Applications of Riemann-Hilbert problems in supercavitation

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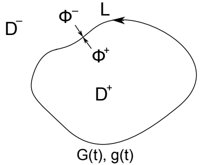
Riemann-Hilbert problem: applications

- Two-dimensional linear elasticity.
- Fluid mechanics: two-dimensional flows of ideal fluid.
- Electromagnetic scattering.
- Random Matrix Theory.
- Integrable systems.
- Nonlinear PDEs.

Riemann-Hilbert problem: the simplest case

Consider a smooth simple closed contour L.



Find all functions $\Phi(z)$ *analytic in the domains* D^+ *and* D^-*, which satisfy on the contour L the following linear equation:*

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in L,$$

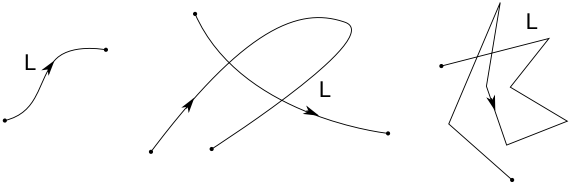
where $G(t)$, $g(t)$ *are Hölder continuous functions*

$$|G(t_2) - G(t_1)| < A|t_2 - t_1|^\lambda, \quad \forall\ t_1, t_2 \in L, \quad 0 < \lambda \leq 1,$$

and $G(t) \neq 0$, $t \in L$.

Riemann-Hilbert problem: complications

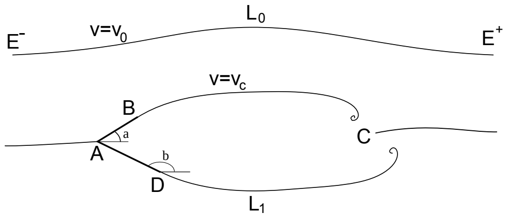
- Complicated contour L.



- $\Phi(z)$ is a vector or a matrix: Matrix Riemann-Hilbert problem.
- $G(t) = 0$ at some points of the contour L.
- Riemann-Hilbert problems on Riemann surfaces.

Cavitational flow around a wedge

Steady inviscid irrotational incompressible flow.



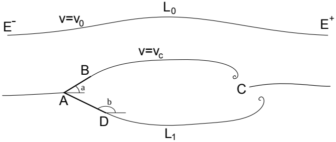
Find an analytic function $w(z) = \varphi(z) + i\psi(z)$ from the conditions

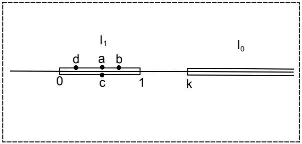
$$\operatorname{Im} w(z) = \begin{cases} \psi_1, & z \in ABCDA, \\ \psi_0, & z \in E^- E^+, \end{cases} \qquad (1)$$

$$\arg \frac{dw}{dz} = \begin{cases} -\alpha, & z \in AB, \\ \pi - \beta, & z \in AD, \end{cases} \qquad \left|\frac{dw}{dz}\right| = \begin{cases} V_c, & z \in BC \cup DC, \\ V_0, & z \in E^- E^+. \end{cases} \qquad (2)$$

Here $\alpha = \alpha_0 + \delta$, $\beta = \beta_0 + \delta$, δ is the yaw angle.

Conformal mapping





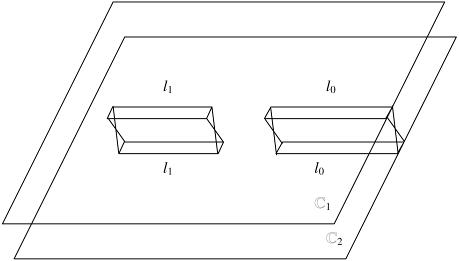
$z = f(\zeta)$ is a conformal mapping from ζ-plane onto the flow domain.
The function $z = f(\zeta)$ can be found from the relation:

$$f'(\zeta) = \frac{dz}{d\zeta} = \frac{dw}{d\zeta} : \frac{dw}{dz}. \tag{3}$$

Need to Find: functions dw/dz and $dw/d\zeta$.

Riemann surface

The functions dw/dz and $dw/d\zeta$ can be found from the solution of two Riemann-Hilbert problems on the Riemann surface $\mathcal{R}$ of the function $u^2 = p(\zeta)$, $p(\zeta) = \zeta(1-\zeta)(\zeta-k)$.



Function $\omega_1(\zeta) = \ln \frac{1}{V_\infty} \frac{dw}{dz}$

Continue the function $\omega_1(\zeta)$ by the symmetry

$$\Phi(\zeta, u) = \begin{cases} -i\omega_1(\zeta), & \zeta \in \mathbb{C}_1, \\ \overline{i\omega_1(\bar{\zeta})}, & \zeta \in \mathbb{C}_2. \end{cases} \tag{4}$$

Riemann-Hilbert problem on $\mathcal{R}$:

Find all the piecewise analytic functions $\Phi(\zeta, u)$ on the Riemann surface $\mathcal{R}$ which satisfy the boundary condition:

$$\Phi^+(\xi, v) = G(\xi, v)\Phi^-(\xi, v) + g(\xi, v), \ \ (\xi, v) \in l_0 \cup l_1,$$

$$G(\xi, v) = \begin{cases} 1, & (\xi, v) \in bcd \cup l_0, \\ -1, & (\xi, v) \in ab \cup da, \end{cases}$$

Function $\omega_1(\zeta) = \ln \frac{1}{V_\infty}\frac{dw}{dz}$ continued

$$g(\xi, v) = \begin{cases} -i\log(\sigma+1), & (\xi, v) \in bcd, \\ 0, & (\xi, v) \in l_0, \\ -2\alpha, & (\xi, v) \in ab, \\ 2(\pi-\beta), & (\xi, v) \in da, \end{cases}$$

and the symmetry condition:

$$\Phi(\zeta, u) = \overline{\Phi(\bar{\zeta}, -u(\bar{\zeta}))},$$

and $\Phi(\zeta, u) = O((\zeta - c)^{-1})$ as $\zeta \to c$; $\Phi(z, u) \to 0$ as $z \to \infty$; $\Phi(\zeta, u)$ is bounded at the points $\zeta = b$ and $\zeta = d$ and may have a logarithmic singularity at the point $\zeta = a$.

Canonical function $X(\zeta, u)$

Find a piecewise meromorphic function $X(\zeta, u)$ which satisfies the homogenious boundary-value condition

$$X^{+}(\xi, v) = -X^{-}(\xi, v), \quad (\xi, v) \in dab, \tag{5}$$

and the symmetry condition

$$X(\zeta, u) = \overline{X(\bar{\zeta}, -u(\bar{\zeta}))}, \quad (\zeta, u) \in \mathcal{R}/L. \tag{6}$$

Take the logarithm of the condition (5):

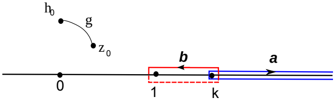
$$\ln X^{+}(\xi, v) = \ln X^{-}(\xi, v) + \ln(-1), \quad (\xi, v) \in dab.$$

One of the solutions:

$$\chi(\zeta, u) = \exp\left\{\frac{1}{4}\int_{dab}\left(1 + \frac{u(\zeta)}{u(\xi)}\right)\frac{d\xi}{\xi - \zeta}\right\}. \tag{7}$$

This function has essential singularity at infinity point of $\mathcal{R}$.

Jacobi inversion problem



$$X(\zeta,u)=\exp\left\{\frac{1}{4}\int_{dab}\left(1+\frac{u}{v}\right)\frac{d\xi}{\xi-\zeta}-\frac{1}{2}\int_{\gamma}\left(1+\frac{u}{v}\right)\frac{d\xi}{\xi-\zeta}\right.$$
$$\left.-\frac{1}{2}\int_{\gamma}\left(1-\frac{u}{\bar{v}}\right)\frac{d\bar{\xi}}{\bar{\xi}-\zeta}-2n_a\int_{l_0^+}\frac{u}{v}\frac{d\xi}{\xi-\zeta}\right\}. \quad (8)$$

Formulation of the Jacobi problem: Find a point $(\zeta_0, u(\zeta_0)) \in \mathcal{R}$ and two integers n_a, n_b which satisfy the condition:

$$\int_0^{\zeta_0}\frac{d\xi}{u(\xi)}+n_a\mathcal{A}+n_b\mathcal{B}=g_0, \quad (9)$$

$$g_0=\frac{1}{4}\int_{dab}\frac{d\xi}{p^{1/2}(\xi)}+\int_0^{\eta_0}\frac{d\xi}{p^{1/2}(\xi)},\quad \mathcal{A}=\oint_{\mathbf{a}}\frac{d\xi}{u(\xi)},\quad \mathcal{B}=\oint_{\mathbf{b}}\frac{d\xi}{u(\xi)}. \quad (10)$$

Function $\Phi(\zeta, u)$

Factorization: $G(\xi, v) = X^+(\xi, v)/X^-(\xi, v)$, $(\xi, v) \in l_0 \cup l_1$.
Write the inhomogeneous problem as:

$$\frac{\Phi^+(\xi, v)}{X^+(\xi, v)} = \frac{\Phi^-(\xi, v)}{X^-(\xi, v)} + \frac{g(\xi, v)}{X^+(\xi, v)}, \quad (\xi, v) \in l_0 \cup l_1 \subset \mathcal{R}. \tag{11}$$

Solution:

$$\Phi(\zeta, u) = X(\zeta, u)[\Psi(\zeta, u) + \Omega(\zeta, u)], \quad (\zeta, u) \in \mathcal{R}, \tag{12}$$

$$\Psi(\zeta, u) = -\frac{\alpha}{2\pi i}\int_{ab}\frac{(1 + u/v)d\xi}{X^+(\xi, u)(\xi - \zeta)}$$

$$+\frac{\pi - \beta}{2\pi i}\int_{da}\frac{(1 + u/v)d\xi}{X^+(\xi, u)(\xi - \zeta)} - \frac{\ln(\sigma + 1)}{4\pi}\int_{bcd}\frac{(1 + u/v)d\xi}{X^+(\xi, u)(\xi - \zeta)}, \tag{13}$$

and $\Omega(\zeta, u)$ is a rational function containing real unknowns M_0, M_1, M_2.

Function $\frac{dw}{d\zeta} = \omega_0(\zeta)$

Introduce a new function

$$\Theta(\zeta, u) = \begin{cases} \omega_0(\zeta) \text{ on } \mathbb{C}_1, \\ \overline{\omega_0(\bar{\zeta})} \text{ on } \mathbb{C}_2. \end{cases} \tag{14}$$

Riemann-Hilbert problem for $\Theta(\zeta, u)$:

$$\Theta^+(\zeta, u) = \Theta^-(\zeta, u), \quad (\zeta, u) \in l_0 \cup l_1, \tag{15}$$

$$\Theta(\zeta, u) = O(\zeta^{-1/2}), \quad \zeta \to \infty, \tag{16}$$

$$\Theta(a, u(a)) = 0, \quad \Theta(c, u(c)) = 0. \tag{17}$$

The solution:

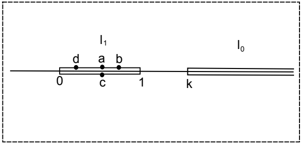
$$\frac{dw}{d\zeta} = N\tilde{\omega}_0(\zeta), \quad \tilde{\omega}_0(\zeta) = \frac{i(\zeta - a)}{p^{1/2}(\zeta)}, \quad a = c, \tag{18}$$

where

$$N = h\left(\mathrm{Im}\int_1^k \tilde{\omega}_0(\xi)d\xi\right)^{-1}. \tag{19}$$

Unknowns and conditions

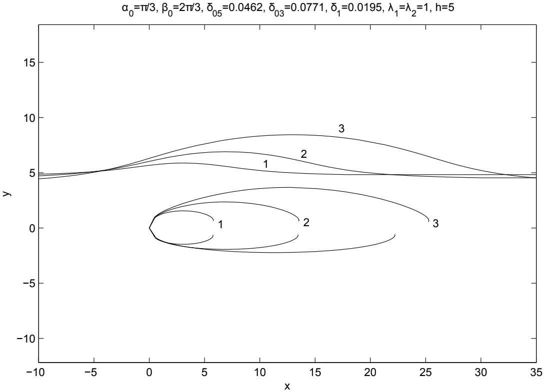
9 real unknowns need to be fixed: a, b, d, k, δ, N, M_0, M_1, M_2.
9 real linear and transcendental conditions exist.
The resulting system is solved by Newton method.



Main difficulties:

- Evaluate the singular and regular integrals with good accuracy and reasonable computational time ($|k-1| \ll 1$).
- Find good initial approximation to the solution.

Numerical Results



The cavity shape and the jet surface for $\alpha_0 = \pi - \beta_0 = \frac{\pi}{3}$, $\lambda_1 = \lambda_2 = 1$, $h = 5$ when $\sigma = 1(1)$, $\sigma = 0.5(2)$, $\sigma = 0.3(3)$.