Orthogonal polynomials with recursion coefficients of generalized bounded variation

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- Let $d\mu$ be a probability measure on the unit circle with $\operatorname{supp} d\mu$ an infinite set
- $d\mu$ defines a sequence of orthonormal polynomials $\varphi_n(z)$ with $\deg \varphi_n = n$ and

$$\int_{\partial\mathbb{D}} \bar{\varphi}_m(z)\varphi_n(z)d\mu(z) = \delta_{m,n}$$

- φ_n obey Szegő recursion relation

$$z\varphi_n(z) = \sqrt{1-|\alpha_n|^2}\varphi_{n+1}(z) + \bar{\alpha}_n\varphi_n^*(z)$$

  with $\varphi_n^*(z) = z^n\overline{\varphi_n(1/\bar{z})}$
- $\alpha_n \in \mathbb{D}$ are called Verblunsky coefficients

Introduction to OPRL

- Let $d\rho$ be a probability measure on $\mathbb{R}$ of infinite support and bounded moments
- $d\rho$ defines a sequence of orthonormal polynomials $p_n(x)$ with $\deg p_n = n$ and

$$\int_{\mathbb{R}} p_m(x) p_n(x) d\rho(x) = \delta_{m,n}$$

- p_n obey the recursion relation

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_{n+1} p_n(x) + a_n p_{n-1}(x)$$

- $a_n > 0$, $b_n \in \mathbb{R}$ are called Jacobi coefficients

Generalized bounded variation

We say a sequence $\{x_n\}_{n=0}^{\infty}$ is of generalized bounded variation with phases $\phi_1, \ldots, \phi_L$, if $\{x_n\}$ can be expressed as a sum of L sequences

$$x_n = \sum_{l=1}^{L} y_n^{(l)}$$

each of which has rotated bounded variation with a respective phase ϕ_l,

$$\sum_{n=0}^{\infty} |e^{i\phi_l} y_{n+1}^{(l)} - y_n^{(l)}| < \infty$$

Examples of generalized b.v. are linear combinations of

$$\frac{e^{in\phi}}{n^{\gamma}}, \quad \frac{\cos(n\phi + \alpha)}{n^{\gamma}}, \quad \text{with } \gamma > 0$$

History and motivation
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Theorems of the type: if the Verblunsky coefficients $\{\alpha_n\}$ of $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_s$ have certain properties, then with an explicit finite set S, we have

(i) $\operatorname{supp} \mu_s \subset S$, so μ has no singular continuous part

(ii) $w(\theta)$ is continuous and strictly positive on $\partial\mathbb{D} \setminus S$

- (Weidmann's theorem)
  If $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$: $\quad S = \{1\}$
- If $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} |e^{i\phi}\alpha_{n+1} - \alpha_n| < \infty$: $\quad S = \{e^{i\phi}\}$
- (Wong, 2009)
  If $\{\alpha_n\} \in \ell^2$ has generalized bounded variation with phases $\phi_1, \dots, \phi_L$: $\quad S = \{e^{i\phi_1}, \dots, e^{i\phi_L}\}$

(Weidmann's theorem)
If $d\rho = f(x)dx + d\rho_s$ has Jacobi parameters $\{a_n, b_n\}_{n=1}^{\infty}$ with $a_n \to 1$, $b_n \to 0$ and $\sum_{n=0}^{\infty} |a_{n+1} - a_n| + \sum_{n=0}^{\infty} |b_{n+1} - b_n| < \infty$ then

(i) $\operatorname{supp} \rho_s \cap (-2, 2) = \emptyset$

(ii) $f(x)$ is continuous and strictly positive on $(-2, 2)$

(Blumenthal-Weyl) If $a_n \to 1$, $b_n \to 0$, then

$$\operatorname{ess\,supp} \rho = [-2, 2]$$

Theorem. *(OPUC) Let* $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_s$ *have* $\{\alpha_n\}_{n=0}^{\infty}$ *in* ℓ^p *with generalized bounded variation with set of phases* $A = \{\phi_1, \ldots, \phi_L\}$*. Then there exists a finite set* $S \subset \partial\mathbb{D}$ *such that*

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OPUC

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The set S *is determined by* p *and the set of phases* A*:*

- *for* $p = 3$, $S = \{\exp(i\eta) | \eta \in A\}$

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- *for* $p = 2q + 1$,
  $S = \{\exp(i\eta) | \eta \in \underbrace{A + \cdots + A}_{q \text{ times}} - (\underbrace{A + \cdots + A}_{q-1 \text{ times}})\}$

Theorem. *(OPRL) Let* $d\rho = f(x)dx + d\rho_s$ *have* $\{a_n^2 - 1\}_{n=1}^{\infty}$ *(or* $\{a_n - 1\}_{n=1}^{\infty}$*) and* $\{b_n\}_{n=1}^{\infty}$ *in* ℓ^p *with generalized bounded variation with set of phases* A*. Then there exists a finite set* $S \subset [-2, 2]$ *such that*

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- *for $p = 3$,*
  $S = \{2\cos(\eta/2) | \eta \in (\tilde{A} + \tilde{A})$ *or* $2\eta \in (\tilde{A} + \tilde{A})\}$

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- *for arbitrary* p,
  $S = \{2\cos(\eta/2) | k\eta \in (\underbrace{\tilde{A} + \cdots + \tilde{A}}_{p-1 \text{ times}}), 1 \le k \le p - 1\}$

Corollary. *(1D discrete Schrödinger operators) Let* $(Hx)_n = x_{n+1} + x_{n-1} + V_n x_n$ *be a discrete Schrödinger operator on a half-line or line, with* $\{V_n\}$ *in* ℓ^p *with generalized bounded variation. Then there exists a finite set* $S \subset (-2, 2)$ *such that*

(i) $\sigma_{\mathrm{sing}}(H) \cap (-2, 2) \subset S$*, so in particular* $\sigma_{\mathrm{sc}}(H) = \emptyset$

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In particular, this is true for the potential $\{V_n\}_{n=1}^{\infty}$ given by

$$V_n = \frac{\cos(n\phi + \alpha)}{n^\gamma}$$

with $\gamma > 0$.