A Markov Chain Approach to Renormalization Group Transformations

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Welcome to the OASIS!

We consider one-dimensional Ising-type Hamiltonian with nearest neighbor interactions and a magnetic field. I will present a Markov chain method for an explicit characterization of the renormalized Hamiltonian after decimation transformation, a commonly used renormalization group (RG) transformation.

- Our original and image configuration spaces are $\mathbb{Z}$.
- $\sigma(\sigma') : \mathbb{Z} \to \{-1, 1\}$, the spin function for the original (image) lattice.
- $\sigma_x(\sigma'_x)$, commonly referred to as the spin at site x.
- $J(J')$ is the nearest neighbor interaction strength, and $m(m')$ is the magnetic field term.
- C normalizing constant.

We consider decimation transformation with arbitrary blocking factor b,

$$e^{C+J'\sigma_0\sigma_b+\frac{m'+m}{2}(\sigma_0+\sigma_b)} = \sum_{\sigma_1,\ldots,\sigma_{b-1}} e^{J\sum_{i=0}^{b-1}\sigma_i\sigma_{i+1}+m\sum_{i=0}^{b}\sigma_i}.$$

As you may have guessed, the direct RG computation gets messy when b gets large. We need to figure out an alternative method.

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The idea is to regard the Ising system as a two-state Markov chain with transition probability matrix

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix},$$

where $p = P(\sigma_1 = 1|\sigma_0 = -1)$ and $q = P(\sigma_1 = -1|\sigma_0 = 1)$. And there is an invariant probability assigned to site 0,
$P(\sigma_0 = -1) = \frac{q}{p+q}$, $P(\sigma_0 = 1) = \frac{p}{p+q}$.
Working with conditional probabilities, we see that for an arbitrary site n,

$$E\sigma_n = \frac{p-q}{p+q},$$

$$\mathrm{Cov}(\sigma_0, \sigma_n) = (1-p-q)^n \frac{4pq}{(p+q)^2}.$$

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As shown in Baxter's
Exactly Solved Models in Statistical Mechanics,

$$E\sigma_0 = 2\phi,$$

$$\mathrm{Cov}(\sigma_0, \sigma_1) = \sin^2(2\phi)\frac{\lambda_2}{\lambda_1},$$

where $0 < \phi < \frac{\pi}{2}$ satisfies

$$\cot(2\phi) = e^{2J}\sinh(m),$$

and λ_1 and λ_2 are the larger and smaller eigenvalues of the transfer matrix V, respectively,

$$V = \begin{pmatrix} e^{J+m} & e^{-J} \\ e^{-J} & e^{J-m} \end{pmatrix}.$$

Therefore

$$\frac{p-q}{p+q} = \frac{e^J \sinh m}{\sqrt{e^{2J}\sinh^2 m + e^{-2J}}},$$

$$(1-p-q)\frac{4pq}{(p+q)^2} =$$

$$\frac{1}{1+e^{4J}\sinh^2 m}\,\frac{e^J\cosh m - \sqrt{e^{2J}\sinh^2 m + e^{-2J}}}{e^J\cosh m + \sqrt{e^{2J}\sinh^2 m + e^{-2J}}}.$$

Let's see what we've got so far,

$$p = \frac{\sqrt{\sinh^2 m + e^{-4J}} + \sinh m}{\cosh m + \sqrt{\sinh^2 m + e^{-4J}}},$$

$$q = \frac{\sqrt{\sinh^2 m + e^{-4J}} - \sinh m}{\cosh m + \sqrt{\sinh^2 m + e^{-4J}}},$$

$$m = \frac{1}{2} \log\left(\frac{1-q}{1-p}\right),$$

$$J = \frac{1}{4} \log\left(\frac{(1-p)(1-q)}{pq}\right).$$

After decimation transformation, we conjecture that the mean and covariance of the system will stay the same.

$$E\sigma_b = \frac{p' - q'}{p' + q'} = \frac{p - q}{p + q},$$

$$\mathrm{Cov}(\sigma_0, \sigma_b) = (1 - p' - q')\frac{4p'q'}{(p' + q')^2} = (1 - p - q)^n \frac{4pq}{(p + q)^2},$$

where we take into consideration that site 0 and site b are nearest neighbors now.

Therefore

$$p' = \frac{p}{p + q}\left(1 - (1 - p - q)^b\right),$$

$$q' = \frac{q}{p + q}\left(1 - (1 - p - q)^b\right).$$

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Putting it all together, we have successfully found an explicit characterization of the renormalized Hamiltonian:

$$(J, m) \mapsto (p, q) \mapsto (p', q') \mapsto (J', m').$$

Mission Accomplished!

Thank You!