

# Universality Limits of a Reproducing Kernel for a Half-Line Schrödinger Operator and Clock Behavior of Eigenvalues

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- clock spacing of zeros
- We extend some recent results of Lubinsky, Levin, Simon, and Totik from measures with compact support to spectral measures of Schrödinger operators on the half-line

# Differential and Difference Equations: Analogues

Difference	Differential
recurrence relation	differential expression
orthogonal polynomials	solutions of eigenvalue equation
$x^n$	$\cos(\sqrt{\xi}x)$
$x^n + a_{n-1}x^{n-1} + \dots$	$\int_0^L f(x) \cos(\sqrt{\xi}x) dx$ with $f \in L^2(0, \infty]$
$x$	energy
$n$	$x$ in $u(\xi, x)$

## Local Statistics

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- What are the distances between consecutive eigenvalues?
  - Poisson (no repulsion) e.g. Molchanov
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- Spacing of eigenvalues is same as spacing of zeros of  $y(\xi, L)$  the solution of the half-line operator with Dirichlet boundary condition

# Reproducing kernel for a Schrödinger operator

## Definition

Let  $A = -\frac{d}{dx^2} + V(x)$  be a Schrödinger operator with the Neumann boundary condition and  $u$  be the solution of the eigenvalue equation  $Au = \xi u$ . We define the reproducing kernel as

$$S_L(\xi, \zeta) = \int_0^L u(\xi, t)u(\zeta, t)dt.$$

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This satisfies the reproducing property with respect to the spectral measure  $d\mu$  of  $A$ , i.e.

$$u(\xi, x)\chi_{[0,L]}(x) = \int S_L(\xi, \zeta)u(\zeta, x)d\mu(\zeta).$$

# Variational Principle for Schrödinger

## Theorem

If  $d\mu$  is a spectral measure of a Schrödinger operator and  $H_L = \left\{ \pi : \pi(\xi) = \int_0^L f(x) \cos(\sqrt{\xi}x) dx, f \in L^2[0, \infty) \right\}$ , then

$$\min\{\|Q\|_{d\mu}^2 : Q \in H_L, Q(\xi_0) = 1\} = S_L(\xi_0, \xi_0)^{-1},$$

and the minimizer is given by

$$\frac{S_L(\xi, \xi_0)}{S_L(\xi_0, \xi_0)}.$$

# Main Result

## Theorem

Let  $A = -\frac{d^2}{dx^2} + p(x) + q(x)$  and let  $d\mu(\xi) = w(\xi)d\xi + d\mu_s$  be its spectral measure. Suppose

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- Let  $I \subset \mathbb{R}$  be a closed and bounded interval such that  $w$  is continuous and non-zero on  $I$  and  $\text{supp}(d\mu_s) \cap I = \emptyset$ .
- Let  $\xi_0 \in I$  and  $a, b, B \in \mathbb{R}$ .

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- Let  $\xi_0 \in I$  and  $a, b, B \in \mathbb{R}$ .

Then uniformly in  $I$  and  $|a|, |b| < B$  as  $L \rightarrow \infty$

$$\frac{S_L(\xi_0 + a/L, \xi_0 + b/L)}{S_L(\xi_0, \xi_0)} \rightarrow \frac{\sin(\pi\rho(\xi_0)(a - b))}{\pi\rho(\xi_0)(a - b)}, \quad (4.1)$$

where  $\rho(\xi)d\xi$  is the density of states.

## Clock Behavior Definition

Fix  $\xi^*$  in an interval  $I$ , and number the zeros  $\xi_N$  of  $u'(-, L)$  with increasing positive integers to the right of  $\xi^*$  and decreasing negative integers to the left so that  $\dots < \xi_{-1} < \xi^* \leq \xi_0 < \dots$ . We say there is **strong clock behavior** of zeros of  $u'$  at  $\xi^*$  on an interval  $I$  if the density of states  $\rho(\xi)d\xi$  is continuous and nonvanishing on  $I$  and for fixed  $n$

$$\lim_{L \rightarrow \infty} L |(\xi_n - \xi_{n+1})| \rho(\xi^*) = 1, \quad (4.2)$$

and we say there is **uniform clock behavior** on  $I$  if the limit in (4.2) is uniform on  $I$  for fixed  $n$ .

**This means**

that, asymptotically, zeros near a chosen point are equally spaced, after a normalization.

## Clock Behavior Theorem

Let  $A$ ,  $\epsilon$ ,  $I$ ,  $\xi_0$  as before. Then there is uniform clock behavior of the zeros of  $u'$  and  $y$  on  $I$ .

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## **Eigenvalues in a box!**

# Inequality for Off-Diagonal Kernels

## Lemma

Let two measures  $d\mu(\xi)$ ,  $d\mu^*(\xi)$  with  $d\mu(\xi) \leq d\mu^*(\xi)$  be spectral measures of Schrödinger operators. Then for any  $\xi, \beta \in \mathbb{R}$ ,

$$\frac{|S_L(\xi, \beta, \mu) - S_L(\xi, \beta, \mu^*)|}{S_L(\xi, \xi, \mu)} \leq \left( \frac{S_L(\beta, \beta, \mu)}{S_L(\xi, \xi, \mu)} \right)^{1/2} \left( 1 - \frac{S_L(\xi, \xi, \mu^*)}{S_L(\xi, \xi, \mu)} \right)^{1/2}.$$

To understand the  $\lim_{n \rightarrow \infty} \frac{S_L(\xi_0 + \frac{a}{n}, \xi_0 + \frac{b}{n})}{S_L(\xi_0, \xi_0)}$ , it is sufficient to

- understand  $S_L^\#(x, y)$  for some model measure  $d\eta^\#$

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- construct a measure  $d\eta^*$  which dominates both  $d\eta$  and  $d\eta^\#$  and a has similarly nice local behavior at  $x_0$  with  $w^*(x_0) = w(x_0)$ ; **scalar multiple** of a spectral measure

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- understand the behavior of a ratio of diagonal kernels - using variational principle. **What happens at infinity in  $\xi$ ?**

## Unperturbed periodic case

We compute directly:

For the unperturbed Schrödinger operator,

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To make this calculation,

- Use a standard formula to express the density of states in terms of the imaginary part of the diagonal Green's function
- Express the Green's function in terms of the solution  $u$  to get the diagonal kernel.
- Take a Taylor series to get the off-diagonal kernel

- This work was done as my Caltech Ph.D. thesis with Barry Simon as my adviser, to whom I owe a great debt of gratitude.
- Maltsev. Universality Limits of a Reproducing Kernel for a Half-Line Schrödinger Operator and Clock Behavior of Eigenvalues.  
**arXiv:0908.1440**, accepted in *Communications of Math Physics*.
- Thank you!