Minimising the time to a majority decision

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Overview

Baby problem

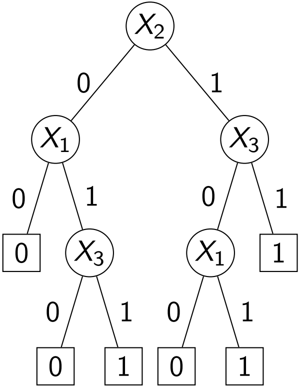
- Let X_1, X_2, X_3 be independent Bernoulli random variables with

$$\mathbb{P}_x(X_i = 1) = x_i, \; x \in [0,1]^3$$

- At least two of the variables agree,
- so they determine a *majority decision*.
- e.g. if $X_1 = 0$, $X_2 = 1$ and $X_3 = 0$ then majority$(X_1, X_2, X_3) = 0$.

The problem: it costs \$1 to observe each variable – what is the cheapest way to find the majority?

Running assumption: $x_1 \leq x_2 \leq x_3$.
Solution: follow the decision diagram



Proof: solve Bellman's optimality equation (see undergraduate lectures on dynamic programming).

Toddler problem: concave observation costs

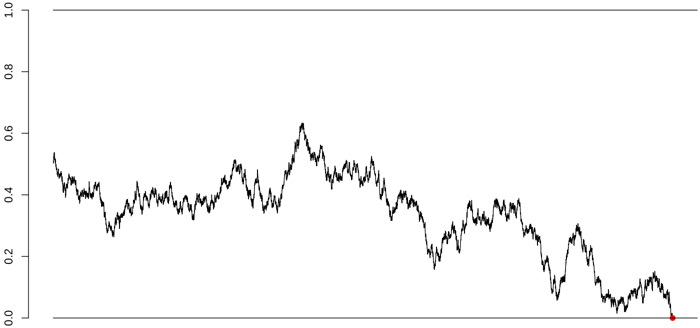
- Suppose now that cost of observing X_i is random with expectation $c_i \geq 0$.
- Algorithm above no longer optimal in general.
- However if $c_i = c(x_i)$ where $c : [0,1] \to [0,\infty)$ is concave and $c(0) = c(1) = 0$, then the algorithm on previous slide *is* optimal.
- the expected cost is

$$\begin{aligned} V^c(x_1, x_2, x_3) &= c(x_2) + x_2\left(c(x_3) + (1-x_3)c(x_1)\right) + \\ &\quad (1-x_2)\left(c(x_1) + x_1 c(x_3)\right) \end{aligned}$$

- Let's consider a model where these concave costs arise naturally – X_i is determined by the boundary at which a diffusion exits an interval.

A diffusive voter

Consider a regular diffusion $(\xi(t), t \geq 0)$ on $[0, 1]$ with absorbing boundaries. A realisation might look like



Expected absorption time is concave in initial position.

more specifically..

- Let $\rho < \infty$ be the absorption time of ξ, i.e.

$$\rho = \inf\{t \geq 0 : \xi(t) \in \{0, 1\}\},$$

- Then $\xi \equiv \xi(\rho) \in \{0, 1\}$ is Bernoulli and without loss,

$$\mathbb{P}_u(\xi = 1) = u \in [0, 1]$$

  ($\xi(0) = u$ under $\mathbb{P}_u$).
- Cost of "revealing" ξ = time "running" the diffusion $= \rho$
- has expectation $c(u) = \mathbb{E}_u[\rho]$ – concave in $u = \mathbb{P}_u(\xi = 1)$

Three diffusive voters..

- Let $(X_i(t); t \geq 0)$, $i = 1, 2, 3$, be three independent copies of ξ, with absorption times ρ_i (and $X_i(0) = x_i$ under $\mathbb{P}_x$).
- Then $X_i \equiv X_i(\rho_i)$, $i = 1, 2, 3$ are independent Bernoulli variables with

$$\mathbb{P}_x(X_i = 1) = x_i,$$

- Revealing value of $X_i \equiv$ running corresponding diffusion until it hits 0 or 1
- i.e., concave observation cost $c(x_i)$.
- We know that if $x_1 \leq x_2 \leq x_3$, we should run X_2 first.

- Acquiring information about $X_i \equiv$ running the corresponding diffusion – begs the question
- What about if we can abandon running a diffusion before it is absorbed?
- i.e. we can run a diffusion for a small time dt and then switch to running another diffusion?
- rest of talk: formulate and answer this.

First of all: intuitive idea

- Recall the (common) generator of the diffusions is

$$\mathcal{G}f(u) = \lim_{t \to 0} \frac{1}{t} \mathbb{E}_u \left[f(\xi(t)) - f(\xi(0)) \right].$$

  E.g. for an Ito diffusion $d\xi(t) = \sigma(\xi(t))dB_t$,
  $\mathcal{G}f(u) = \frac{1}{2}\sigma^2(u)\frac{\partial^2}{\partial u^2}f(u)$.
- Can show $c(u) = \mathbb{E}_u[\rho_i]$ is in $\mathcal{D}(\mathcal{G})$ and $\mathcal{G}c = -1$ on $(0,1)$.
- Thus, if $x_1 < x_2 < x_3$,

$$\mathcal{G}_2 V(x_1, x_2, x_3) = \mathcal{G}c(x_2) = -1,$$

  while

$$\begin{aligned}
\mathcal{G}_1 V(x_1, x_2, x_3) &= x_2(1 - x_3)\mathcal{G}c(x_1) + (1 - x_2)\mathcal{G}c(x_1) \\
&= (1 - x_2 x_3)\,\mathcal{G}c(x_1) = -(1 - x_2 x_3) > -1.
\end{aligned}$$

  and similarly $\mathcal{G}_3 V > -1$.

What does this mean?

Intuitively: if $x_1 < x_2 < x_3$ and we're allowed to "run" one of the diffusions for a small time dt, we should pick X_2.

what is a strategy?

We switch between the diffusions according to a $[0,\infty)^3$-valued process

$$c = (c_1(t), c_2(t), c_3(t); t \geq 0)\,.$$

"$c_i(t)$ is the amount of time spent on X_i by calendar time t", i.e.

- c_i is increasing, $c_i(0) = 0$.
- $c_1(t) + c_2(t) + c_3(t) = t$, $\forall t \geq 0$
- c is *non-anticipating* (makes choices based only on what has been revealed). I.e. for every $\eta \in [0,\infty)^3$ define

$$\mathcal{F}(\eta) = \sigma(X_i(u_i);\ 0 \leq u_i \leq \eta_i,\ i = 1,2,3).$$

We require $\{c(t) \preceq \eta\} \in \mathcal{F}(\eta)$ for each $t \geq 0$.

- The process we observe is

$$X^c = (X_1(c_1(t)), X_2(c_2(t)), X_3(c_3(t)); t \geq 0).$$

- The *decision time* is the first time that at least two components of X^c coincide at a common endpoint of $[0,1]$:

$$\tau^c = \inf\{t \geq 0 : X_i^c(t) = X_j^c(t) \in \{0,1\}, \text{ for distinct } i,j\}.$$

- for any sensible strategy, τ^c is a.s. finite.
- Problem: Find a strategy $c^\star$ with minimal decision time, i.e.

$$\mathbb{E}[\tau^{c^\star}] = \inf_c \mathbb{E}[\tau^c]$$

Optimising over multi-parameter time changes!

Theorem

There exists an a.s. unique strategy $c^\star$ such that $c_i^\star$ increases only at times $t \geq 0$ (i.e. $c_i^\star(s) > c_i^\star(t)$ for every $s > t$) when

$$X_j^{c^\star}(t) \leq X_i^{c^\star}(t) \leq X_k^{c^\star}(t)$$

under some labelling $\{i, j, k\} = \{1, 2, 3\}$ of the processes.

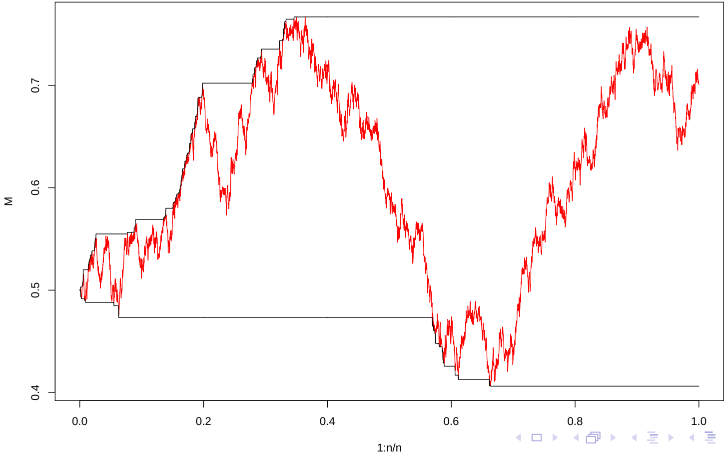
That is, if $t_0 \geq 0$ is such that

$$X_1^{c^\star}(t_0) < X_2^{c^\star}(t_0) < X_3^{c^\star}(t_0)$$

then "$c^\star$ runs X_2 exclusively until it hits $X_1^{c^\star}(t_0)$ or $X_3^{c^\star}(t_0)$."

What does the observation process look like?

The middle valued component behaves as a diffusion between the minimum and maximum, where it receives a singular perturbation: it is a “doubly perturbed diffusion”.



Optimality theorem

Actually $c^\star$ is optimal in a very strong sense: $\tau^{c^\star}$ is a stochastic minimum, i.e.

Theorem

The decision time $\tau^{c^\star}$ of the strategy $c^\star$ satisfies

$$\mathbb{P}_x(\tau^{c^\star} > t) = \inf_c \mathbb{P}_x(\tau^c > t), \text{ for every } t \geq 0 \text{ and } x \in [0,1]^3.$$

where the infimum is taken over all strategies.

Homework

- Have only proven Theorem for Ito diffusions with continuous diffusivities (i.e. solutions of “nice” SDEs).
- It “must” be true for general diffusions and,
- there “should” be an elegant proof.
- Similar idea should work for majority of $m = 2k + 1$ diffusive voters

Thanks!