Magnetic response of quantum gases

Stochastic and Analytic Methods in Mathematical Physics

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The system

Consider a ideal quantum gas in $\mathbb{R}^3$ composed of identical, spinless, non-relativistic, charged particles, obeying either the Bose-Einstein or the Fermi-Dirac statistics.

The gas particles moves in a medium

The gas is subjected to an external magnetic field $\mathbf{B} := (0, 0, B), B \geq 0$,

The system is at thermal equilibrium.

A disordered medium

The non-negative Poisson random field.

$$V^{(\omega)}(\mathbf{x}) = \int_{\mathbb{R}^3} \boldsymbol{\mu}^{(\omega)}(\mathrm{d}\mathbf{y})\, u(\mathbf{y}-\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^3,\ \omega \in \Omega,$$

where $\boldsymbol{\mu}^{(\omega)}$ denotes a random Poisson measure on $\mathbb{R}^3$ and $u \in L_0^\infty(\mathbb{R}^3)$; $u \geq 0$, we have $\mathbb{P}$-a.s.

$$(M1) \quad \forall \mathbf{x} \in \mathbb{R}^3, 0 \leq V^{(\omega)}(\mathbf{x}) \leq c(\omega) \ln(1 + |\mathbf{x}|).$$

see e.g. Germinet, Hislop and Klein.

Question: what is the influence of the disorder on the diamagnetism?

Hermann Schulz-Baldes, Stefan Teufel, Orbital polarization and magnetization for independent particles in disordered media, arXiv:1201.4812

See also:

R. Seiringer, J. Yngvason, V. A. Zagrebnov, Disordered Bose Einstein Condensates with Interaction in One Dimension arXiv:1207.7054

Eric Cancès, Salma Lahbabi, Mathieu Lewin, Mean-field models for disordered crystals, arXiv:1203.0402

Crystal

Here the interaction V is periodic, and satisfies

$$(M2) \quad V \in L^p_{\mathrm{loc}}(\mathbb{R}^3), p > 3.$$

- $V \sim \frac{1}{|x|^{1-\alpha}}; 1 > 3\alpha > 0$

-Probabilistic technique $\rightarrow$ V define an $\mathbb{R}^3$-ergodic random field.

Intermediate case : Anderson potential

Let

$$V^{(\omega)}(\mathbf{x}) = \sum_{\mathbf{j}\in\mathbb{Z}^3} \lambda_{\mathbf{j}}(\omega) u(\mathbf{x}-\mathbf{x_j}) \quad \mathbf{x}\in\mathbb{R}^3,\ \omega\in\Omega, \quad g\in\mathbb{R}.$$

$\{\lambda_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^3}$ are i.i.d. random variables and $\sum_{\mathbf{j}\in\mathbb{Z}^3}\left(\int_{\Lambda_1(\mathbf{j})} \mathrm{d}\mathbf{x}\,|u(\mathbf{x})|^p\right)^{\frac{1}{p}} < \infty$, $p > 3$, $\Lambda_1(\mathbf{j})$ denotes the unit cube centered on site $\mathbf{j}$.

$V^{(\omega)}$ is a $\mathbb{Z}^3$-ergodic random field $\rightarrow$ $\mathbb{R}^3$-ergodic random field.

Further models: The random displacements model on $\mathbb{R}^3$ or the quasi-periodic case, see e.g. Figotin-Pastur.

Stable interactions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathbb{E}[\cdot\,] := \int_\Omega \mathbb{P}(\mathrm{d}\omega)(\cdot\,)$ be the associated expectation. We consider random electric potentials,:

(E) $V^{(\omega)}$ is an $\mathbb{R}^3$-ergodic random field

(R) The realizations of $V^{(\omega)}$ are given by:

$$V^{(\omega)}(\mathbf{x}) = V_1^{(\omega)}(\mathbf{x}) + V_2^{(\omega)}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^3,\, \omega \in \Omega,$$

where $\mathbb{P}$-a.s. on Ω:

(R1) $V_1^{(\omega)} \in L^p_{\mathrm{uloc}}(\mathbb{R}^3)$ with $p > 3$.

(R2) $V_2^{(\omega)}$ obeys the conditions:

$$0 \leq V_2^{(\omega)}(\mathbf{x}) \leq c_\alpha(\omega)(1 + |\mathbf{x}|^\alpha) \quad \text{with } \alpha \in (0, \frac{1}{4}) \text{ and } c_\alpha(\omega) > 0.$$

Recall that the space $L^p_{\mathrm{uloc}}(\mathbb{R}^3)$ consists of measurable functions $f : \mathbb{R}^3 \to \mathbb{C}$ satisfying:

$$\|f\|_{1\leq p<\infty,\mathrm{uloc}} := \sup_{\mathbf{x}\in\mathbb{R}^3} \left(\int_{|\mathbf{x}-\mathbf{y}|<1} \mathrm{d}\mathbf{y}\, |f(\mathbf{y})|^p \right)^{\frac{1}{p}} < \infty,$$

and

$$\|f\|_{\infty,\mathrm{uloc}} := \mathrm{ess}\sup_{\mathbf{x}\in\mathbb{R}^3} |f(\mathbf{x})| < \infty.$$

Hamiltonians

The 'one-particle' operator in $L^2(\mathbb{R}^3)$. Define on $\mathcal{C}_0^\infty(\mathbb{R}^3)$:

$$H_\infty(b,\omega) := \frac{1}{2}(-i\nabla - b\mathbf{a})^2 + V^{(\omega)}, \quad b \in \mathbb{R}.$$

Here $B\mathbf{a}(\mathbf{x}) := \frac{B}{2}\mathbf{e}_3 \wedge \mathbf{x} = \frac{B}{2}(-x_2, x_1, 0)$ and $b := \frac{qB}{c} \in \mathbb{R}$.

Then $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$, $H_\infty(b,\omega)$ are essentially self-adjoint operators. Also $H_\infty(b,\omega) := \frac{1}{2}(-i\nabla - b\mathbf{a})^2 + V^{(\omega)}, b \in \mathbb{R}$ is an ergodic family of operators.

Further there exists $E_0 > -\infty$ s.t. $\mathbb{P}$-a.s., $\forall b \in \mathbb{R}$,

$$H_\infty(b,\omega) \geq E_0.$$

If $\Lambda_L = (-L/2, L/2)^3$, $L > 0$. Then we denote by $H_L(b,\omega)$, the restriction of $H_\infty(b,\omega)$ in $L^2(\Lambda_L)$

The pressure

Let $\beta := (k_B T)^{-1} > 0$ and

$$\mathcal{D}_{-1} := \mathbb{C} \setminus [\mathrm{e}^{\beta E_0}, +\infty), \quad \mathcal{D}_{+1} := \mathbb{C} \setminus (-\infty, -\mathrm{e}^{\beta E_0}].$$

$\epsilon = -1$ refers to the bosonic case, $\epsilon = +1$ to the fermionic case.
For $b \in \mathbb{R}$ and $z \in \mathcal{D}_\epsilon \cap \mathbb{R}_+^*$, the finite-volume pressure :

$$P_L^{(\omega)}(\beta, b, z, \epsilon) := \frac{\epsilon}{\beta |\Lambda_L|} \mathrm{Tr}_{L^2(\Lambda_L)} \ln(\mathbb{I} + \epsilon z \mathrm{e}^{-\beta H_L(b,\omega)})$$

Then $\mathbb{P}$-a.s. on Ω, $\forall \beta > 0$, $P_L^{(\omega)}$ is an analytic w.r.t. $(z, b) \in \mathcal{D}_\epsilon \times \mathbb{R}$.
The finite-volume magnetization and orbital susceptibility are :

$$\chi_{L,n}^{(\omega)}(\beta, b, z, \epsilon) := \left(\frac{q}{c}\right)^n \frac{\partial^n P_L^{(\omega)}}{\partial b^n}(\beta, b, z, \epsilon) \quad n = 1, 2.$$

Integrated density of states.

Introduce the integrated density of states (IDS). Let $N(E), E \in \mathbb{R}$ denotes the number of eigenvalues of the operator $H_L(b,\omega)$ smaller than E counting with their multiplicity and

$$\rho(E) := \lim_{L\to\infty} \frac{N(E)}{|\Lambda_L|}.$$

Define $(\xi, z) \mapsto \mathfrak{f}_\epsilon(\beta, z; .) := \ln\left(1 + \epsilon z e^{-\beta\xi}\right)$. Under general conditions, it holds :

$$P_\infty(\beta,\omega,z,\epsilon) := \lim_{L\to\infty} P_L(\beta,\omega,z,\epsilon) = -\frac{\epsilon}{\beta}\int_{\mathbb{R}} \mathfrak{f}'_\epsilon(\xi,z)\rho(\xi)d\xi$$

$\rightarrow$ open question : regularity of ρ w.r.t. b

The pressure via a Dunford-integral formula

Let $\beta > 0$, $b \in \mathbb{R}$, $z \in \mathcal{D}_\epsilon$, and $K \subset \mathcal{D}_\epsilon$ be a compact set containing z. Let Γ_K be a contour around $[E_0, \infty)$ and

$$\mathcal{L}_{X,0}^{(\omega)}(\beta, b, z, \epsilon) := \frac{i}{2\pi} \int_{\Gamma_K} \mathrm{d}\xi \, \mathfrak{f}_\epsilon(\beta, z; \xi) R_X(b, \omega, \xi),$$

$R_X(b, \omega, \xi), := (H_X(b, \omega) - \xi)^{-1}$, $X = L$ or $X = \infty$. We get

$$P_L^{(\omega)}(\beta, b, z, \epsilon) = \frac{\epsilon}{\beta |\Lambda_L|} \mathrm{Tr}_{L^2(\Lambda_L)} \mathcal{L}_{L,0}^{(\omega)}(\beta, b, z, \epsilon)$$

Theorem

Under our assumptions. Then:
i) $\mathbb{P}$-a.s. on Ω, $\forall b \in \mathbb{R}$, $\forall 0 < \beta_1 < \beta_2$ and for any compact subset K of $\mathcal{D}_\epsilon$:

$$P_\infty(\beta, b, z, \epsilon) := \lim_{L \to \infty} P_L^{(\omega)}(\beta, b, z, \epsilon) = \frac{\epsilon}{\beta} \mathbb{E}\left[\mathcal{L}_{\infty,0}^{(\omega)}(\mathbf{0}, \mathbf{0}; \beta, b, z, \epsilon)\right]$$

uniformly in $(\beta, z) \in [\beta_1, \beta_2] \times K$.

The magnetic response

The gauge invariant perturbation theory (G. Nenciu)$\to$ Introduce

$$\mathcal{L}_{X,1}^{(\omega)}(\beta, b, z, \epsilon) := -\frac{i}{2\pi}\int_{\Gamma_K} \mathrm{d}\xi\ \mathfrak{f}_\epsilon(\beta, z; \xi) R_X(b,\omega,\xi) T_{1,X}(b,\omega,\xi),$$

$$\mathcal{L}_{X,2}^{(\omega)}(\beta, b, z, \epsilon) := \frac{i}{\pi}\int_{\Gamma_K} \mathrm{d}\xi\, \mathfrak{f}_\epsilon(\beta, z; \xi) R_X(b,\omega,\xi)\big((T_{1,X}(b,\omega,\xi))^2 - T_{2,X}(b,\omega,\xi)\big)$$

$$T_{1,X}(\mathbf{x},\mathbf{y}; b,\omega,\xi) := \mathbf{a}(\mathbf{x}-\mathbf{y})\cdot(i\nabla_\mathbf{x} + b\mathbf{a}(\mathbf{x}))R_X(\mathbf{x},\mathbf{y}; b,\omega,\xi),$$

$$T_{2,X}(\mathbf{x},\mathbf{y}; b,\omega,\xi) := \frac{1}{2}\mathbf{a}^2(\mathbf{x}-\mathbf{y})R_X(\mathbf{x},\mathbf{y}; b,\omega,\xi), \quad (\mathbf{x},\mathbf{y})\in\mathbb{R}^6\setminus D_\infty.$$

$$\mathcal{L}_{L,0}^{(\omega)}(b) \sim \sum_{n=0}^{2} b^n \mathcal{L}_{L,n}^{(\omega)}(0) + O(b^3) \quad \text{if} \quad b\sim 0$$

.

Magnetisation/susceptibility

we have:

$$\mathcal{X}_{L,n}(\beta, b, z, \epsilon) = \left(\frac{q}{c}\right)^{n} \frac{\epsilon}{\beta|\Lambda_L|} \mathrm{Tr}_{L^2(\Lambda_L)} \mathcal{L}^{(\omega)}_{L,n}(\beta, b, z, \epsilon) \quad n = 1, 2.$$

Magnetisation

Theorem

i) $\mathbb{P}$*-a.s. on* Ω*,* $\forall b \in \mathbb{R}$*,* $\forall 0 < \beta_1 < \beta_2$ *and for any compact subset* K *of* $\mathcal{D}_\epsilon$*, we have :*

$$\mathcal{X}_{\infty,1}(\beta, b, z, \epsilon) := \lim_{L\to\infty} \mathcal{X}_{L,1}^{(\omega)}(\beta, b, z, \epsilon) = \left(\frac{q}{c}\right)\frac{\epsilon}{\beta}\mathbb{E}\big[\mathcal{L}_{\infty,1}^{(\omega)}(\mathbf{0}, \mathbf{0}; \beta, b, z, \epsilon)\big]$$

uniformly in $(\beta, z) \in [\beta_1, \beta_2] \times K$*.*
ii) $\mathbb{P}$*-a.s. on* Ω*,* $\forall \beta > 0$*,* $\forall z \in \mathcal{D}_\epsilon$*,* $P_\infty(\beta, \cdot, z, \epsilon) \in \mathcal{C}^1(\mathbb{R})$*, and* $\mathcal{X}_{\infty,1}(\beta, b, z, \epsilon) = (\frac{q}{c})\frac{\partial P_\infty}{\partial b}(\beta, b, z, \epsilon)$*.*

Zero field susceptibility

Theorem

i) $\mathbb{P}$*-a.s. on* Ω*,* $\forall 0 < \beta_1 < \beta_2$ *and for any compact subset* K *of* $\mathcal{D}_\epsilon$ *:*

$$\mathcal{X}_{\infty,2}(\beta, 0, z, \epsilon) := \lim_{L\to\infty} \mathcal{X}_{L,2}^{(\omega)}(\beta, 0, z, \epsilon) = \left(\frac{q}{c}\right)^2 \frac{\epsilon}{\beta} \mathbb{E}\big[\mathcal{L}_{\infty,2}^{(\omega)}(\mathbf{0}, \mathbf{0}; \beta, 0, z, \epsilon)\big],$$

uniformly in $(\beta, z) \in [\beta_1, \beta_2] \times K$*.*
ii) $\mathbb{P}$*-a.s. on* Ω*,* $\forall \beta > 0$ *and* $\forall z \in \mathcal{D}_\epsilon$*,* $P_\infty(\beta, \cdot, z, \epsilon)$ *is a* $\mathcal{C}^2$*-function near* $b = 0$*, and* $\mathcal{X}_{\infty,2}(\beta, 0, z, \epsilon) = (\frac{q}{c})^2 \frac{\partial^2 P_\infty}{\partial b^2}(\beta, 0, z, \epsilon)$*.*

Back to the Anderson /periodic case

Let $\Omega := \Lambda_1(\mathbf{0})$. For the Anderson model $\rightarrow$

$$\mathcal{X}_{\infty,n}(\beta, b, z, \epsilon) = \left(\frac{q}{c}\right)^n \frac{\epsilon}{\beta|\Omega|} \int_\Omega \mathrm{d}\mathbf{x}\, \mathbb{E}\big[\mathcal{L}^{(\omega)}_{\infty,n}(\mathbf{x}, \mathbf{x}; \beta, b, z, \epsilon)\big] \quad n = 0, 1, 2.$$

and for the periodic case $\rightarrow$

$$\mathcal{X}_{\infty,n}(\beta, b, z, \epsilon) = \left(\frac{q}{c}\right)^n \frac{\epsilon}{\beta|\Omega|} \int_\Omega \mathrm{d}\mathbf{x}\, \mathcal{L}^{(\omega)}_{\infty,n}(\mathbf{x}, \mathbf{x}; \beta, b, z, \epsilon) \quad n = 0, 1, 2.$$

More on the periodic case when $\epsilon = 1$

V is C^∞ periodic electric potential.

Let $k \in 2\pi\Omega \to E_j(k)$, $j = 1, 2..$ be Bloch curves associated to $H_\infty = H_\infty(b = 0)$ and the Bloch bands:

$$\mathcal{E}_j := [\min E_j(k), \max E_j(k)].$$

$$\to \sigma(H_\infty) = \sigma_{ac}(H_\infty) = \overline{\bigcup_{j=1}^{\infty} \mathcal{E}_j}$$

The fermi level

Recall $z = e^{\beta\mu}$. Let

$$\rho_\infty(\beta, z, b) := \beta z \frac{\partial P_\infty}{\partial z}(\beta, z, b) = \lim_{L\to\infty} \beta z \frac{\partial P_L}{\partial z}(\beta, z, b)$$

Let $\rho_0 > 0$ be fixed. If $\mu_\infty(\beta, \rho_0)$ is the unique real solution of

$$\rho_\infty\left(\beta, e^{\beta\mu}, 0\right) = \rho_0.$$

Then the Fermi energy, $\mathcal{E}_F(\rho_0) := \lim_{\beta\to\infty} \mu_\infty(\beta, \rho_0)$.

Suppose the solution of

$$\rho(\mathcal{E}) = \rho_0$$

$\mathcal{E} \in (\min \mathcal{E}_N, \max \mathcal{E}_N)$ for some $N > 0 \to$ the metallic case and

$$\mathcal{E}_F(\rho_0) = \mathcal{E}$$

.

The Landau Peierls formula

Theorem

Let $\rho_0 > 0$. Suppose that $\exists! N \geq 1$ s.t. $\mathcal{E}_F(\rho_0) \in (\min \mathcal{E}_N, \max \mathcal{E}_N)$ and $\mathcal{S}_F := \{\mathbf{k} \in \Omega^* \,:\, E_N(\mathbf{k}) = \mathcal{E}_F(\rho_0)\}$ is smooth and non-degenerate. Then

$$\mathcal{X}_{\mathrm{M}}(\rho_0) := \lim_{\beta \to \infty} \mathcal{X}(\beta, \rho_0) = -\left(\frac{e}{c}\right)^2 \frac{1}{12} \frac{1}{(2\pi)^3}$$

$$\left\{ \int_{\mathcal{S}_F} \frac{\mathrm{d}\sigma(\mathbf{k})}{\left|\nabla E_N(\mathbf{k})\right|} \left[\frac{\partial^2 E_N(\mathbf{k})}{\partial k_1^2} \, \frac{\partial^2 E_N(\mathbf{k})}{\partial k_2^2} - \left(\frac{\partial^2 E_N(\mathbf{k})}{\partial k_1 \partial k_2} \right)^2 + \mathcal{R}(\rho_0) \right\}$$

Small density regim

Theorem

Let $k_F := (6\pi^2\rho_0)^{\frac{1}{3}}$ be the Fermi wave vector. Then when $\rho_0 \to 0$ we get the Landau-Peierls formula:

$$\mathcal{X}_{\mathrm{M}}(\rho_0) = -\frac{e^2}{24\pi^2 c^2}\frac{(m_1^* m_2^* m_3^*)^{\frac{1}{3}}}{m_1^* m_2^*} k_F + o(k_F);$$

here $\left[\frac{1}{m_i^*}\right]_{1\leq i\leq 3}$ are the eigenvalues of the positive definite Hessian matrix $\{\partial_{ij}^2 E_1(\mathbf{0})\}_{1\leq i,j\leq 3}$.

semi-conductor case

Let $\rho_0 > 0$ be fixed. Suppose that there exists some $N \in \mathbb{N}^*$ such that $\rho_0 = \rho(\mathcal{E})$ for all $\mathcal{E} \in [\max \mathcal{E}_N, \min \mathcal{E}_{N+1}]$. Then:

$$\mathcal{E}_F(\rho_0) = \frac{\max \mathcal{E}_N + \min \mathcal{E}_{N+1}}{2}.$$

semi-conductor case

Theorem

Assume that the Fermi energy is in the middle of a non-trivial gap. Then there exist 2N functions $\mathfrak{c}_j(\cdot), \mathfrak{d}_j(\cdot)$, with $1 \leq j \leq N$, defined on Ω^* :

$$\mathcal{X}_{\mathrm{SC}}(\rho_0) := \lim_{\beta\to\infty} \mathcal{X}(\beta, \rho_0) = \left(\frac{e}{c}\right)^2 \frac{1}{2} \frac{1}{(2\pi)^3} \int_{\Omega^*} \mathrm{d}\mathbf{k} \sum_{j=1}^{N} \Big\{ \mathfrak{c}_j(\mathbf{k}) + \big\{ E_j(\mathbf{k}) - \mathcal{E}_F(\rho_0) \big\} \mathfrak{d}_j(\mathbf{k}) \Big\}.$$

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