

# Lifshitz tail for Schrödinger Operators with random $\delta$ magnetic fields

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# Random magnetic Schrödinger operators

We consider the Schrödinger operators on  $\mathbb{R}^2$  with random magnetic fields:

$$\mathcal{L}_\omega = \left( \frac{1}{i} \nabla - a_\omega \right)^2,$$

where  $\omega$  is an element of some probability space  $(\Omega, \mathbf{P})$ . The vector-valued function  $a_\omega = (a_{\omega,x}, a_{\omega,y})$  is the magnetic vector potential, and the magnetic field corresponding to  $a_\omega$  is given by

$$\text{curl } a_\omega = \partial_x a_{\omega,y} - \partial_y a_{\omega,x}.$$

## Random $\delta$ magnetic fields

We assume

$$\operatorname{curl} a_\omega = \sum_{\gamma \in \Gamma_\omega} 2\pi \alpha_\gamma(\omega) \delta_\gamma \quad (1)$$

in the distribution sense, where  $\Gamma_\omega$  is a discrete set in  $\mathbb{R}^2$ ,  $\alpha(\omega) = \{\alpha_\gamma(\omega)\}_{\gamma \in \Gamma_\omega}$  is a sequence of real numbers satisfying  $0 \leq \alpha_\gamma(\omega) < 1$ , and  $\delta_\gamma$  is the Dirac measure concentrated on the point  $\gamma$ .

## Assumptions on $\Gamma_\omega, \alpha_\omega$

**(A1)** For any Borel set  $E$  in  $\mathbb{R}^2$ , the following functions are measurable with respect to  $\omega$ .

$$n_\omega(E) = \#(\Gamma_\omega \cap E), \quad \Phi_\omega(E) = \sum_{\gamma \in \Gamma_\omega \cap E} \alpha_\gamma(\omega)$$

**(A2)** Let  $Q_0 = \{z = x + iy \mid -\frac{1}{2} \leq x < \frac{1}{2}, -\frac{1}{2} \leq y < \frac{1}{2}\}$ . For any Borel set  $E \subset Q_0$ , the random variables  $\{n(E + n)\}_{n \in \mathbb{Z}^2}$  and  $\{\Phi(E + n)\}_{n \in \mathbb{Z}^2}$  are i.i.d. random variables.

## Assumptions on $\Gamma_\omega, \alpha_\omega$

**(A3)**  $0 < \mathbf{E}[\Phi(Q_0)] < \infty, \mathbf{V}[\Phi(Q_0)] < \infty.$

**(A4)** For any  $\epsilon > 0,$

$$\mathbf{P}\{n(Q_0) \leq 1 \text{ and } \Phi(Q_0) < \epsilon\} > 0.$$

**(A5)** For some  $\delta$  with  $0 < \delta < 1,$

$$\mathbf{P}\{n(Q_0) = n(\delta Q_0) = 1\} > 0.$$

## Examples

**Ex.1 Small perturbation of a lattice.**  $\Gamma_\omega = \{n + f_n(\omega)\}_{n \in \mathbb{Z}^2}$ , where  $\{f_n\}$  are i.i.d.,  $\mathbb{R}^2$ -valued random variables satisfying  $|f_n(\omega)| < \delta/2$  for some deterministic constant  $\delta$  with  $0 < \delta < 1$ .  $\{\alpha_\gamma\}$  are  $[0, 1)$ -valued i.i.d. random variables independent of  $\{f_n\}$ , satisfying  $\mathbf{E}[\alpha_\gamma] > 0$  and

$$\mathbf{P}\{\alpha_\gamma < \epsilon\} > 0 \quad \text{for any } \epsilon > 0. \quad (2)$$

In this example,  $f_n$  may be constant (ordinary lattice).

## Examples

**Ex.2 Poisson model.**  $\Gamma_\omega$  is the Poisson point process on  $\mathbb{R}^2$  with intensity measure  $\rho dx dy$  for some positive constant  $\rho$ , i.e.,

- (1) For disjoint  $E_1, \dots, E_k$ ,  $N(E_1), \dots, N(E_k)$  are independent,
- (2)  $\mathbb{P}\{N(E) = j\} = \frac{(\rho|E|)^j}{j!} e^{-\rho|E|}$  for  $j = 0, 1, 2, \dots$ , where  $|E|$  is the Lebesgue measure of  $E$ .

$\{\alpha_\gamma\}$  are i.i.d. random variables independent of  $\Gamma_\omega$  and satisfying  $\mathbf{E}[\alpha_\gamma] > 0$  (the assumption (2) in Ex.1 is not necessary;  $\alpha_\gamma$  can be constant ).

## Construction of vector potentials

For given  $\Gamma_\omega$  and  $\alpha_\omega$ , we indentify  $z = (x, y)$  as the complex number  $z = x + iy$  and put

$$\begin{aligned} a_\omega &= (\operatorname{Im} \phi_\omega, \operatorname{Re} \phi_\omega), \\ \phi_\omega(z) &= \frac{\alpha_0(\omega)}{z} + \sum_{\gamma \in \Gamma_\omega \setminus \{0\}} \alpha_\gamma(\omega) \left( \frac{1}{z - \gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2} \right), \quad (3) \end{aligned}$$

where we put  $\alpha_0(\omega) = 0$  if  $0 \notin \Gamma$ . Under the above assumptions, the sum (3) converges almost surely.

## Construction of vector potentials

Let  $C$  be a counter-clockwise, closed, simple smooth curve not intersecting with  $\Gamma$ , and  $D$  its interior. The magnetic flux through  $D$  is calculated as

$$\begin{aligned} \int_D (\partial_x a_{\omega,y} - \partial_y a_{\omega,x}) dx dy &= \int_C (a_{\omega,x} dx + a_{\omega,y} dy) \\ &= \int_C (\operatorname{Im} \phi_\omega dx + \operatorname{Re} \phi_\omega dy) = \operatorname{Im} \int_C \phi_\omega dz \\ &= 2\pi \sum_{\gamma \in D \cap \Gamma_\omega} \alpha_\gamma, \end{aligned}$$

by the Stokes theorem and the residual theorem. Thus (1) holds.

## Boundary conditions

We define a self-adjoint extension of  $\mathcal{L}_\omega$  by

$$\begin{aligned} H_\omega u &= \mathcal{L}_\omega u, \\ D(H_\omega) &= \{u \in L^2(\mathbb{R}^2) \mid \mathcal{L}_\omega u \in L^2(\mathbb{R}^2), \\ &\quad \limsup_{z \rightarrow \gamma} |u(z)| < \infty \quad \forall \gamma \in \Gamma_\omega\}. \end{aligned}$$

At the point  $\gamma$  with  $0 < \alpha_\gamma < 1$ , the condition  $\limsup_{z \rightarrow \gamma} |u(z)| < \infty$  can be replaced by  $\lim_{z \rightarrow \gamma} u(z) = 0$ . This means there is infinitesimally thin, electrically shielded solenoid at the point  $\gamma$ .

## Integrated density of States

Let  $Q_k = \{z = x + iy \mid -k - \frac{1}{2} \leq x < k + \frac{1}{2}, -k - \frac{1}{2} \leq y < k + \frac{1}{2}\}$  for  $k = 0, 1, 2, \dots$ . Let  $H_\omega^k$  be the self-adjoint realization of the operator  $\mathcal{L}_\omega$  on  $L^2(Q_k)$  with the Neumann boundary conditions  $(\frac{1}{i}\nabla - a_\omega) u \cdot \mathbf{n} = 0$  on  $\partial Q_k$  and the same boundary conditions at  $\gamma \in \Gamma \cap Q_k$  as above. For  $E \in \mathbb{R}$ , we define

$$N_\omega^k(E) = \# \{ \text{eigenvalues of } H_\omega^k \text{ less than or equal to } E \},$$

$$N(E) = \lim_{k \rightarrow \infty} \frac{1}{|Q_k|} N_\omega^k(E),$$

where  $|\cdot|$  denotes the Lebesgue measure. As is well-known,  $N(E)$  exists and is independent of  $\omega$  almost surely.

## Lifshitz tail

We can prove that  $\sigma(H_\omega) = [0, \infty)$  under the assumptions (A1)–(A4), so  $\inf \sigma(H_\omega) = 0$ . Our main result is the following.

**Theorem.** Under assumptions (A1)–(A5), there exist some constants  $C > 0$  and  $E_0 > 0$  such that

$$N(E) \leq e^{-CE^{-1}} \quad (4)$$

for any  $E$  with  $0 < E < E_0$ .

The estimate (4) is called the **Lifshitz tail**, after the name I. M. Lifshitz, who derived this type of estimate in 1964 by some physical consideration.

# History

The Lifshitz tail is proved mainly for Schrödinger operators with random **scalar** potentials, or possibly with a background homogeneous magnetic field. For random **magnetic** fields, we know:

- **Ueki '00** (Gaussian random magnetic fields)
- **Nakamura '00** (Regular random magnetic fields)
- **Borg–Pule '04** (Smooth approximation of  $\delta$  magnetic fields)

## Known result for random $\delta$ magnetic fields

In 2006, **James Borg** considered the Schrödinger operators with random  $\delta$  magnetic fields in his thesis, with  $\Gamma_\omega = \mathbb{Z}^2$  (fixed). He obtained the representation of the Laplace transform of the density of states  $dN$ , in terms of the **winding number** of the Brownian motion (the Lifshitz tail was not yet proved in his thesis).

## Two strategies to prove the Lifshitz tail

**Tauberian theorem.** Prove the Laplace transform of  $dN$

$$\rho(t) = \int_0^\infty e^{-t\lambda} dN(\lambda)$$

has the asymptotics  $\rho(t) \sim t^{1/2}$  as  $t \rightarrow \infty$ . Then, the Minlos–Povzner Tauberian theorem gives the Lifshitz tail.

**First Eigenvalue.** Prove the probability  $\mathbb{P}\{\lambda_1(H_\omega^k) \leq ck^{-2}\}$  is small. Then a simple consideration gives the Lifshitz tail.

## Feynman–Kac–Ito formula

Usually, the Feynman–Kac–Ito formula requires  $a \in L^2_{\text{loc}}$ , however, our potential does not satisfy this assumption. But the set of singularities  $\Gamma$  is a countable set in  $\mathbb{R}^2$ , which is the **polar set**, i.e.,

$$\mathbb{P}_x\{w : \text{Brownian path starting from } x \mid w_t \in \Gamma \text{ for some } t\} = 0$$

for any  $x \in \mathbb{R}^2$ , where  $\mathbb{P}_x$  denotes the Wiener measure of the Brownian paths starting from  $x$ . Using this fact, Borg proved the following.

## Feynman–Kac–Ito formula

**Theorem.** For any  $\psi \in L^2(\mathbb{R}^2)$  and  $t \geq 0$ , we have

$$e^{-tH_\omega} \psi(x) = \mathbb{E}_x \left[ \psi(w(t)) e^{-i \int_0^t a_\omega(w(s)) \cdot dw} \right],$$

where the expectation  $\mathbb{E}_x$  is taken over all the Brownian paths  $w$  starting from  $x$ , and the integral  $\int_0^t a_\omega(w(s)) \cdot dw$  is interpreted in the sense of the Ito stochastic integral.

## Winding number

For a closed path  $w = (u, v)$  from  $x$  to itself which avoids  $\Gamma$ ,

$$\begin{aligned}\int_0^t a_\omega(w(s)) \cdot dw &= \int_0^t (\operatorname{Im} \phi_\omega(w(s)) du + \operatorname{Re} \phi_\omega(w(s)) dv) \\ &= \operatorname{Im} \int_C \phi_\omega(z) dz \\ &= \sum_{\gamma \in \Gamma} 2\pi \alpha_\gamma(\omega) n(w, \gamma),\end{aligned}$$

where  $C = \{u_s + iv_s \mid 0 \leq s \leq t\}$ , and  $n(w, \gamma)$  the winding number of the path  $w$  around  $\gamma$  ( $n(w, \gamma) = 0$  except finite  $\gamma$ 's).

## The formula for Laplace transform of DS

By the ergodic theorem and above formulas, we have

$$\begin{aligned}
 \rho(t) &= \int_0^\infty e^{-t\lambda} dN(\lambda) = \mathbb{E}_\omega[\text{Tr}(\chi_{Q_0} e^{-tH_\omega} \chi_{Q_0})] \\
 &= \frac{1}{4\pi t} \mathbb{E}_\omega \int_{Q_0} \mathbb{E}_{0,z,t,z} [e^{-i \int_0^t a_\omega(w(s)) \cdot dw}] dx dy \\
 &= \frac{1}{4\pi t} \mathbb{E}_\omega \int_{Q_0} \mathbb{E}_{0,z,t,z} [e^{-2\pi i \alpha_\gamma \sum_{\gamma \in \Gamma} n(w,\gamma)}] dx dy,
 \end{aligned}$$

where  $\mathbb{E}_{0,z,t,z}$  is taken over all the closed Brownian paths from  $z$  at time 0, to  $z$  at time  $t$ .

## IDS and the first eigenvalue

By the subadditivity of  $N_\omega^k(E)$ , it follows that  $N(E) = \inf_k \frac{1}{|Q_k|} \mathbb{E}[N_\omega^k(E)]$  a.s.. Thus we have

$$\begin{aligned} N(E) &\leq \frac{1}{|Q_k|} \int_{\Omega} N_\omega^k(E) \chi_{\lambda_1(H_\omega^k) \leq E} d\mathbb{P}(\omega) \\ &\leq CE_+ \int_{\Omega} \chi_{\{\lambda_1(H_\omega^k) \leq E\}} d\mathbb{P}(\omega) \\ &= CE_+ \mathbb{P}\{\lambda_1(H_\omega^k) \leq E\}. \end{aligned}$$

Then, we take  $E \sim ck^{-2}$  for small  $c$  and estimate the last probability, by using the large deviation technique.

## Avron–Herbst–Simon inequality

**Theorem.** (Avron–Herbst–Simon '78)

Let  $a \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ . Then,

$$\int_{\mathbb{R}^2} \left| \left( \frac{1}{i} \nabla - a \right) u \right|^2 dx dy \geq \int_{\mathbb{R}^2} (\operatorname{curl} a) |u|^2 dx dy$$

for any  $u \in C_0^\infty(\mathbb{R}^2)$ .

## Nakamura's idea

In Nakamura '00, by using the AHS inequality and the Kato inequality

$$\left( \left( \frac{1}{i} \nabla - a \right)^2 u, u \right) \geq (-\Delta |u|, |u|)$$

he obtained the lower bound for the lowest eigenvalue  $\lambda_1$  of  $H$

$$\lambda_1(H_\omega) \geq \lambda_1 \left( \frac{1}{2} (-\Delta + \operatorname{curl} a_\omega) \right).$$

This enables us to reduce the problem to the scalar potential case. However, this method is **not** applicable for our operator, since  $\operatorname{curl} a_\omega = 0$  a.e.

## Hardy inequality

For  $d \geq 3$ , there exists a positive constant  $C_d$  such that

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \geq C_d \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} dx \quad (5)$$

for any  $u \in C_0^\infty(\mathbb{R}^d)$ . This inequality is called **the Hardy inequality**. The inequality (5) fails when  $d = 2$ . However, **Laptev and Weidl '99** proved that a similar inequality holds if there exists a  $\delta$  magnetic field at the origin.

## Hardy type inequality

**Theorem.**(Laptev–Weidl '99) Let  $\alpha \in \mathbb{R}$  and put  $a_\alpha(z) = \left(\operatorname{Im} \frac{\alpha}{z}, \operatorname{Re} \frac{\alpha}{z}\right)$  (so  $\operatorname{curl} a_\alpha = 2\pi\alpha\delta_0$ ). Then, we have

$$\int_{|z| \leq R} \left| \left( \frac{1}{i} \nabla - a_\alpha \right) u(z) \right|^2 dx dy \geq \rho(\alpha) \int_{|z| \leq R} \frac{|u(z)|^2}{|z|^2} dx dy \quad (6)$$

for any  $R > 0$  and any  $u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ . Here

$$\rho(\alpha) = \min_{n \in \mathbb{Z}} |n - \alpha|^2.$$

## Proof of main theorem

Let us return to our model. Let  $\delta$  be the constant given in (A5) of our assumption. Then, the probability of the event

$$n_\omega(Q_0 + n) = n_\omega(\delta Q_0 + n) = 1 \quad (7)$$

is positive for any  $n \in \mathbb{Z}^2$ . When (7) holds, we denote  $\Gamma_\omega \cap (Q_0 + n) = \{\gamma_n(\omega)\}$ ,  $\alpha_n(\omega) = \alpha_{\gamma_n(\omega)}(\omega)$ . For  $z \in n + Q_0$ , define

$$V_\omega(z) = \begin{cases} \frac{4}{(1-\delta)^2} \rho(\alpha_n(\omega)) & \text{if (7) holds and } |z - \gamma_n(\omega)| < \frac{1-\delta}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

## Proof of main theorem

By using the Laptev–Weidl inequality together with some appropriate gauge transform, we have

$$\int_{n+Q_0} \left| \left( \frac{1}{i} \nabla - a_\omega \right) u(z) \right|^2 dx dy \geq \int_{n+Q_0} V_\omega(z) |u(z)|^2 dx dy \quad (8)$$

for any  $n \in \mathbb{Z}^2$  and any  $u \in C_0^\infty(\mathbb{R}^2 \setminus \Gamma_\omega)$ .

## Proof of main theorem

We have

$$\nabla|u| = \operatorname{Re}(\operatorname{sgn} \bar{u} \nabla u) = \operatorname{Re}(\operatorname{sgn} \bar{u} (\nabla - ia_\omega)u) \quad \text{a.e.},$$

where  $\operatorname{sgn} z = z/|z|$  for  $z \neq 0$  and  $\operatorname{sgn} 0 = 0$ . Thus

$$\left| \left( \frac{1}{i} \nabla - a_\omega \right) u \right|^2 \geq \left| \nabla|u| \right|^2 \quad \text{a.e.} \quad (9)$$

## Proof of main theorem

By (8) and (9), we have the following inequality:

$$\int_{Q_k} \left| \left( \frac{1}{i} \nabla - a_\omega \right) u \right|^2 dx dy \geq \frac{1}{2} \int_{Q_k} \left( |\nabla |u||^2 + V_\omega |u|^2 \right) dx dy$$

for any  $u \in C_0^\infty(\mathbb{R}^2 \setminus \Gamma_\omega)$ . This implies  $\lambda_1(H_\omega) \geq \lambda_1\left(\frac{1}{2}(-\Delta + V_\omega)\right)$  by the min-max principle. So we can reduce the problem to the scalar potential case.

## Remained problems

- (1) Can one deduce the Lifshitz tail from the stochastic formula for the Laplace transform of DS? (Control of infinite winding numbers of the Brownian motions...)
- (2) Does the Anderson localization occur for our operator? (Wegner estimate, etc..)