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joint work with Paolo Buttà
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VIII International Conference in Mathematical Physics in Armenia:
”Analytic and Probabilistic Methods in Mathematical Physics”
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1. INTRODUCTION

The classical incompressible Navier-Stokes (NS) equations in a domain $D \subset \mathbb{R}^d$, $d = 2, 3$, in absence of external forces, are

$$\begin{align*}
\partial_t u + u \cdot \nabla u &= \nu \Delta u - \nabla p, \\
\nabla \cdot u &= 0
\end{align*}$$

where $u = u(x, t), x \in D, t \geq 0$, is the velocity field, $\nu$ is the viscosity, $p$ the pressure, and $u(0)$ the initial data.

The problem is completed by the no-slip (Dirichlet) boundary conditions $u \mid_{\partial D} = 0$.

The pressure can be eliminated by going over to the vorticity field $\omega(x, t)$, the "curl" or "rotation" of $u$:

$$\omega(x, t) = \text{curl} u(x, t) = \nabla \times u(x, t).$$
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Batchelor (1967) argued on physical grounds that one can take the Laplacian in Eq. (2) as the Neumann Laplacian, and restore the no-slip condition by adding a singular vorticity production term at the boundary.
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Eq. (2) would be replaced by the formal equation

$$\partial_t \omega + (u \cdot \nabla) \omega = \nu \Delta \omega + f \delta_{\partial D},$$  \hspace{1cm} (3)

where $\delta_{\partial D}$ is a $\delta$-function, and the function $f$ ("vorticity production") on $\partial D$ is determined by the boundary conditions.
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The explicit control of the vorticity near the boundary could also shed some light on the problem of the convergence of the solutions of the NS equations to solutions of the Euler equations as $\nu \to 0$. 

The first (and only) rigorous results are due to Benfatto and Pulvirenti who considered the vorticity production scheme for the half-plane. They proved existence, uniqueness, equivalence to the original NS problem, and convergence of the Chorin iteration method. We consider here the case when the domain $D$ is the flat cylinder $C := T \times [0, \pi]$, the only bounded 2-dimensional region with smooth and flat boundary.
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The outline of our approach is as follows.

By Fourier expansion the problem is reduced to a discrete infinite set of integro-differential equations for the Fourier modes of the vorticity. The boundary conditions determine the vorticity production term as a function of $\omega$. To the system thus defined we can apply elementary methods inspired by the recent work of Dinaburg, Li, Sinai:

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2. THE NS EQUATIONS ON THE FLAT CYLINDER WITH VORTICITY PRODUCTION

We denote the coordinates on the flat cylinder $C = \mathbb{T} \times [0, \pi]$ as $(x_1, x_2)$, where $x_1 \in \mathbb{T}$ is periodic and $x_2 \in [0, \pi]$.

The boundary is not connected and made of two copies of $\mathbb{T}$, at $x_2 = 0$ and $x_2 = \pi$.

The equations for the velocity field are

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\begin{aligned}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u &= \Delta u - \nabla p, \\
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The vorticity takes the form

\[ \omega(x, t) := \nabla^\perp \cdot u = \partial_{x_1} u_2 - \partial_{x_2} u_1, \quad \nabla^\perp = (-\partial_{x_2}, \partial_{x_1}). \]
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Solenoidality (\(\nabla \cdot u = 0\)) and the boundary conditions imply that the average of the vorticity vanishes

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u = \nabla^\perp \Delta_N^{-1} \omega. \quad (4b)\]
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We get the following problem for $\omega$

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where $f$ takes care of the boundary condition.
As the boundary is made of two pieces we have

\[ f \delta_{\partial C}(x, t) = f_1(x_1, t)\delta(x_2) + f_2(x_1, t)\delta(x_2 - \pi). \]
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Absence of external forces and \( \int_{\mathcal{C}} \omega(x, t) \, dx = 0 \) imply

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We expand \( \omega(x, t) \) in the Neumann basis \( \{ e^{ik_1 x_1} \cos k_2 x_2 \} \):

\[ \omega(x, t) = \sum_{k_1 \neq 0} \omega_{k_1, 0}(t) e^{ik_1 x_1} + 2 \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \geq 1} \omega_{k_1, k_2}(t) e^{ik_1 x_1} \cos(k_2 x_2), \]
For convenience we extend $\omega$ by parity: $\hat{\omega}_{k_1,k_2} = \omega_{k_1,k_2}$, so that

$$\omega(x, t) = \sum_{k \in \mathbb{Z}^2 \atop k \neq 0} \hat{\omega}_{k_1,k_2}(t) e^{i k \cdot x}.$$
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$$\omega(x, t) = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \hat{\omega}_{k_1,k_2}(t) e^{i k \cdot x}. $$

The boundary condition now is

$$\sum_{(0,0) \neq k \in \mathbb{Z}^2} \frac{k_1}{k^2} \hat{\omega}_{k_1,k_2} e^{i k_1 x_1} \cos(k_2 x_2) \bigg|_{\partial C} = 0.$$
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where $\sum_{s,+} a_s = a_0 + 2 \sum_{i \geq 1} a_{2i}$, and $\sum_{s,-} a_s = 2 \sum_{i \geq 1} a_{2i-1}$. 
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$$N_{k_1, k_2}[\omega] = i \sum_{h_2 \in \mathbb{Z}} \frac{\delta_{\text{odd}}(h_2 + k_2)}{\pi} \frac{2h_2}{h_2^2 - k_2^2} R_{k_1, h_2}.$$
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We obtain an infinite set of coupled ODE’s (we drop the hat of \( \hat{\omega}_{ij} \)). For all \( k_1 \in \mathbb{Z}, \ k_2 \geq 0 \),

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\dot{\omega}_{k_1,k_2}(t) + N_{k_1,k_2}[\omega(t)] = -k^2 \omega_{k_1,k_2}(t) + f_{\pm,k_1}(t), \tag{7}
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where \( f_{\pm,k_1}(t) = f_{1,k_1}(t) \pm f_{2,k_1}(t) \), the + [resp. –] sign is chosen for \( k_2 \) even [resp. odd], and

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Observe that if $\omega$ satisfies the relations (6) then $N_{0,0}[\omega] = 0$, and, as $f_{+,0}(t) = 0$ and $\omega_{0,0}(0) = 0$, we have $\omega_{0,0}(t) \equiv 0$. 
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The following technical lemma shows that the functions $f_{\pm,k_1}$ are uniquely determined in terms of $\omega$. 

Lemma 1. The infinite system of Volterra equation of the first kind for $a(t)$, 

$$
\sum_{k_2} f_{\pm,k_2} \int_0^t ds e^{-k_2(t-s)} a(s) = b(t), \quad k_1 \neq 0,
$$

where $b(t)$ is a bounded differentiable function with $b(0) = 0$, has a unique solution which can be represented as 

$$
a(t) = \int_0^t ds G_{\pm,k_1}(t-s) b'(s) + \int_0^t ds H_{\pm,k_1}(t-s) b(s).
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The functions $G_{\pm,k_1}$ and $H_{\pm,k_1}$, $k_1 \neq 0$ have the following properties:
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The functions $G_{k_1}^{\pm}$ and $H_{k_1}^{\pm}$, $k_1 \neq 0$ have the following properties:
\[ G_{k_1}^\pm(t) := \frac{2}{\pi} d_{\pm}(k_1) \left[ \delta(t) + \frac{e^{-k_1^2t}}{\sqrt{t}} \sum_{n=1}^{4} \frac{d_{\pm}(k_1)^n}{\Gamma(n/2)} t^{(n-1)/2} \right], \]

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\[ H_{k_1}^\pm(t) \leq B_\gamma |k_1|^3 \exp \left[ -(1 - \gamma) k_1^2 t \right], \]

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3. LOCAL EXISTENCE AND REGULARIZATION

By rather elementary estimates we prove local (in time) existence, in a suitable class of functions, with regularization in the periodic variable for $t > 0$.

Theorem 2. Let $\omega_{k_1, k_2}(0)$ satisfy for all $k_1 \in \mathbb{Z}$, $k_2 \geq 0$, $k \neq (0, 0)$, and for some $1 < \alpha < 2$, $\beta \geq 0$, and $D_0 > 0$, the inequalities

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Then there exist a time $T_0 = T_0(D_0, \alpha, \beta)$ and a constant $D_2 = D_2(D_0, \alpha, \beta)$ such that there is a unique solution of the problem (6), (8) for $t \in [0, T_0]$ which satisfies the inequalities

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Moreover, if $D_0$ is sufficiently small, the solution is global and the estimate above is valid for any $t \geq 0$. 

Sketch of the proof.

The proof is based on an iteration scheme with a contraction argument in the Banach space $\Omega_{\alpha,\beta,T}$, the space of the functions $\{\omega_{k_1}, k_2(s) : (k_1, k_2) \neq (0, 0), s \in [0, T]\}$ with norm $\|\omega\|_{\alpha,\beta,T}:= \sup_{s \in [0, T]} \sup_{k_1, k_2} |\omega_{k_1, k_2}(s)| e^{(1+ |k_1|)s/4} |k|^{\alpha} (1 + |k_1|^{\beta})$.

Some properties of the terms of the equation (8) are needed.

We first consider the transport term.

Lemma 3. There is a constant $C_N > 0$ such that, for any $\omega, \tilde{\omega} \in \Omega_{\alpha,\beta,T}$ satisfying Eq.s (7) $|N_{k_1, k_2}[\omega(t)] - N_{k_1, k_2}[\tilde{\omega}(t)]| \leq C_N e^{-(1+ |k_1|)t/4} R(\omega, \tilde{\omega}) \|\omega - \tilde{\omega}\|_{\alpha,\beta,T}$

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We then iterate by setting, for each integer $n \geq 1$, 

$$
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$$

where $N(n)_{k_1, k_2}(t) := N(k_1, k_2)[\omega(n-1)(t)]$ and $f(n)_{\pm, k_1}(t)$ is the solution of 

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\]

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\[
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$$
The theorem easily follows from the following lemma.

**Lemma 5.** Under the assumptions above, for all \( n \geq 1 \):

i) There is some \( T_1 = T_1(D_0, \alpha, \beta) > 0 \) such that
\[
\| \omega(n) \|_{\alpha, \beta, T} \leq \frac{1}{2} \| \omega(n) \|_{\alpha, \beta, T} - \| \omega(n-1) \|_{\alpha, \beta, T},
\]

ii) There is some \( T_0 = T_0(D_0, \alpha, \beta), 0 < T_0 \leq T_1 \), such that
\[
\| \omega(n+1) - \omega(n) \|_{\alpha, \beta, T} < \frac{1}{2} \| \omega(n) \|_{\alpha, \beta, T},
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\[
\| \omega(n) \|_{\alpha, \beta, T} \leq D_2 := 2(1 + 2C_f)D_0,
\]

\( 0 \leq T \leq T_1 \), \( C_f \) being the constant occurring in the inequalities for the vorticity production term in Lemma 3.

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For $n = 0$ we have $\| \omega^{(0)} \|_{\alpha, \beta, T} \leq D_0 < D_2$. 
For $n = 0$ we have $\|\omega^{(0)}\|_{\alpha, \beta, T} \leq D_0 < D_2$. Proceed by induction and assume that assertion i) holds for any $0 \leq n' < n$ up to some time $T_1$. As $\omega_{k_1, k_2}^{(n-1)}(0) = \omega_{k_1, k_2}(0)$ for all $n \geq 1$, we have,

$$
\|\omega^{(n-1)}\|_{\alpha, \beta, 0} \leq D_0, \quad \|\omega^{(n-1)}\|_{\alpha, \beta, T} \leq D_2.
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\[
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\]

Furthermore we have,

\[
\left| \omega^{(n)}_{k_1, k_2}(t) \right| \leq \frac{D_0 e^{-k^2 t}}{|k|\alpha (1 + |k_1|\beta)} + \\
\int_0^t ds \ e^{-k^2(t-s)} \left[ |f^{(n-1)}_{\pm, k_1}(s)| + |N^{(n-1)}_{k_1, k_2}(s)| \right].
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\]

By applying the inequalities of the previous lemmas we get

\[
\left| \omega^{(n)}_{k_1, k_2}(t) \right| \leq \frac{e^{-(1+|k_1|t)/4}}{|k|^{\alpha} (1 + |k_1|^{\beta})} \left[ D_0 + 2 \frac{1 - e^{-k^2 t/2}}{|k|^{2-\alpha}} \times \left( C_f D_0 |k_1|^{2-\alpha} + C_f D_2^2 + C_N D_2^2 \right) \right]
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$$\left\|\omega^{(n-1)}\right\|_{\alpha, \beta, 0} \leq D_0, \quad \left\|\omega^{(n-1)}\right\|_{\alpha, \beta, T} \leq D_2.$$  

Furthermore we have,

$$\left|\omega^{(n)}_{k_1, k_2}(t)\right| \leq \frac{D_0 e^{-k^2 t}}{|k|^\alpha (1 + |k_1|^\beta)} +$$

$$+ \int_0^t ds \ e^{-k^2 (t-s)} \left[ \left| f^{(n-1)}_{\pm, k_1}(s) \right| + \left| N^{(n-1)}_{k_1, k_2}(s) \right| \right].$$

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As on the right we have $D_2^2$, and $D_2$ is proportional to $D_0$, we see that if $D_0$ is small enough, the term in square brackets is less than $D_2$ for all times, and we have a global solution.
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$$
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$$

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$$
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Assertion ii) is proved in a similar way.

The proof of Theorem 2 is now easy. In fact, by Lemma 5, \{\omega(n)\} is a uniformly bounded Cauchy sequence in \(\Omega_{\alpha,\beta,T}\) for \(0 < T < T_0\). This proves existence and uniqueness of the solution in \(\Omega_{\alpha,\beta,T}\) for \(0 < T < T_0\). ■

For what follows we need the following simple remark.

Remark. If we omit the decay factor \(e^{-(1+|k_1|)t/4}\) and take \(\beta = 0\) the proof of Theorem 2 goes through with minor changes.

If we assume that the initial data satisfy the boundary conditions and \(|\omega_{k_1,k_2}(0)| \leq D_0 |k|^{\alpha}\) \(\forall k_1 \in \mathbb{Z}, k_2 \geq 0, k \neq (0,0)\), with \(1 < \alpha < 2\), then there exist a time \(T_0 = T_0(D_0,\alpha)\) and a unique solution \(\{\omega_{k_1,k_2}(t)\}; k_1 \in \mathbb{Z}, k_2 \geq 0\) of equations (8) such that \(\|\omega\|_{\alpha,T_0} < \infty\), where \(\|\omega\|_{\alpha,t} := \sup_{s \in [0,t]} \sup_{k_1 \in \mathbb{Z}} \sup_{k_2 \geq 0} |\omega_{k_1,k_2}(s)|^{\alpha}\).
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\[
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We can now see that the local solutions just obtained are weak solutions of the original NS problem.
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**Lemma 6.** Let \( \{\omega_{k_1,k_2}(t); k_1 \in \mathbb{Z}, k_2 \geq 0\}, \ t \in [0, T], \) be a solution of equations (8) such that \( \|\omega\|_{\alpha,T} < \infty, \ 1 < \alpha < 2, \) and

\[
\omega(x, t) := \sum_{k \in \mathbb{Z}^2} \hat{\omega}_{k_1,k_2}(t) e^{ik \cdot x}.
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\]

Then, the velocity field \( u(x, t) := \nabla^\perp \Delta_N^{-1} \omega(x, t) \) coincides, for \( t \in [0, T], \) with a weak solution to the NS system with Dirichlet boundary conditions.
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**Lemma 6.** Let \( \{ \omega_{k_1,k_2}(t); k_1 \in \mathbb{Z}, k_2 \geq 0 \}, t \in [0, T], \) be a solution of equations (8) such that \( \| \omega \|_{\alpha, T} < \infty, 1 < \alpha < 2, \) and

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**Proof.** We have

\[
u(x, t) = \sum_{k \in \mathbb{Z}^2} u_{k_1,k_2}(t) e^{i k \cdot x}, \quad u_{k_1,k_2}(t) := -ik^\perp \frac{\hat{\omega}_{k_1,k_2}(t)}{k^2},
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Hence $u \in L^2([0, T]; V)$, where $V$ is the space of solenoidal vector fields in $H^1_0(\mathcal{C})^2$. 
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Hence \( u \in L^2([0, T]; V) \), where \( V \) is the space of solenoidal vector fields in \( H^1_0(C)^2 \).

In order to show that \( u \) is a weak solution we need to check that, for any solenoidal \( C^\infty \) vector field \( \Phi \) of compact support in \( C \) we have

\[
\frac{d}{dt} \int_C dx \, \Phi(x) \cdot u(x, t) = \int_C dx \, \Delta \Phi(x) \cdot u(x, t) - \int_C dx \, \Phi(x) \cdot [u(x, t) \cdot \nabla] u(x, t).
\]
So that we have the estimate

\[
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Hence \( u \in L^2([0, T]; V) \), where \( V \) is the space of solenoidal vector fields in \( H^1_0(\mathcal{C})^2 \).

In order to show that \( u \) is a weak solution we need to check that, for any solenoidal \( C^\infty \) vector field \( \Phi \) of compact support in \( \mathcal{C} \) we have

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\frac{d}{dt} \int_\mathcal{C} dx \, \Phi(x) \cdot u(x, t) = \int_\mathcal{C} dx \, \Delta \Phi(x) \cdot u(x, t) -
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**Theorem.** Let $\omega_{k_1,k_2}(0)$ satisfy the boundary conditions, and, for any $k \neq (0,0)$, the inequalities

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with $1 < \alpha < 2$, $\beta \geq 0$, and $D_0 > 0$.

Then there is a unique solution $\{\omega_{k_1,k_2}(t)\}$ of equations (8) which satisfies for all $t \geq 0$ the boundary conditions, with the following properties.

i) There are constants $D_1, \nu > 0$ (depending on $D_0, \alpha, \beta$), such that for all $k \neq 0$ the following inequalities hold

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**Proof.** The previous results amount, up to some details, to a proof of assertion i) for \( \beta = 0 \) and \( \nu = 0 \).
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tends to 0 exponentially fast, we prove the result for \( \beta > 0 \) and find some \( \nu > 0 \) by using the global result for small initial data. Assertion ii) is proved by some kind of simple bootstrap argument.

Assertion iii) is proved exactly as in Lemma 6. ■
As the flat cylinder has a smooth boundary, we know, by the general results on the 2-d NS equations that for \( t > 0 \) the solutions with \( u_0 \in V \) are infinitely differentiable in time and space.
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Nevertheless, making use of the properties of the heat kernel, we can prove the following result.
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Nevertheless, making use of the properties of the heat kernel, we can prove the following result.
Corollary 8. For $t > 0$ the expression for the solution of the equations (8)

$$\omega(x, t) = \sum_k \omega_{k_1, k_2}(0) e^{-k^2 t} e^{i k \cdot x} + \sum_k \int_0^t dse^{-k^2(t-s)} N_{k_1, k_2}[\omega(s)] e^{i k \cdot x} +$$

$$+ F_+(x, t) + F_-(x, t)$$
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where

$$F_\pm(x, t) = \sum_{k_1 \neq 0} \sum_{k_2, \pm} e^{i k_1 x_1} \cos(k_2 x_2) \int_0^t ds e^{-k^2 (t-s)} f_{\pm, k_1}(s)$$

are differentiable term by term in $x_2$ and infinitely differentiable in $x_1$ with continuous derivatives up to the boundary.
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