Convergence of Particle Filtering Method for Nonlinear Estimation of Vortex Dynamics

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February 20, 2010

Outline

1. Background
   - Nonlinear Filtering
   - Stochastic Vortex Model
   - Particle Filter

2. Our Results
   - Solvability of Zakai Equation
   - Convergence of Solutions to Zakai Equation

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Introduction to Nonlinear Filtering

- We begin with a complete probability space $(\Omega, \mathcal{F}, P)$ on which our stochastic process will be defined. Consider the stochastic differential equation for the signal process X_t:

$$dX_t = f(X_t)dt + \sigma(X_t)dW_t \tag{1}$$

- Observation process is defined as:

$$dY_t = h(X_t)dt + dB_t \tag{2}$$

where W_t and B_t are uncorrelated noises.

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Introduction to Nonlinear Filtering

The nonlinear filtering problem is to calculate the following conditional expectation

$$\pi_t(\varphi) = E[\varphi(\mathrm{X}_t)|\mathcal{Y}_t] \tag{3}$$

which is the least square estimate for $\varphi(\mathrm{X}_t)$ given $\mathcal{Y}_t$ and satisfies a nonlinear stochastic differential equation, called the Fujisaki-Kallianpur-Kunita [FKK] equation. It is shown that $\pi_t(x)$ can be represented as

$$\pi_t(x) = \rho_t(x) / \int_{\mathrm{R}^d} \rho_t(x) dx \tag{4}$$

with $\rho_t(x)$, unnormalized filtering density, satisfies a linear stochastic differential equation, called the Zakai equation.

Biot-Savart Law

By the Green function technique, one can express the relation between **u** and ω as Biot-Savart Law:

$$\mathbf{u} = \mathbf{u}_\infty + K * \omega. \tag{5}$$

where K is the rotational part of the Green function,

$$K(x) = (2\pi|x|^2)^{-1}(-x_2, x_1). \tag{6}$$

Numerically, it is difficult to handle because of its singularity.

Stochastic Vortex Model

Denote $X_i(t)$ as the position for the i-th point vortice with initial data ξ_i, then

$$X_i(t) = \xi_i + \int_0^t \mathbf{u}_{\epsilon,s}(X_i(s))ds + \int_0^t \sigma(X_i(s))dW_s, \quad \text{for} \quad i = 1, \cdots, N \tag{7}$$

with

$$\mathbf{u}_{\epsilon,t}(x) = \sum_{j=1}^{N} \alpha_j K_\epsilon(x - x_j(t)), \qquad \forall x \in \mathrm{R}^2. \tag{8}$$

Equation (7) will be the signal process in the nonlinear filtering problem.

Idea of Particle Filter

- Numerical method for solving Zakai equation.
- The Monte-Carlo method with correction step.
- Remove the unlikely particles and multiply those situated in the right areas.
- Particle filter is also called the sequential Monte-Carlo method.

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Uniqueness Result

Definition

An Y_t-adapted stochastic process μ_t taking value in $\mathcal{M}(\mathrm{R}^{2N})$ is said to be a measure valued solution to the Zakai equation corresponding to the initial condition $\mu_0(dx) = P(x_0 \in dx|\mathcal{Y}_0)$, if $\langle|\mu_\cdot|, 1\rangle \in \mathrm{L}_2([0, T] \times \Omega; dt \times dP)$, for every $t \leq T$ and $T < \infty$, $\langle|\mu_t|, 1\rangle \in \mathrm{L}_2(\Omega, dP)$ and for any $\psi \in \mathrm{C}_b^2(\mathrm{R}^{2N})$ the following equality holds P-almost surely.

$$\begin{aligned}\langle\mu_t, \psi\rangle = \langle\mu_0, \psi\rangle &+ \int_0^t \langle\mu_s, \mathrm{L}\psi\rangle ds \\ &+ \int_0^t \langle\mu_s, \mathrm{M}\psi\rangle dY_s, \quad \forall t \in [0, T].\end{aligned} \tag{9}$$

Uniqueness Result

Theorem

Assume $a(x) = \frac{1}{2}\sigma(x)\sigma^T(x)$ *and* $h(x)$ *are locally Lipschitz in* x *and both have quadratic growth, then the solution to the signal process (7) exists and satisfies*

$$E \sup_{0\le t\le T} |X(t)|^{2d} \le k_T < \infty, \quad \textit{for} \quad d = 1, 2, \cdots . \tag{10}$$

Moreover, the measure valued solution to the Zakai equation is unique.

Moment Estimate

Lemma

For any $t \geq 0$ *and* $p \geq 2$*, if* $|h(X_j^n(t))| \leq g(t)$ *with*

$$\int_0^t |g(s)|^2 ds < \infty \quad \textit{for} \quad \textit{each} \quad t > 0. \tag{11}$$

Then there exists a constant $c_1^{t,p}$ *such that*

$$\tilde{E}[(a_j^n(t))^p] \leq c_1^{t,p}, \quad j = 1, \cdots, n, \tag{12}$$

where

$$a_j^n(t) = 1 + \sum_{k=1}^m \int_{i\varepsilon}^t a_j^n(s) h^k(X_j^n(s)) dY_s^k. \tag{13}$$

Main Theorem

- Define the $\mathcal{F}_t$-adapted random variable $\psi^n = \{\psi^n_t, \quad t \geq 0\}$ by

$$\psi_t^n := \Big(\prod_{i=1}^{[t/\varepsilon]} \frac{1}{n}\sum_{j=1}^{n} a_j^{n,i\varepsilon}\Big)\Big(\frac{1}{n}\sum_{j=1}^{n} a_j^n(t)\Big). \tag{14}$$

- Let $\rho^n = \{\rho^n_t, \quad t \geq 0\}$ be the measure-valued process defined by

$$\rho_t^n := \frac{\psi^n_{[t/\varepsilon]\varepsilon}}{n}\sum_{j=1}^{n} a_j^n(t)\delta_{X_j^n(t)} \tag{15}$$

- ρ^n_t approximates the solution to the Zakai equation ρ_t and formula (15) is the approximation of Kallianpur-Striebel formula.

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If the coefficients σ and f are globally Lipschitz and have finite initial data. h satisfies the condition in the previous lemma. Then for any $T \geq 0$, there exists a constant c_3^T independent of n such that for any positive $\phi \in C_b(\mathbb{R}^{2n})$, we have

$$\tilde{E}[(\rho_t^n(\phi) - \rho_t(\phi))^2] \leq \frac{c_3^T}{n} \|\phi\|_\infty^2, \qquad t \in [0, T]. \tag{16}$$

In particular, for all $t \geq 0$, ρ_t^n converges in expectation to ρ_t.

Summary

- Unique solvability of the Zakai equation is obtained for unbounded *h*.
- Particle filter convergence of Zakai equation is generalized for *h* with deterministic $\mathbb{L}^2$ functional bound.
- *Outlook*
  - Stochastic vortex model could be analyzed on some Riemannian manifold(i.e. sphere), where coefficients of the model have nice properties.
  - It is interesting to analyze the regularized kernel as ϵ approaches 0 and find its relation with the singular kernel in the Euler equation.

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