Dynamical description of gravitational collapse

Alessandro Michelangeli

LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN and UNIVERSITY OF CAMBRIDGE

(a joint work with B. Schlein)

Arizona School of Analysis with Applications
March 2010

Many-body system of gravitating particles

$$H = \sum_{i=1}^{N} \left(\sqrt{m^2c^4 + c^2p_i^2} - mc^2\right) - \sum_{i<j}^{N} \frac{Gm^2}{|x_i - x_j|}$$

$$p = i\hbar\nabla$$

on $L^2_{\text{sym}}(\mathbb{R}^{3N}) = \bigotimes_{i=1}^{N}{}_s \, L^2(\mathbb{R}^3)$ (BOSONS)

or $L^2_{\text{asym}}(\mathbb{R}^{3N}) = \bigwedge_{i=1}^{N} L^2(\mathbb{R}^3)$ (FERMIONS)

$G \simeq 6.673\ 10^{-11}$ (S.I.), $N \approx 10^{57}$(for neutron stars)

Boson star: Stability and scaling [Lieb-Yau 1987]

$$H_N = \sum_{i=1}^{N} \sqrt{1-\Delta_i} - \frac{G}{N} \sum_{i<j}^{N} \frac{1}{|x_i - x_j|}$$

$$\boxed{\frac{\inf \sigma(H_N)}{N}} \xrightarrow{N\to\infty} \begin{cases} > -\infty & \text{if } G < G_{\text{crit}} \\ = -\infty & \text{if } G > G_{\text{crit}} \end{cases}$$

(**gravitational collapse**)

$$\left(1.3 \simeq \frac{4}{\pi} \leqslant \boxed{G_{\text{crit}}} \lesssim 2.7\right)$$

Dynamical analysis ($G < G_{\text{crit}}$)

$$i\partial_t \Psi_{N,t} = H_N \Psi_{N,t}$$

$$\begin{tikzcd}
\Psi_N \arrow[r, "\text{partial trace}"] \arrow[d, swap, "\begin{array}{c}\text{many-body}\\ \textbf{linear}\text{ dynamics}\end{array}"] & \gamma_N^{(k)} \arrow[r, "N\to\infty"] \arrow[d] & |\varphi^{\otimes k}\rangle\langle\varphi^{\otimes k}| \arrow[d, "\begin{array}{c}\textbf{non linear}\\ \text{Hartree}\end{array}"] \\
\Psi_{N,t} \arrow[r] & \gamma_{N,t}^{(k)} \arrow[r, "N\to\infty"] & |\varphi_t^{\otimes k}\rangle\langle\varphi_t^{\otimes k}|
\end{tikzcd}$$

effective dynamics

$$i\partial_t\varphi_t = \sqrt{1-\Delta}\,\varphi_t - \left(\frac{G}{|x|}*|\varphi_t|^2\right)\varphi_t$$

global well-posedness [Lenzmann, 2005]

$$\operatorname{Tr}\left|\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\right| \xrightarrow{N\to\infty} 0 \qquad \forall t\in\mathbb{R}$$

[Elgart-Schlein, 2005]

$$\operatorname{Tr}\left|\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\right| \lesssim \frac{e^{ct}}{\sqrt{N}}$$

[Knowles-Pickl, 2009]

and also: [Erdős-Yau, 2000], [Rodnianski-Schlein, 2007]

The super-critical regime $G > G_{\text{crit}}$

➠ blow-up in Hartree eqn.: $\|\varphi_t\|_{H^{1/2}} \xrightarrow{t \to T_{\text{blow-up}}} +\infty$
[Frölich-Lenzmann, 2005]

➠ does this **blow-up** describe the **collapse** of boson stars?

i.e., can the **many-body dynamics** be approximated by the **Hartree dynamics** also in the *super-critical* regime? (this was an UNPROVED assumption until today!)

our analysis ⇒ affirmative answer

Collapse and loss of regularity [M-Schlein,2010]

IF $$H_N^{(\boldsymbol{\alpha})} = \sum_{i=1}^{N} \sqrt{1-\Delta_i} - \frac{G}{N} \sum_{i<j}^{N} \frac{1}{|x_i - x_j| + \boldsymbol{\alpha_N}} \qquad \begin{matrix} G > G_{\text{crit}} \\ \alpha_N \sim N^{-k} \end{matrix}$$

$$\Psi_N = \varphi^{\otimes N} \quad \text{➠} \quad \Psi^{(\boldsymbol{\alpha})}_{N,t} \quad \text{➠} \quad \widetilde{\gamma}^{(1)}_{N,t}$$

$$\varphi \in H^2(\mathbb{R}^3) \text{ s.t. } \sup_{|t|\leqslant T} \|\varphi_t\|_{H^{1/2}} =: \kappa(T) < \infty \text{ and } \kappa(T) \xrightarrow{T \to T_{\text{blow-up}}} +\infty$$

THEN $$\mathrm{Tr}\left|(1-\Delta)^{1/4}\Big(\widetilde{\gamma}^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\Big)(1-\Delta)^{1/4}\right| \lesssim \frac{1}{\boldsymbol{N^{1/4}}} + \sqrt{\boldsymbol{\alpha_N}}$$

$\forall |t| \leqslant T$ (the first **effective** control **in energy** ever)

WHENCE $\exists\, N(t) \to \infty$, as $t \to T_{\text{blow-up}}$, such that

$$\left\|(1-\Delta_1)^{\frac{1}{4}}\,\Psi_{N(t),t}\right\|^2_{L^2(\mathbb{R}^{3N})} = \mathrm{Tr}\,(1-\Delta)^{1/2}\widetilde{\gamma}^{(1)}_{N(t),t} \;\approx\; \|\varphi_t\|^2_{H^{1/2}} \xrightarrow{\boldsymbol{t \to T}_{\text{blow-up}}} \infty$$

EXTRAS

The strategy:

$$\mathrm{Tr}\left|(1-\Delta)^{1/4}\big(\tilde{\gamma}^{(1)}_{N,t}-|\varphi_t\rangle\langle\varphi_t|\big)(1-\Delta)^{1/4}\right|$$

$$\leqslant\ \mathrm{Tr}\left|(1-\Delta)^{1/4}\big(\tilde{\gamma}^{(1)}_{N,t}-|\varphi_t^{(\boldsymbol{\alpha})}\rangle\langle\varphi_t^{(\boldsymbol{\alpha})}|\big)(1-\Delta)^{1/4}\right|$$

$$+\ \big\|\varphi_t-\varphi_t^{(\boldsymbol{\alpha})}\big\|_{H^{1/2}}$$

$$\lesssim N^{-1/4}\ \textbf{(II)}$$

$$\lesssim\ \boldsymbol{\alpha}^{1/2}\ \textbf{(I)}$$

$$\Rightarrow \mathrm{Tr}\,(1-\Delta)^{1/2}\tilde{\gamma}^{(1)}_{N(T),T}\approx\|\varphi_t\|^2_{H^{1/2}}\to+\infty$$

as $T\to T_{\text{blow up}}$

I. Closeness between Hartree and Hartree$^{(\alpha)}$

$$i\partial_t\varphi_t = \sqrt{1-\Delta}\varphi_t - \Big(\frac{G}{|x|} * |\varphi_t|^2\Big)\varphi_t$$

$$i\partial_t\varphi_t^{(\alpha)} = \sqrt{1-\Delta}\varphi_t^{(\alpha)} - \Big(\frac{G}{|x|+\alpha} * \big|\varphi_t^{(\alpha)}\big|^2\Big)\varphi_t^{(\alpha)}$$

$$\varphi_t\big|_{t=0} = \varphi_t^{(\alpha)}\big|_{t=0} = \varphi \in H^2, \qquad G \in \mathbb{R}$$

and assuming *only* $\sup_{|t|\leqslant T} \|\varphi_t\|_{H^{1/2}} = \kappa < +\infty$

$$\Rightarrow \ \big\|\varphi_t - \varphi_t^{(\alpha)}\big\|_{H^{1/2}} \lesssim \kappa\sqrt{\alpha} \qquad \forall |t| \leqslant T$$

II. Fluctuations in Fock's space

$$\{\Psi_{n,t}\}_n^\infty =: \Phi_t \in \mathcal{F} = \bigoplus_{n\geqslant 0} L^2_{\mathrm{sym}}(\mathbb{R}^{3n})$$

$$\{H_n\}_n^\infty \leftrightarrow \mathcal{H}_N \text{ acting on } \mathcal{F}, \qquad \Phi_t = e^{-it\mathcal{H}_N}\Phi$$

$$\mathcal{H}_N = \int \mathrm{d}x\, (1-\Delta_x)^{1/4} a_x^* \,(1-\Delta_x)^{1/4} a_x \qquad (=: \mathcal{K})$$

$$-\frac{G}{2N}\int \mathrm{d}x\,\mathrm{d}y\, \frac{1}{|x-y|+\alpha}\, a_x^* a_y^* a_y a_x \qquad (=: \mathcal{V})$$

$$\gamma_\Phi^{(1)}(x,y) = \frac{1}{\langle \Phi, \mathcal{N}\Phi\rangle}\langle \Phi, a_x^* a_y \Phi\rangle \qquad \mathcal{N} = \int \mathrm{d}x\, a_x^* a_x$$

$\widetilde{\gamma}_{N,t}^{(1)}(x,y) - \overline{\varphi}_t^{(\alpha)}(x)\varphi_t^{(\alpha)}(y) =$ fluctuations in $\mathcal{F}$ with
non-fixed number of particles

Expansion in coherent states

initial datum: $\{0,\dots,0,\varphi^{\otimes N},0,0,\dots\} = \frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega \in \mathcal{F}$

$$= \underbrace{\frac{\sqrt{N!}}{N^{N/2}e^{-N/2}}}_{\simeq N^{1/4}} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} e^{i\theta N} W(e^{-i\theta}\sqrt{N}\varphi)\Omega$$

coherent state $:= W(f)\Omega = e^{-\frac{\|f\|_2^2}{2}} \sum_{n\geqslant 0} \frac{(a^*(\varphi))^n}{\sqrt{n!}}\Omega$

$$\Rightarrow \int \mathrm{d}x\,\mathrm{d}y \left|(1-\Delta_x)^{1/4}(1-\Delta_y)^{1/4}\Big(\widetilde{\gamma}^{(1)}_{N,t}(x,y) - \varphi_t^{(\alpha)}(x)\overline{\varphi}_t^{(\alpha)}(y)\Big)\right|^2$$

$$\leqslant \frac{\|\varphi_t^{(\alpha)}\|_{H^{1/2}}}{\sqrt{N}} \int_0^{2\pi} \mathrm{d}\theta \left\langle \mathcal{U}_N^\theta(t)\Omega, \mathcal{K}\,\mathcal{U}_N^\theta(t)\Omega \right\rangle \lesssim \frac{\mathbf{1}}{\sqrt{\boldsymbol{N}}}$$

$\boldsymbol{\mathcal{U}_N^\theta} = W^*(e^{-i\theta}\sqrt{N}\varphi_t)\, e^{-it\mathcal{H}_N} W(e^{-i\theta}\sqrt{N}\varphi)$ **dynamics of fluctuations**

- Thus, the **core** is the (**challenging**) estimate

$$\langle \mathcal{U}_N^\theta(t)\Omega, \mathcal{K}\,\mathcal{U}_N^\theta(t)\Omega\rangle \leqslant C\Big(\|\varphi_t^{(\alpha)}\|_{H^2}\Big) \lesssim 1$$

(a' la Rodnianski-Schlein,
more difficult: $(-\Delta) \mapsto (1-\Delta)^{1/2}$)

(higher regularity)

- the result is: $\forall |t| \leqslant T \quad (< T_{\text{blow-up}})$

$$\Big\|(1-\Delta)^{1/4}\Big(\widetilde{\gamma}_{N,t}^{(1)} - |\varphi_t^{(\alpha)}\rangle\langle\varphi_t^{(\alpha)}|\Big)(1-\Delta)^{1/4}\Big\|_{\text{HS}} \leqslant \frac{C(\kappa,T)}{N^{1/4}}$$

- last, (a trick with density matrices)

Hilbert-Schmidt control ⇒ **trace class** control