Geometric influence

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Arizona, 15th March, 2010

Joint work with Nathan Keller and Elchanan Mossel

Influences of boolean functions on hypercube

- $f : \{0,1\}^n \to \{0,1\}$. The influence of j-th coordinate on f

$$I_j(f) := \mathbb{P}\{x \in \{0,1\}^n : f(x) \neq f(x \oplus e_j)\},$$

  where $x \oplus e_j$ denotes the point obtained from x by replacing x_j by $1 - x_j$. Here $\mathbb{P}$ = product Bernoulli measure on $\{0,1\}^n$.

- Central concept in Discrete Harmonic Analysis - applications in Combinatorics, Theoretical Computer Science, Percolation, Social Choice Theory, ...

- Identify the subset $A \subseteq \{0,1\}^n$ with the indicator function 1_A.

Why useful?

- Geometric meaning: For set $A \subseteq \{0,1\}^n$,

$$\sum_{j=1}^{n} I_j(A) = 2^{-(n-1)} \#\{ \text{ edges between } A \text{ and } A^c \}.$$

- Lower bound on max influence by Kahn-Kalai-Linial (KKL) '88:

$$\max_{1 \le j \le n} I_j(A) \ge ct(1-t)\frac{\log n}{n}, \quad t = \mathrm{Ber}^{\otimes n}(A).$$

- Russo's Lemma (Margulis '74, Russo '82): If $A \subseteq \{0,1\}^n$ increasing, then

$$\frac{d\mathrm{Ber}(p)^{\otimes n}(A)}{dp} = \sum_{j=1}^{n} I_j(A).$$

Influences in continuous probability space

- How can we define influences for $f : (\mathbb{R}^n, \nu^{\otimes n}) \to \{0,1\}$?
  Several existing definitions!

- $A \subseteq \mathbb{R}^n$, $A_j^x := \left\{y \in \mathbb{R} : (x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_n) \in A\right\}$.

- Bourgain et.al. '92:

$$I_j(A) = \mathbb{P}\left\{x \in \mathbb{R}^n : 1_{A_j^x} \text{ is not constant function}\right\}.$$

  - Artificial
  - KKL bound still holds!

- Mossel et.al. '09:

$$I_j(A) = \mathbb{E}\left[\mathrm{Var}(1_{A_j^x} \middle| x \setminus \{x_j\})\right].$$

  - Reasonable
  - No KKL type bound.

- All of them lack geometric interpretation for continuous probability spaces.

The *geometric influence* of the j-th coordinate on $A \subseteq (\mathbb{R}^n, \nu^{\otimes n})$ is

$$I_j^{\mathcal{G}}(A) := \mathbb{E}[m_\nu(A_j^x)].$$

where

$$m_\nu(A_j^x) := \liminf_{r \downarrow 0} \frac{\nu(A_j^x + [-r, r]) - \nu(A)}{r}.$$

is lower Minkowski content of the fibre A_j^x.

- We always assume ν has a density.

Lemma

Let ν be a probability measure on $\mathbb{R}$ with a 'nice' density. Let $A \subset \mathbb{R}^n$ be a convex set. Then

$$\lim_{r \downarrow 0} \frac{\nu^{\otimes n}(A + [-r, r]^n) - \nu^{\otimes n}(A)}{r} = \sum_{i=1}^{n} I_i^{\mathcal{G}}(A).$$

- In literature, $\liminf_{r \downarrow 0} \frac{\nu^{\otimes n}(A+[-r,r]^n) - \nu^{\otimes n}(A)}{r}$ is sometimes called 'boundary under uniform enlargement.'
- Not true for general sets, e.g. $\mathbb{Q}^n$.

Main Result : KKL-type bound

Theorem

Consider the product Gaussian measure $\Phi^{\otimes n}$ on $\mathbb{R}^n$. Then for any Borel-measurable set $A \subset \mathbb{R}^n$ with $\Phi^{\otimes n}(A) = t$

$$\max_{1 \le i \le n} I_i^{\mathcal{G}}(A) \ge ct(1-t)\frac{\sqrt{\log n}}{n},$$

where $c > 0$ is a universal constant.

- Dependence on n is tight.
- Similar statements for measures whose isoperimetric function $\mathcal{I}(t)$ satisfy

$$\mathcal{I}(t) \ge K \min\{t, 1-t\} \left(\log \frac{1}{\min\{t, 1-t\}}\right)^{\delta}$$

for all $t \in [0, 1]$ and for some $\delta > 0$.

Russo's formula and sharp threshold

Let ν be a probability measure on $\mathbb{R}$ with 'nice' continuous density λ.
Let $\{\nu_\theta : \theta \in \mathbb{R}\}$ denote a family of probability measures with ν_θ has a density $\lambda_\theta(x) = \lambda(x - \theta)$.

Lemma

Let $A \subseteq \mathbb{R}^n$ be increasing. Then the function $\theta \to {\nu_\theta}^{\otimes n}(A)$ is differentiable and its derivative is given by

$$\frac{d{\nu_\theta}^{\otimes n}(A)}{d\theta} = \sum_{j=1}^{n} I_j^{\mathcal{G}}(A)$$

Corollary

Let $\Phi_\theta = N(\theta, 1)$. Let $A \subset \mathbb{R}^n$ be an increasing and transitive

$$\Phi_{\theta_0}^{\otimes n}(A) > \epsilon \quad \Rightarrow \quad \Phi_{\theta_1}^{\otimes n}(A) > 1 - \epsilon$$

where $\theta_1 - \theta_0 = c \log(1/2\epsilon)(\log n)^{-1/2}$.

Gaussian isoperimetric inequality under symmetry

Theorem (Sudakov & Tsirelson '74, Borell '75)

For any $A \subseteq R^n$

$$\liminf_{r \downarrow 0} \frac{\Phi^{\otimes n}(A + [-r, r]^n) - \Phi^{\otimes n}(A)}{r} \geq \underbrace{\phi(\Phi^{-1}(t))}_{\text{Gaussian isoperimetric function}}$$

where $t = \Phi^{\otimes n}(A)$.

Theorem

Then for any transitive set $A \subset \mathbb{R}^n$ *we have*

$$\liminf_{r \downarrow 0} \frac{\Phi^{\otimes n}(A + [-r, r]^n) - \Phi^{\otimes n}(A)}{r} \geq ct(1 - t)\sqrt{\log n},$$

where $t = \Phi^{\otimes n}(A)$ *and* $c > 0$ *is a universal constant .*

A statistical application

- Let $X_1, X_2, \ldots, X_n$ i.i.d. $N(\theta, 1)$.

$$H_0 : \theta = \theta_0 \text{ vs } H_1 : \theta = \theta_1 \quad (\theta_1 > \theta_0)$$

- The most efficient test:

$$\text{Reject } H_0 \text{ if } \bar{X}_n > K \text{ where } \mathbb{P}_{\theta_0}\{\bar{X}_n > K\} = 5\%$$

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- How bad we would perform if we use the test:

$$\text{Reject } H_0 \text{ if } f(X_1, \ldots, X_n) > K \text{ where } \mathbb{P}_{\theta_0}\{f(X_1, \ldots, X_n) > K\} = 5\%$$

where $f : \mathbb{R}^n \to \mathbb{R}$ satisfies

(i) f is nondegenerate.
(ii) f is transitive.
(iii) The function f is monotonically increasing in each of its coordinates.

- Answer: $\boxed{\theta_1 > \theta_0 + \Omega(\frac{1}{\sqrt{\log n}})}$ ensures 95% power for all such f.