On random matrices with weakly confining potentials

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- THE STIELTJES-WIGERT RANDOM MATRIX MODEL

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- summary

- Let $H = (H_{jk})_{j,k=1}^{N}$ be a square $N \times N$ matrix with randomly distributed elements H_{jk}. This is a random matrix with respect to a probability distribution, defined by:

$$P_{\beta}^{(N)}(H) \propto \exp\left(-\beta \mathrm{Tr} \mathrm{V}(H)\right),$$

- The first and most studied ensembles are the Gaussian ensembles, $V(H) = H^2$. It can be shown that the previous expression is automatically restricted to the form

$$P(H) = \exp\left(-a \mathrm{Tr} H^2 + b \mathrm{Tr} H + c\right), \qquad a > 0,$$

if one postulates statistical independence of the matrix elements H_{ij}. There are three different ensembles defined depending on the values of the parameter $\beta = 1, 2$ or 4.

Ensembles of random $N \times N$ matrices H are defined by the following demands:

1. The probability $P(H)d\left[H\right]$ is invariant under any transformation $H \rightarrow U^{-1}HU$, where U is either an orthogonal ($\beta = 1$), unitary ($\beta = 2$) or symplectic ($\beta = 4$) matrix. That is to say, if $H' = U^{-1}HU$ where U belongs to the unitary group $U(N;\beta)$, then $P(H')d\left[H'\right] = P(H)d\left[H\right]$.

2. The matrix elements which are not related by the symmetry of the matrix are statistically independent (Gaussian ensembles)

- Diagonalization: for each matrix H there is a matrix U that maps it onto its eigenvalues. The Jacobian of the transformation is $J_\beta(\{x_i\}) = \prod_{i<j} |x_i - x_j|^\beta$. The resulting expression is

$$P(x_1, \ldots, x_N) = \mathrm{C_N} \prod_{i<j} |x_i - x_j|^\beta \prod_{i=1}^{N} \mathrm{e}^{-\frac{\beta}{2} V(x_i)}.$$

  The potential $V(x) = \log^2 x$ (log-normal weight function $\omega(x) = \mathrm{e}^{-\log^2 x}$) is at the center of most developments in this talk.
- The main relevant quantities are m-partial integrations over the previous N-dimensional probability density function

- A central and powerful result in random matrix theory is that m-point correlation function can be computed from the two-point kernel as follows (simplest case of a Hermitian ($\beta = 2$) ensemble)

$$R_m^{(N)}(x_1, \ldots, x_m) = \det\left(K_N\left(x_i, x_j\right)\right)_{1 \leq i,j \leq m}$$

- Orthogonal polynomials method $\Longrightarrow$ explicit expressions for $K_N\left(x_i, x_j\right)$. Let $p_N(x) = c_N x^N + \ldots$ the Nth orthogonal polynomial associated to $\mathrm{e}^{-V(x)}$, the two-point kernel is

$$\begin{aligned} K_N\left(x, y\right) &= \mathrm{e}^{-\frac{V(x)+V(y)}{2}} \sum_{i=0}^{N-1} p_i\left(x\right) p_i\left(y\right) \\ &= \frac{c_{N-1}}{c_N} \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x-y} \mathrm{e}^{-\frac{V(x)+V(y)}{2}} \end{aligned}$$

Examples of weakly confining potentials

- The Wigner-Dyson paradigm refers to strongly confining potentials like $V(x) = x^2$ (Gaussian), other classical ensembles (like Laguerre and Jacobi ensembles), or polynomial potentials.
- Models with growth $\lim_{x\to\infty} V(x) < \mathrm{e}^{-|x|}$ when $x \in (-\infty, \infty)$ or $\lim_{x\to\infty} V(x) < \mathrm{e}^{-\sqrt{x}}$ when $x \in (0, \infty)$ are weakly confining (moment problem).
- We study even weaker potentials that behave as $V(x) \sim k^2 \log^2 x$ for large x. These models have a two-point kernel ($q = \mathrm{e}^{-2a}$)

$$K(x-y) = \frac{a \sin\left(\pi\left(x-y\right)\right)}{\pi \sinh\left(a\left(x-y\right)\right)}$$

- We consider Chern-Simons theory on a three-manifold M and for a gauge group G, with action

$$S(A) = \frac{k}{4\pi} \int_M \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

where A is a connection on M.

- Witten showed in 1989, that the partition function of Chern-Simons theory

$$Z_k(M) = \int \mathcal{D}A e^{iS_{\mathrm{CS}}(A)},$$

defines a topological invariant.

- Chern-Simons theory is of interest in the study of topological strings, the fractional quantum Hall effect, ...
- The partition function of CS theory on certain manifolds has very simple expressions (M. Mariño, Comm. Math. Phys. 253, 25 (2004)).The simplest case is S^3 and gauge group $U(N)$

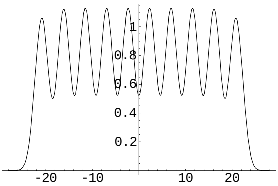
$$Z_{\mathrm{CS}}(S^3) = \int_{-\infty}^{\infty} \prod_i \frac{du_i}{2\pi} \mathrm{e}^{-\frac{u_i^2}{2g_s}} \prod_{i<j} \left(2\sinh\left(\frac{u_i - u_j}{2}\right)\right)^2$$

- Thus, we have N-dimensional integral expressions for Chern-Simons partition functions whose expression resemble that of random matrix theory.

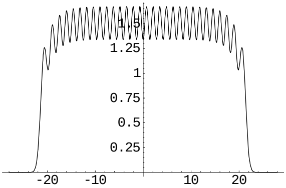
- The models with a $\log^2 x$ behavior can be solved with q-orthogonal polynomials (polynomials in the q-Askey scheme). In particular, the model $\omega(x) = \exp\left(-k^2 \log^2 x\right)$ is solved with Stieltjes-Wigert polynomials
- The Gaussian model with a sinh Vandermonde is easily mapped into a Hermitian model with a $\exp\left(-k^2 \log^2 x\right)$ weight function. Hence, the partition function of Chern-Simons theory on S^3 and gauge group $U(N)$ is given by

$$
\begin{aligned}
Z_{\mathrm{CS}}(S^3) &= \int [dM] \mathrm{e}^{-\frac{1}{2g_s}\mathrm{Tr}(\log M)^2} \\
&= \mathrm{e}^{\frac{1}{4} i\pi N^2} (k+N)^{-N/2} \prod_{j=1}^{N-1} \left(2 \sin \frac{\pi j}{k+N}\right)^{N-j}
\end{aligned}
$$

- There are expressions for other CS observables that also can be computed, like $< \mathfrak{s}_\lambda(M) >_w = \int [dM]\mathfrak{s}_\lambda(M) \mathrm{e}^{-\frac{1}{2g_s}\mathrm{Tr}(\log M)^2}$.
- The density of states can also be computed with the orthogonal polynomials and the behavior is very different from the one of the Gaussian ensembles (Wigner's semicircle).



$N = 10$ and $q = 0.3$



$N = 30$ and $q = 0.5$

Conclusions and Outlook

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- The models behave differently from the better known classical random matrix ensembles (e.g. Gaussian models).
- There exist connections to other problems, like the study of N non-intersecting Brownian motions or with $1D$ integrable models, like the Sutherland model (arXiv:hep-th/0406093, arXiv:0808.1079, arXiv:1003.1228).