Gaussian Fluctuations of Eigenvalues in Wigner Random Matrices

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Wigner Matrices

Real Symmetric Wigner Matrix

Let n be a large number. A Wigner real symmetric matrix (of size n) is defined as a random real symmetric $n \times n$ matrix $M_n = (m_{ij})_{1 \leq i,j \leq n}$ where

- For $1 \leq i < j \leq n$, m_{ij} are i.i.d. real random variables.
- For $1 \leq i \leq n$, m_{ii} are i.i.d. real random variables.
- The entries m_{ij} have exponential decay i.e. there exists constants $C, C' > 0$ such that $\mathbb{P}\left(|m_{ij}| \geq t^C\right) \leq \exp(-t)$, for all $t \geq C'$.

- A similar definition holds for complex Hermitian Wigner matrices
- We can order the n real eigenvalues so that $x_1 \leq x_2 \leq \ldots \leq x_n$

The Gaussian Orthogonal Ensemble (GOE) ($\beta = 1$)

Probability Distribution

$$\mathbb{P}(\mathrm{d}H) = C_n^{(\beta)} e^{-\frac{\beta}{2}\mathrm{Tr}H^2} \mathrm{d}H$$

- Consists of $n \times n$ real symmetric matrices
- There are $\frac{n(n+1)}{2}$ distinct entries:

$$\{H_{ij}; 1 \leq i \leq j \leq n\}$$

- Each entry is a Gaussian random variable with mean zero and variance $\frac{1+\delta_{ij}}{2}$

The Gaussian Unitary Ensemble (GUE) ($\beta = 2$)

Probability Distribution

$$\mathbb{P}(\mathrm{d}H) = C_n^{(\beta)} e^{-\frac{\beta}{2}\mathrm{Tr}H^2}\mathrm{d}H$$

- Consists of $n \times n$ complex Hermitian matrices
- There are n^2 distinct entries:

$$\{\mathrm{Re}H_{ij}; 1 \leq i \leq j \leq n, \mathrm{Im}H_{ij}; 1 \leq i < j \leq n\}$$

- Each entry is a Gaussian random variable with mean zero and variance $\frac{1+\delta_{ij}}{4}$

The Largest Eigenvalue in the GUE

- Consider an $n \times n$ GUE matrix
- The largest eigenvalue grows like $\sqrt{2n}$ and has a standard deviation of order $n^{-\frac{1}{6}}$
- The centered and rescaled eigenvalue

$$\left(x_{\max} - \sqrt{2n}\right) \sqrt{2} n^{1/6}$$

converges in distribution as $n \to \infty$ to the Tracy-Widom distribution

Fluctuations of Eigenvalues

- We wish to study the fluctuations of the kth largest eigenvalue, x_k, from the GOE when $k \to \infty$ and $n - k \to \infty$.

Examples

Example 1 (bulk): $k = n/2$
Example 2 (edge): $k = n - \log n$

- Gustavsson proved that in the GUE, the limiting distribution is normal.
- We extend this result to the GOE and then apply a universality result of Tao and Vu to extend the result to a class of real symmetric Wigner matrices.

The Expected Value of x_k

A Consequence of Wigner's Semicircle Law

Define

$$G(t) = \frac{2}{\pi} \int_{-1}^{t} \sqrt{1 - x^2} \mathrm{dx} \qquad -1 \leq t \leq 1.$$

Let $t = t(k, n) = G^{-1}(k/n)$

- We expect that x_k should be near $t\sqrt{2n}$
- If $k = \frac{n}{2}$, then $t = 0$

Main Results in the Bulk

Let $x_1 < x_2 < \cdots < x_n$ be the ordered eigenvalues from a random matrix drawn from the GOE, GUE, or GSE. Consider $\{x_{k_i}\}_{i=1}^m$ such that $0 < k_i - k_{i+1} \sim n^{\theta_i}$, $0 < \theta_i \leq 1$, and $\frac{k_i}{n} \to a_i \in (0,1)$ as $n \to \infty$. Define $s_i = s_i(k_i, n) = G^{-1}(k_i/n)$ and set

$$X_i = \frac{x_{k_i} - s_i\sqrt{2n}}{\left(\frac{\log n}{2\beta(1-s_i^2)n}\right)^{1/2}} \quad i = 1, \ldots, m$$

where $\beta = 1, 2, 4$ corresponds to the GOE, GUE, or GSE. Then as $n \to \infty$,

$$\mathbb{P}[X_1 \leq \xi_1, \ldots, X_m \leq \xi_m] \longrightarrow \Phi_\Lambda(\xi_1, \ldots, \xi_m)$$

where Φ_Λ is the cdf for the m-dimensional normal distribution with covariance matrix $\Lambda_{i,j} = 1 - \max\{\theta_k : i \leq k < j < m\}$ if $i < j$ and $\Lambda_{i,i} = 1$.

Main Results for the Edge

Let $x_1 < x_2 < \cdots < x_n$ be the ordered eigenvalues from a random matrix drawn from the GOE, GUE, or GSE. Consider $\{x_{n-k_i}\}_{i=1}^m$ such that $k_1 \sim n^\gamma$ where $0 < \gamma < 1$ and $0 < k_{i+1} - k_i \sim n^{\theta_i}$, $0 < \theta_i < \gamma$. Set

$$X_i = \frac{x_{n-k_i} - \sqrt{2n}\left(1 - \left(\frac{3\pi k_i}{4\sqrt{2}n}\right)^{2/3}\right)}{\left(\left(\frac{1}{12\pi}\right)^{2/3} \frac{2\log k_i}{\beta n^{1/3} k_i^{2/3}}\right)^{1/2}} \quad i = 1, \ldots, m$$

where $\beta = 1, 2, 4$ corresponds to the GOE, GUE, or GSE. Then as $n \to \infty$,

$$\mathbb{P}[X_1 \leq \xi_1, \ldots, X_m \leq \xi_m] \longrightarrow \Phi_\Lambda(\xi_1, \ldots, \xi_m)$$

where Φ_Λ is the cdf for the m-dimensional normal distribution with covariance matrix $\Lambda_{i,j} = 1 - \frac{1}{\gamma}\max\{\theta_k : i \leq k < j < m\}$ if $i < j$ and $\Lambda_{i,i} = 1$.

Result for Real Symmetric Wigner Matrices

- Using a recent universality result by Tao and Vu known as the Four Moment Theorem, we have the following corollary.

Corollary

The conclusions of the previous two Theorems also hold with $\beta = 1$ when $x_1 \leq x_2 \leq \ldots \leq x_n$ are the ordered eigenvalues of any other real symmetric Wigner matrix $M_n = \left(m_{ij}\right)_{1 \leq i,j \leq n}$ where m_{ij} has mean 0 and variance $\frac{1+\delta_{ij}}{2}$ for $1 \leq i \leq j \leq n$ and $\mathbb{E}(m_{ij}^3) = 0$, $\mathbb{E}(m_{ij}^4) = 3/4$ for $1 \leq i < j \leq n$.

Thanks!!

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