach, like [MF] , considers only part of them.

In general for the case of N particles with contact interactions the form $\mathcal{Q}$ is not semibouded and cannot be associated to a self-adjoint operator.
However its restriction to a suitable subspace may define a self-adjoint operator; through [MF] we know that this is the case e.g. when $N = 3$ and the subspace is chosen to be a subset of the set of functions invariant under rotations and under permutation of the particle index.
If the restriction of the form $\mathcal{Q}_\Sigma$ to a suitable space Σ is closed and semi-bounded it defines a self-adjoint operator which we shall denote by K_Σ.
We shall denote by H_Σ the hamiltonian $H_0 + G_0 K_\Sigma G_0$.
The theory of contact interaction as presented in [DFT] is a model in which all relevant quantities (spectrum, scattering data, spectral shift function,..) can be evaluated explicitly. Its relevance for Physics comes from the fact that this simple tool is useful, as we shall see , to capture the main features of N-body interactions when the range of the potential is very short and the scattering length very large (some subsystem is near a zero-energy resonance).
These feature do not depend on the details of the shape of the interaction potential and therefore the model has some features of *universality*.

We will prove the following.
Given a N-body system in R^3 with two body interactions and hamiltonian $H_\epsilon^N = H_0 + \sum_{i\neq j} V_{i,j}^\epsilon$, where the potentials are scaled as before.
If all two-body channels (two-body subsystems) have positive energy spectrum and in at least two of them there is zero-energy resonance, the the bilinear form associated to the difference of resolvents of H^ϵ and of H_0 in the formalism of Krein and Kato (we will describe it presently) converges weakly to the bilinear form used in [DFT] to describe point interactions.

If the corresponding point interaction Hamiltonian H_Σ^N exists in some subspace Σ, one has in Σ *strong resolvent convergence* when $\epsilon \to 0$ of the family of hamiltonians H_ϵ^N to H_Σ^N.

Strong resolvent convergence *is the strongest result* one may expect for N-body systems. It implies in particular convergence of all scattering quantities.
This result makes the contact interaction model a very useful tool in the case of interactions with a very small range and a very large scattering length (i.e. in presence of zero energy resonances in some channels).
Notice that this *is not a perturbative result* (the presence of a resonance is fragile under perturbation) but rather *an approximation,* using as small parameters the range of the potential and the inverse of the scattering length.

In general, *the singular set* $\Gamma = \cup \Gamma_{i,j}$ *is given by the singularities of the potential* in the limit $\epsilon \to 0$ and the self-adiont extension obtained *depends only on the shapes of the zero-energy resonances* which are present in the subsystems.

We will see that the specific assumptions on the zero energy resonances we have imposed lead to a specific subset of self-adjoint extensions which traditionally are labeled by the name Skorniakov- Ter-Martirosian [ST]

Other self-adjont extension are obtained by choosing potentials such that the N-body system has a different set of channels with a zero-energy resonance.

For example, if the N-body system has a positive hamiltonian with a zero energy resonance and no resonances in the M-body channels for $M < N$, then the limit when $\epsilon \to 0$ of H^N_ϵ is the self-adjiont extension described (in term of a positive quadratic form) by Albeverio et al. [AHKS].

This may correspond to a *effective* three-body interaction and can be related to a Wigner-Wiesskopf resonance for an effective $\frac{1}{R^2}$ potential, as often stated in the Theoretical Physics literature.
For this extension Frank and Seiringer [FS] have proved recently the Lieb-Thirring inequality; the proof could also be given proving that the inequality holds uniformly in ϵ.
In view of the result by Vugalter and Zishlin a natural candidate for a restricted space Σ is the space of functions that are antisymmetric with respect to the exchange of a subset of coordinates.
Since there is no contact interaction between two identical fermions (the wave function is zero at coinciding points) a *natural candidate* is the system of two species of (scalar) fermions, or equivalently a collection of spin $\frac{1}{2}$ fermions.
It must of course be proved that this *natural candidate* gives indeed an self-adjoint extension: one must prove that the corresponding bilinear form is bounded below for all N.
In view of our convergence result, the proof that H_Σ^N is bounded below if Σ is the singularity set for a family of N spin $\frac{1}{2}$ fermions could be given by proving that the lower bound of the energy for the corresponding system with a regular potential is uniform in ϵ.

REMARK 1

The existence of self-adjoint extensions different from those in [ST] has been advocated recently by Minlos for the case of three particles but their structure has not been studied so far.
There is no strong evidence (although some hints may be found) that these other extensions are a sign of a three body *effective potential.*
It is worth noticing that some of the functions in the domain of this effective hamiltonian *are not* in the regularity class on which the M-F extensions are constructed.

♣

REMARK 2

It is interesting to notice that the quadratic form given in [DFT] is the result one obtains if one does perturbation expansion *at the lowest relevant order* and uses formally as potential the sum of the δ functions which characterize the hyperplanes.
This is the prescription Fermi used to describe point interaction for two particles.
This has the following "explanation".
A *contact interaction* ,when described by a hamiltonian, can be viewed as limit in resolvent sense of the interaction through a family of two body potentials $V^{\epsilon}_{i,j}$ which have a zero-energy resonance and are such that $lim_{\epsilon\to 0}\|V^{\epsilon}_{i,j}\|_1 = 0$ but $\frac{1}{\epsilon}V^{\epsilon}_{i,j}$ converge in distribution to $c_{i,j}\delta(x_i - x_j)$.
The family of potentials has uniformly bounded Rollnik norms and therefore perturbation theory gives a uniformly convergent series for the resolvent.
Since $lim_{\epsilon\to 0}\|V^{\epsilon}_{i,j}\|_1 \to 0$ if there are no resonances, each term of order ≥ 1 in the series converges to zero.
The zero-energy resonance provides a factor proportional to ϵ^{-1}; as a result *the terms of order one in* ϵ have now a finite limit.
They correspond *formally* to *a delta potential* since $\epsilon^{-1}V^{\epsilon}_{i.j}$ converges in distribution to $c_{i,j}\delta(x_i - x_j)$

♣

In the remaining part of my talk I will give some details of the approximation of contact interactions by a sequence of potentials of decreasing support which have a zero energy resonance.

There are two main steps in the proof.

One is *localization at the boundary* and it is clearly seen in the two body case [AHK].

The other is *decoupling of the channels* in the Konno-Kuroda formalism.

We review first the case $N = 2$ as described in [AHK].

Consider the family of hamiltonians

$$H_\epsilon = -\Delta + V_\epsilon(x) \qquad V_\epsilon(x) = \frac{\lambda(\epsilon)}{\epsilon^2} V(\frac{x}{\epsilon}) \tag{1}$$

Set

$$v_\epsilon(x) = \sqrt{|V_\epsilon(x)|} \qquad u_\epsilon(x) = sign(V_\epsilon(x))\sqrt{|V_\epsilon(x)|} \tag{2}$$

In the following *we take $\lambda(\epsilon)$ to be constant.*
If it is not constant, the limit will depend on $\lambda'(0)$.

One uses now the general resolvent formula (Krein-Milman, Kato, Konno-Kuroda [KK])

$$\frac{1}{(-\Delta + V) - k^2} - \frac{1}{-\Delta - k^2} = G_k v \frac{1}{1 + uG_k v} u G_k \tag{3}$$

The scaling of space is obtained with unitary operator U_ϵ defined by $U_\epsilon V(x) U_\epsilon^* = V(\frac{x}{\epsilon})$. One has, with

$$U_\epsilon \Delta U_\epsilon^* = \epsilon^2 \Delta, \quad U_\epsilon G_k U_\epsilon^* = \frac{1}{\epsilon^2} G_{\frac{k}{\epsilon}} \quad G_k = \frac{1}{H_0 - k^2} \quad k^2 \in C - R^+ \tag{4}$$

Notice that

$$(\epsilon^{\frac{1}{2}} U_\epsilon G_{\epsilon k} v f)(x) = \epsilon^{\frac{3}{2}} \int U_\epsilon G_{\epsilon k}(\epsilon x - \epsilon x') v(x') f(x') d^3x' =$$

$$\int G_k(x - \epsilon x') v(x') f(x') d^3x' \tag{5}$$

When $\epsilon \to 0$ this provides *localization at the boundary*

$$lim_{\epsilon \to 0} \epsilon^{\frac{1}{2}} U_\epsilon G_{\epsilon z} v \to_{weakly} |G_0 >< v| \tag{6}$$

To complete the analysis we have to control the term

$$\frac{\epsilon\lambda}{1 + uG_{\epsilon k}v} \tag{7}$$

Assume that uG_0v has an eigenvalue -1 with eigenfunction ϕ.
Then $\psi = G_0v\phi$ satisfies $(-\Delta + V)\psi = 0$ and if $\psi \notin L^2$ it represents a zero energy resonance.
If the resonance is simple, as we will suppose, the eigenspace of uG_0v to the eigenvalue -1 is one dimensional.
Since uG_0v is compact when V is of Rollnick class, -1 is an isolated eigenvalue and therefore

$$\frac{\epsilon}{1 + uG_0v + \epsilon} = |\phi)(\phi| + \frac{\epsilon}{1 + uG_0v + \epsilon}(1 - |\phi)(\phi|) \rightarrow_{\epsilon\rightarrow 0} |\phi)(\phi| \tag{8}$$

Putting all things together one has for $Imz \neq 0$

$$lim_{\epsilon\rightarrow 0}\frac{1}{H^{\epsilon} - k^2} = \frac{1}{H_0 - k^2} + |G_k)\frac{i|k|}{4\pi}(G_k| \tag{9}$$

The right-hand side is the resolvent of a self-adjoint extension of the symmetric operator $-\Delta$ defined on functions that have support away from the origin.
Since the limit is of rank one, the limit is actually in norm!

We will consider now the many partices case, with two-body resonances.
We consider first the case $N = 3$ expanding on the comments that are given in [AHW].
Particles 1 and 2 interact with particle 0 through a two body potential V.
Faddaev's equation for the resolvent are

$$R(z) = R_0(z) - R_0(z)T(z)R_0(z)$$

$$T(z) = \sum_{\alpha,\beta=1,2} M^{\alpha,\beta}, \qquad M^{\alpha,\beta} = \delta_{\alpha,\beta} - T^{\alpha}(z)R_0(z) \sum_{\gamma\neq\alpha} M^{\gamma,\beta}$$

where $T^{\alpha}(z),\ \alpha = 1, 2$ is the solution of

$$T^{\alpha}(z) = V(x_{\alpha}) - V(x_{\alpha})R_0(z)T^{\alpha}(z) = V(x_{\alpha}) - T^{\alpha}(z)R_0(z))V(x_{\alpha}) \tag{10}$$

Eq. (10) can be written, at least formally, as

$$T^{\alpha}(z) = (1 + R_0(z)V(x_{\alpha})^{-1}V(x_{\alpha})$$

If one wants to have a non-zero limit for T_{α} when scaling the potential as in the case of two particles one must have for each α a solution of the equation

$$R_{\epsilon}(0)V(x_{\alpha})\phi_{\epsilon} = -\phi_{\epsilon} + O(\epsilon)$$

Therefore one must have a zero-energy resonance.
But Faddaev's system is not symmetric and requires further manipulations to find the explicit form of the limit.

A more symmetric presentation is due to Kato and Konno-Kuroda [KK] who generalize previous work by Krein and Birman, for hamiltonians that can be written in the form

$$H = H_0 + H_{int} \qquad H_{int} = B^* A \tag{11}$$

where $B, \; A$ are densely defined closed operators with $D(A) \cap D(B) \subset D(H_0)$ such that , for every z in the resolvent set of H_0, the operator $A \frac{1}{H_0 - z} B^*$ has a bounded extension, denoted by $Q(z)$.
One has then for the resolvent $R(z)$ of H the following expression

$$R(z) - R_0(z) = [R_0(z) B^*](1 - Q(z))^{-1} [A R_0(z)] \tag{12}$$

The total interaction potential is $V(y_1, y_2) = V(y_1) + V(y_2)$ where $y_i = (x_i - x_0)$
We remark that this more symmetric formulation is also used in [Ya] and in the proof of the Efimov effect in [Ta].
One is tempted to use (12) with

$$B_\epsilon = \sqrt{|V_\epsilon|}, \; A_\epsilon = sign V_\epsilon \; \sqrt{|V_\epsilon|}$$

as we have done in the case of two particles. But

$$\sqrt{V(y_1) + V(y_2)} \neq \sqrt{V(y_1)} + \sqrt{V(y_2)} \tag{13}$$

and in doing so one loses separation between the two channels.

Notice however that, if V_1 and V_2 are of Rollnick class in R^3, under the scaling $V \to V^\epsilon(x) = \epsilon^{-2}V(\frac{x}{\epsilon})$ and setting

$$W^\epsilon = [\sqrt{V_1^\epsilon(y_1) + V_2^\epsilon(y_2)} - \sqrt{V_1^\epsilon(y_1)} - \sqrt{V_2^\epsilon(y_2)}] \quad (14)$$

one has

$$lim_{\epsilon\to 0}\|W^\epsilon_{1,2}\| = 0. \quad (15)$$

This is due, roughly speaking, to the fact that the "overlap" vanishes when $\epsilon \to 0$.
Therefore we con consider $W^\epsilon_{1,2}$ as a regular perturbation which goes to zero when $\epsilon \to 0$.
We shall neglect it altogether.
In this way we recover *separation between the two channels* and write the Konno-Kuroda formula with

$$A_\epsilon = \sqrt{V_1^\epsilon} + \sqrt{V_2^\epsilon} \qquad B_\epsilon = signV_1^\epsilon\sqrt{V_1^\epsilon} + signV_2^\epsilon\sqrt{V_2^\epsilon} \quad (16)$$

This is the second main step in the proof.

Remark

Eq. (15) shows that the neighborhood of the multiple coincidence sets *does not play a special role in the limit.* This is coherent with the fact that it plays no special role in the constructions given in [M-F] and in [DFT].

♣

After these preliminaries, we consider the problem of convergence when $\epsilon \to 0$.
We have

$$R_\epsilon(z) - R_0(z) = R_0(z)[\epsilon^{-\frac{1}{2}} B_\epsilon^*] \epsilon (1 - Q_\epsilon(z))^{-1} [\epsilon^{-\frac{1}{2}} A_\epsilon] R_0(z) \tag{16}$$

with $Q_\epsilon(z) = A_\epsilon \frac{1}{H_0 - \epsilon z} B_\epsilon^*$
We can now use the same unitary operators (change if integration variables) as in the case of two particles.
This leads in the limit to localize to the two coincidence hyperplanes the factors $R_0(z)$ on the right and on the left of the right hand side of (16).
The localization on two hyperplanes gives rise to four terms, that we shall call respectively diagonal and off-diagonal.
The factor $\epsilon(1 - Q_\epsilon(z))^{-1}$ vanishes in the limit unless

$$1 - Q_\epsilon(\epsilon z) = \epsilon K + o(\epsilon)) \tag{17}$$

for some invertible operator K.

By assumption there two zero energy "resonant elements"

$$\psi_1 = \xi(x_1 - x_0).1, \qquad \psi_2 = 1.\xi(x_2 - x_0)$$

where $\xi(x)$ are the resonance in two channels.
We assume that there are no further zero energy resonances in the system.
It follows that $\phi_i \equiv B_i \psi_i$ satisfies $(1 - Q(0))\phi_i = 0$
The operator K is therefore the projection over the Hilbert space spanned by ϕ_1 and ϕ_2.
Recalling that $(\psi_i, V_i \psi_i) = (\psi_i, H_0 \psi_i)$ and that $\phi = B_i \psi$ one verifies that the kernel of the operator K is represented by $G_0(x_1, x_2; y_1, y_2)$ where $G_z = \frac{1}{H_0 - z}$
It follows that in the limit $\epsilon \to 0$ the kernel of the operator

$$B_\epsilon^* (1 - Q_\epsilon(z))^{-1} A_\epsilon \tag{18}$$

converges weakly to a bilinear form defined on functions on the coincidence hyperplanes with kernel $G_0(x_1, x_2; y_1, y_2)$.
It is easy to verify that this bilinear form is precisely the bilinear form on the "charges" in the formulation of contact interaction through quadratic forms given in [DFT].
Since the kernels coincide, *if they determine an operator* this operator is the limit, in strong resolvent sense, of the operators H_ϵ.

COMMENTS AND EXTENSIONS

Notice that the kernel of the bilinear form for the charges is invariant along the coincidence hyperplanes; this is due to the fact that the the eigenfunctions corresponding to the zero energy resonance are functions of one variable only. As a consequence the form is unbounded below .

In the case $N=3$ the convergence we have shown above does not imply resovent convergence of the operators, because the limit form does not define an operator.

One has rather, as we know from M-F, a one parameter family of operators $\tilde{H}_\theta, \ \theta \in [0,1)$.

A different limit would be obtained in the case $N = 3$ if there are no two-body resonance but instead one has a genuine three body resonance.

In this case the Konno-Kuroda formalism would give a contribution to the limit coming from the projection corresponding to this "three-body resonance."

The kernel of the bilinear form depends now on the structure of the resonance, is not invariant under translation along the coincidence hyperplanes and may lead to a self-adjoint extension bounded below.

For example the contact interaction described in [AHKS] with positive hamiltonian $H_{contact}$ correspond to the choice of hamiltonians with of two-body potentials which have a N-body zero energy resonance.
In this case it easier to give a sequence of hamiltonians H_ϵ with support and inverse of the scattering length which tend to zero with ϵ.
In this case also the limit contact interaction has a zero energy resonance with resonance function proportional to $\sum_{j \neq i} \frac{1}{|x_i - x_j|}$ (this is also the limit of the zero energy resonance functions $\phi_\epsilon(X)$ for $\epsilon \neq 0$).

Remark

The analysis we have given can be extended to the case of charged particles subject to to a magnetic field $B(x)$ under the (Landau) scaling $B^\epsilon(x) = \frac{1}{\epsilon^2} B(\frac{x}{\epsilon})$.

♣

The analysis given above for the case of three particles interacting pairwise through a family of potentials $V_{i,j}^{\epsilon}$ with a zero-energy resonance and scaled suitably, can be extended to the case of N particles with obvious minor modifications.

In analogy with the case $N = 2$ the *compensator* W_N^{ϵ} is defined by

$$W_N^{\epsilon} = \sum [\sqrt{V_i^{\epsilon}(y_i) + V_j^{\epsilon}(y_j)} - \sum \sqrt{V_i^{\epsilon}(y_i)} - \sqrt{V_j^{\epsilon}(y_j)}]$$

As before, $lim_{\epsilon \to 0} \|W_N^{\epsilon}\| = 0$.

One uses as before the Konno-Kuroda formula (which hods in general) and proves that the kernel of the associated operator converges weakly to the kernel given in the construction made in [DFT] of the N body point interaction. If, possibly restricted to an invariant subspace, this kernel represents a hamiltonian $H_{contact}^{N}$ (of contact interaction among N particles), as remarked before, one has weak (and therefore strong) resolvent convergence of the family of Hamiltonians H_N^{ϵ} to $H_{contact}^{N}$.

A natural restriction to an invariant subspace is obtained dividing the N particles in two species and requiring antisymmetry under exchange of the coordinates of particles of the same specie .
In view of the results of Vugaltier and Zihslin for the three particle case, one can expect that the restriction to this subspace forces the energy to go up, and provides a quadratic form bounded below (and therefore a limit hamiltonian) .
In this case the limit operator $H_{contact}$ is well defined as a self-adjoint operator bounded below.
One has strong resolvent convergence as $\epsilon \to 0$ of the family H^{ϵ} to $H_{contact}$ on the subspace of functions that are antisymmetric under permutation of the coordinates of the identical particles.
This has been proved so far in the case a pair of identical (spinless) fermions of unit mass interacting with a different particle of mass μ ([MiM] and the talk of Sandro Teta) if μ is not too small (1 is allowed) . In this case the limit form is closed and bounded below and defines therefore a self-adjoint operator
I expect that one can generalize this result to the case of a Fermi gas of N spin $\frac{1}{2}$ massive particles for arbitrary finite N.

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