Ferromagnetic Ordering of Energy Levels

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The 1-D Spin $\frac{1}{2}$ Heisenberg Ferromagnet

The length L spin chain is defined over the Hilbert space

$$\mathcal{H} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{L}$$

with $\mathfrak{sl}_2$ invariant Hamiltonian

$$H = -\sum_{i=1}^{L-1} J_{i,i+1}(4\vec{S}_i \cdot \vec{S}_{i+1} - \mathbb{1})$$

Let

$$E(H,S) = \min \operatorname{spec} H|_{\mathcal{H}^{(S)}}$$

where $\mathcal{H}^{(S)} \subset \mathcal{H}$ is the subspace of total spin S.

Ordering of Energy Levels

H is ferromagnetic, and one can show that

$$E(H,S) \geq E(H,S') \text{ whenever } S < S'$$

For a more general $\mathfrak{sl}_2$ invariant Hamiltonian $\widetilde{H}$, we say that $\widetilde{H}$ satisfies the Ferromagnetic Ordering of Energy Levels (FOEL) property if

$$E(\widetilde{H},S) \geq E(\widetilde{H},S') \text{ whenever } S < S'$$

For higher spin models, we can show that

$$\widetilde{H} = -\sum_{i=1}^{L-1} \sum_{\ell_i} J_{i,i+1}^{\ell_i} \left(\frac{1}{s_i s_{i+1}} \vec{S}_i \cdot \vec{S}_{i+1} - \mathbb{1} \right)^{\ell_i}$$

satisfies FOEL, if the coupling coefficients $J_{i,i+1}^{\ell_i}$ satisfy a system of inequalities.

Sufficient Conditions for FOEL

We provide an argument for the spin $\frac{1}{2}$ case which generalizes to $\widetilde{H}$. Consider the sequence of Hilbert spaces

$$\mathcal{H}_1 = \mathbb{C}^2, \mathcal{H}_2 = \mathbb{C}^2 \otimes \mathbb{C}^2, \ldots, \mathcal{H}_L = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{L} = \mathcal{H}$$

and the sequence of Hamiltonians $H_k : \mathcal{H}_k \to \mathcal{H}_k$

$$H_1 = 0, H_2 = -J_{1,2}(4\vec{S}_1 \cdot \vec{S}_2 - \mathbb{1}), \ldots, H_L = -\sum_{i=1}^{L-1} J_{i,i+1}(4\vec{S}_i \cdot \vec{S}_{i+1} - \mathbb{1})$$

Sufficient Conditions for FOEL

The proof is by induction. Clearly H_1 satisfies FOEL. Suppose that H_k satisfies FOEL. Then

$$\begin{array}{ccccc} E(H_k, S) & \geq & E(H_k, S+1) & \geq & E(H_k, S+2) \\ \text{IV} & \nwarrow\!\!\!/ & \text{IV} & \nwarrow\!\!\!/ & \text{IV} \\ E(H_{k+1}, S+\frac{1}{2}) & \geq & E(H_{k+1}, S+\frac{3}{2}) & \geq & E(H_{k+1}, S+\frac{5}{2}) \end{array}$$

Hence, H_{k+1} satisfies FOEL.

$$E(H_{k+1}, S+\frac{1}{2}) \geq E(H_k, S+1)$$

follows from the induction hypothesis, and from the fact that each piece of the Hamiltonian added at step $k+1$, $-J_{k,k+1}(4\vec{S}_k \cdot \vec{S}_{k+1} - \mathbb{1})$, is positive semi-definite.

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$$
\begin{array}{ccccc}
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\rotatebox{-90}{$\geq$} & \rotatebox{45}{$\geq$} & \rotatebox{-90}{$\geq$} & \rotatebox{45}{$\geq$} & \rotatebox{-90}{$\geq$} \\
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Sufficient Conditions for FOEL

$$E(H_k, S) \geq E(H_{k+1}, S + \frac{1}{2})$$

follows from the following extension of the Perron-Frobenius Theorem:

Theorem

Let A be an $n \times n$ matrix and let B be an $m \times m$ matrix such that $n \leq m$ and all off-diagonal elements of A and B are non-positive and $B_{i,j} \leq A_{i,j}$ for $1 \leq i, j \leq n$. Then

$$\inf \operatorname{spec}(B) \leq \inf \operatorname{spec}(A)$$

Now choose $A = H_k|_{\mathcal{H}}^{(S)}$ and $B = H_{k+1}|_{\mathcal{H}}^{(S+\frac{1}{2})}$ with the dual canonical basis (or Hulthén bracket basis). One can perform the necessary calculations by realizing the Hamiltonian H as a sum of elements in the Temperley-Lieb algebra.