SCHRÖDINGER OPERATORS WITH POTENTIAL $V(n) = n^{-\gamma}\cos(2\pi n^{\rho})$

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ABSTRACT. Let H be the Schrödinger operator with potential $V(n) = n^{-\gamma}\cos(2\pi n^{\rho})$, where $\rho \in (1,2)$ and $\gamma \in (0, \frac{1}{2} - \frac{\rho-1}{2})$. I show that for almost every boundary condition H has pure-point spectrum.

1. Introduction

In this short note, I wish to show that already the sequence $n^{\rho} \pmod 1$ generates sufficient randomness to expect results for decaying potentials similar to the case of independent, identically distributed random variables as discussed by Simon in [12]. Let ρ and γ satisfy

$$\rho \in (1,2), \quad \gamma \in \left(0, \frac{1}{2} - \frac{\rho - 1}{2}\right) \tag{1.1}$$

and introduce the potential

$$V_{\gamma,\rho}(n) = \frac{1}{n^{\gamma}}\cos(2\pi n^{\rho}). \tag{1.2}$$

For $\beta \in \mathbb{R}$, we introduce the half-line Schrödinger operator $H^{\beta}_{\gamma,\rho} : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ by

$$(H^{\beta}_{\gamma,\rho} u)(n) = \begin{cases} u(n+1) + u(n-1) + V^{\gamma,\rho}(n)u(n), & n \geq 2; \\ u(2) + (V_{\gamma,\rho}(1) + \beta)u(1), & n = 1. \end{cases} \tag{1.3}$$

Here β plays the role of a boundary condition. Since $V_{\gamma,\rho}(n)$ converges to 0 as $n \to \infty$, we have that $\sigma_{\mathrm{ess}}(H^{\beta}_{\gamma,\rho}) = [-2,2]$. We will show

Theorem 1.1. *Let ρ and γ obey* (1.1). *Then for almost every β, $H^{\beta}_{\gamma,\rho}$ has pure point spectrum in $[-2,2]$.*

This result is motivated by the work of Lukic [10], in which he showed that for $V(n) = \frac{1}{n^{\gamma}}\cos(2\pi\alpha n)$ with $\gamma > 0$ and α irrational the operator H has absolutely continuous spectrum in $[-2,2]$. Furthermore, he shows that the possible singular spectrum $[-2,2]$ is contained in a finite set of points.

The condition on β in Theorem 1.1 is optimal. In fact it is known (see Section 12.4. in [13]) that for a dense G_{δ} set of β the operator $H^{\beta}_{\gamma,\rho}$ has singular continuous spectrum.

Date: June 29, 2010.
1991 *Mathematics Subject Classification.* Primary 81Q10; Secondary 47B36.
Key words and phrases. Schrödinger Operators, pure point spectrum, decaying potentials.
H. K. was supported by an Erwin Schrödinger Junior Research Fellowship.

Remark 1.2. *It is possible to consider more general potentials of the form*

$$V(n) = \frac{1}{n^\gamma}\cos(2\pi n^\rho) + \frac{f(n)}{n^{\gamma+\varepsilon}}, \tag{1.4}$$

where γ and ρ obey (1.1)*, $f(n)$ is a bounded sequence, and $\varepsilon > 0$. The proof of the theorem is essentially the same.*

Remark 1.3. *The proof shows that for some $\varepsilon > 0$, we have for all generalized eigenfunctions ψ*

$$|\psi(n)| \leq \exp(-|n|^\varepsilon) \tag{1.5}$$

for n large enough. It be possible to give an upper bound similar to [12] *by a simple extension of the proof.*

Let me now discuss, if the range of γ and ρ given in (1.1) is optimal. By the results of Christ and Kiselev [2], Deift and Killip [4], or Remling [11], we have that $H^\beta_{\gamma,\rho}$ has absolutely continuous spectrum for $\gamma > \frac{1}{2}$. Furthermore, the results of Stolz from [15] imply that $H^\beta_{\gamma,\rho}$ has purely absolutely continuous spectrum for $\rho \in (0,1)$ and $\gamma > 0$. The already mentioned result by Lukic [10] imply absolutely continuous spectrum for $\rho = 1$ and $\gamma > 0$.

However, I would expect that in the complement of this range, that is $\rho > 1$ and $\gamma \in (0, \frac{1}{2})$ the operator $H^\beta_{\gamma,\rho}$ has pure point spectrum for almost every β. The main reason for this is that the Lyapunov exponent $L_\lambda(E)$ associated to the potential $V(n) = 2\lambda\cos(2\pi n^\rho)$ is expected to behave like $L_\lambda(E) \geq \gamma\lambda^2$ for some $\gamma > 0$ as $\lambda \to 0$. See the work of Bourgain [1], and my own in [7] and [8] for some positive results in this direction.

Naive computations with transfer matrices actually suggest the following more general picture. Assume that V is a bounded potential and that the associated Lyapunov exponent behaves like $L_\lambda(E) \sim \lambda^p$ as $\lambda \to 0$. Then one would expect absolutely continuous spectrum for $\gamma > \frac{1}{p}$ and $\gamma < \frac{1}{p}$. Somewhat supportive of this is Question 4.7. in the review article [14] of Simon.

2. Proof of Theorem 1.1

Introduce for α irrational, $\omega \in [0,1]$, and $\lambda > 0$ the Almost–Mathieu operator by

$$\begin{aligned} &\widehat{H}_{\lambda,\alpha,\omega} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \\ &\qquad \widehat{H}_{\lambda,\alpha,\omega} u(n) = u(n+1) + u(n-1) + 2\lambda\cos(2\pi(\omega + n\alpha))u(n). \end{aligned} \tag{2.1}$$

We will need the following fact about the spectrum of this operator.

Theorem 2.1. *Let $\delta > 0$, $\lambda > 0$, α irrational, and $\omega \in [0,1]$. There exists a constant $\kappa = \kappa(\delta) > 0$ such that for*

$$2\cos(\pi\alpha) \in [-2+\delta, -\delta] \cup [\delta, 2-\delta], \tag{2.2}$$

we have

$$\operatorname{dist}(\pm 2\cos(\pi\alpha), \sigma(\widehat{H}_{\lambda,\alpha,\omega})) \geq \lambda\kappa. \tag{2.3}$$

Proof. See [1], [5], [7], [9]. □

Given an interval $[a,b] \subseteq \mathbb{Z}$ with $a \geq 2$, we denote by $H_{\gamma,\rho}^{[a,b]}$ the restriction of $H_{\gamma,\rho}^{\beta}$ to $\ell^2([a,b])$. We have dropped the β from the notation, since $H_{\gamma,\rho}^{[a,b]}$ no longer depends on it. We will use the assertion of that the Almost–Mathieu operator has gaps, to conclude that also appropriate finite restrictions of $H_{\gamma,\rho}^{\beta}$ do.

Before stating this, let us introduce some notation. For $k \geq 2$, we introduce

$$\Lambda_k = [2^{k-1}, 2^{k+1} + 2^{k-1}], \tag{2.4}$$

$$\Lambda_k^c = [2^k, 2^{k+1}], \tag{2.5}$$

$$\Lambda_k^- = [2^{k-1}, 2^k - 1], \tag{2.6}$$

$$\Lambda_k^+ = [2^{k+1} + 1, 2^{k+1} + 2^{k-1}]. \tag{2.7}$$

We also define

$$\varepsilon = \frac{1}{6}(2 - \rho - 2\gamma). \tag{2.8}$$

We note $\varepsilon > 0$ if (1.1) holds.

Proposition 2.2. *Let $\delta > 0$ and $k \geq k_1(\delta)$. There exists a set $\mathcal{E}_{\delta,k}^1$ such that*

(i) $|\mathcal{E}_{\delta,k}^1| \leq \frac{1}{2}\frac{1}{k^2}$.

(ii) *Let*

$$E \in ([-2+\delta, -\delta] \cup [\delta, 2-\delta]) \setminus \mathcal{E}_{\delta,k}^1. \tag{2.9}$$

Then there exist intervals $I^{\pm} \subseteq \Lambda_k^{\pm}$ satisfying

(a) $\#(I^{\pm}) \geq 2 \cdot 2^{(\gamma+2\varepsilon)k} + 1$.

(b) $\operatorname{dist}(E, \sigma(H_{\gamma,\rho}^{I_{\pm}})) \geq 2^{-(\gamma+\varepsilon)k}$.

The proof will be given in one of the following sections. I note that this proposition is the main step and one could for example finish the proof by an adaptation of the results of Stolz in [16]. I have decided to include a proof to present some ideas of [8] in a more accessible framework. For $x, y \in \Lambda \subseteq \mathbb{Z}_+ \setminus \{1\}$, the Green's function is introduced as

$$G_{\gamma,\rho}^{\Lambda}(E, x, y) = \langle e_x, (H_{\gamma,\rho}^{\Lambda} - E)^{-1} e_y \rangle \tag{2.10}$$

with e_x the standard basis of $\ell^2(\mathbb{Z}_+)$.

Corollary 2.3. *Let $\delta > 0$ and $k \geq k_2(\delta)$. There exists a set $\mathcal{E}_{\delta,k}^2$ such that*

(i) $|\mathcal{E}_{\delta,k}^2| \leq \frac{1}{k^2}$.

(ii) *Let $y \in \Lambda_k^c$, $x \in \{2^{k-1}, 2^{k+1} + 2^{k-1}\}$, and*

$$E \in ([-2+\delta, -\delta] \cup [\delta, 2-\delta]) \setminus \mathcal{E}_{\delta,k}^2. \tag{2.11}$$

We have

$$|G_{\Lambda_k}(E, x, y)| \leq \exp\left(-2^{\frac{1}{5}\varepsilon k}\right). \tag{2.12}$$

We now proceed to derive Theorem 1.1. The strategy of proof is often called *spectral averaging*. See for example Section 12.3. in the book [13] by Simon for another implementation of this strategy. Fix some γ, ρ satisfying (1.1). For $\delta > 0$, introduce

$$\mathcal{E}_{\delta} = \bigcap_{\ell \geq k_2(\delta)} \left(\bigcup_{k \geq \ell} \mathcal{E}_{\delta,k}^2 \right). \tag{2.13}$$

The Borel–Cantelli argument shows that $|\mathcal{E}_\delta| = 0$. In particular also

$$\mathcal{E} = \bigcup_{j\geq 2} \mathcal{E}_{\frac{1}{j}} \tag{2.14}$$

has zero measure. It is well-known that there exists a unique probability measure μ^β that satisfies

$$\int \frac{1}{t-z} d\mu^\beta(t) = \langle e_1, (H^\beta_{\gamma,\rho} - z)^{-1} e_1\rangle \tag{2.15}$$

for $\mathrm{Im}(z) > 0$. This measure is known a the *spectral measure*. The main ingredient in spectral averaging is

Lemma 2.4. *There exists a set $\mathcal{B}$ such that $|\mathbb{R} \setminus \mathcal{B}| = 0$ and for $\beta \in \mathcal{B}$, we have $\mu^\beta(\mathcal{E}) = 0$.*

Proof. By Theorem 11.8. in [13], we have that $\int \mu^\beta d\beta$ is the Lebesgue measure. Thus

$$\int \mu^\beta(\mathcal{E}) d\beta = 0.$$

Since $\mu^\beta(\mathcal{E}) \geq 0$, the claim follows. □

We will now show that for $\beta \in \mathcal{B}$, H^β has pure point spectrum. For μ^β almost every $E \in (-2, 2) \setminus \{0\}$ there exists a generalized eigenfunction, that is a nonzero solution u of

$$H^{\gamma,\rho}_\beta u = Eu \tag{2.16}$$

interpreted as a difference equation satisfying $|u(n)| \leq n$ for $n \geq 1$ and $u(0) = 0$. See for example Lemma 3.1. in [17].

Proof of Theorem 1.1. Let $\beta \in \mathcal{B}$ and u some generalized eigenfunction belonging to the generalized eigenvalue E. We will show that $u \in \ell^2(\mathbb{Z}_+)$, which implies that μ^β must be a pure point.

By construction of $\mathcal{E}$, we can choose $\delta > 0$ and $\ell \geq 1$ such that

$$E \in [-2 + \delta, -\delta] \cup [\delta, 2 - \delta].$$

and for $k \geq \ell$

$$E \in \mathcal{E}_{\delta,k}.$$

For $x \in \Lambda_k^c = [2^k, 2^{k+1}]$, we have

$$u(x) = -G^{\Lambda_k}_{\gamma,\rho}(E, x, 2^k) u(2^{k-1} - 1) - G^{\Lambda_k}_{\gamma,\rho}(E, x, 2^{k+1} + 2^{k-1}) u(2^{k+1} + 2^{k-1} + 1).$$

Since by Corollary 2.3 the Green's functions are exponentially small in x, we obtain that $|u(x)| \leq \frac{1}{x^2}$ for k large enough. This implies that $u \in \ell^2(\mathbb{Z}_+)$ finishing the proof. □

3. Proof of Proposition 2.2

We note that $(m+n)^{\rho} = m^{\rho} + n\cdot\rho m^{\rho-1} + O(\frac{n^2}{m^{2-\rho}})$. Motivated by this, we introduce $\alpha_m = \rho m^{\rho-1}, \quad \omega_m = m^{\rho}$, and the potential

$$V_m(n) = \frac{1}{m^{\gamma}}\cos(2\pi(\alpha_m n + \omega_m)). \tag{3.1}$$

We introduce the whole line Schrödinger operator $H_m = \Delta + V_m$, where Δ is the discrete Laplacian. This is just the Almost–Mathieu operator. In fact, we have

$$H_m = \widehat{H}_{\frac{1}{2}m^{-\gamma},\alpha_m,\omega_m}. \tag{3.2}$$

We furthermore have that $V_m(n) - V_{\gamma,\rho}(n)$ is small for $|n-m|$ small. We make this precise

Lemma 3.1. *Assume* (1.1) *and define* ε *as in* (2.8). *For* $|n| \leq m^{\gamma+2\varepsilon}$, *we have*

$$|V(m+n) - V_m(n)| \leq \frac{C_1}{m^{\gamma+2\varepsilon}}, \tag{3.3}$$

where $C_1 = C_1(\gamma,\rho)$ *is a constant only depending on* γ *and* ρ.

Proof. We compute

$$V(m+n) - V_m(n) = \left(\frac{1}{(m+n)^{\gamma}} - \frac{1}{m^{\gamma}}\right)\cos(2\pi(m+n)^{\rho}) + \frac{1}{m^{\gamma}}\left(\cos(2\pi(m+n)^{\rho}) - \cos(2\pi(\alpha_m n + \omega_m))\right).$$

So

$$|V(m+n) - V_m(n)| \leq \frac{n}{(m-n)^{\gamma+1}} + \frac{2\pi\rho(\rho-1)}{(m-n)^{\gamma+2-\rho}}n^2.$$

The claim follows. □

Define

$$N = \left\lfloor 2^{k(\gamma+2\varepsilon)}\right\rfloor, \quad E_m = 2\cos(\pi\alpha_m). \tag{3.4}$$

Lemma 3.2. *Let* $C > 0$ *and assume* (1.1). *For* $k \geq k_0(C) \geq 1$, *define*

$$S = \left\lfloor \frac{2\cdot 2^{\gamma k}}{C}\right\rfloor. \tag{3.5}$$

There exist $\{m_s^{\pm}\}_{s=-S}^{S}$ *such that*

$$E_{m_s^{\pm}} \in \left[\frac{C}{2^{\gamma k}}(s-1), \frac{C}{2^{\gamma k}}(s+1)\right] \tag{3.6}$$

and

$$m_s^{\pm} \in [2^{k\pm 1} + N + 1, 2^{k\pm 1} + 2^{k-1} - N - 1]. \tag{3.7}$$

Proof. Since the map $x \mapsto 2\cos(\pi x)$ is Lipschitz, it suffices to find a sequence $\hat{m}_s^{\pm}$ such that $\alpha_{\hat{m}_s^{\pm}}$ (mod 1) is $\frac{C}{2\pi}\frac{1}{2^{\gamma k}}$ dense in $[0,1]$. Then we can choose $m_s^{\pm}$ as a refinement.

For this observe first that

$$\alpha_{m+1} - \alpha_m \leq \frac{\rho(\rho-1)}{m^{2-\rho}} \leq \frac{\rho(\rho-1)}{m^{\gamma}}\cdot\frac{1}{m^{\gamma}},$$

where we used (1.1). Second, we can compute that

$$\alpha_{2^{k\pm1}+2^{k-1}-N-1}-\alpha_{2^{k\pm1}+N+1}=\rho 2^{(k\pm1)(\rho-1)+N+1}\left((2^{k-1}-2N-2)^{\rho-1}-1\right)\geq 2$$

for k large enough. The claim follows. □

We apply this lemma with $C=\frac{\kappa}{4}$, where κ is the constant from Theorem 2.1. We introduce the energy intervals

$$I_s^{\pm}=\left[E_{m_s}^{\pm}+\frac{\kappa}{4}\frac{1}{2^{\gamma k}},E_{m_s}^{\pm}-\frac{\kappa}{4}\frac{1}{2^{\gamma k}}\right]. \tag{3.8}$$

We also have for $k\geq 1$ large enough that for any

$$E\in[-2+\delta,-\delta]\cup[\delta,2-\delta] \tag{3.9}$$

there exists s such that $E\in I_s^+$ and t such that $E\in I_t^-$. Define

$$Q_s^{\pm}=[m_s^+-N,m_s^++N]. \tag{3.10}$$

We now obtain

Lemma 3.3. *For $k\geq 1$ large enough. We have that*

$$\#\left(\sigma(H_{\gamma,\rho}^{Q_s^{\pm}})\cap I_s^{\pm}\right)\leq 4. \tag{3.11}$$

Proof. Let $m=m_s^{\pm}$, $\alpha=\alpha_m$, $\omega=\omega_m$, and $Q=Q_s^{\pm}$. By Theorem 2.1, we have

$$\sigma(\widehat{H}_{\frac{1}{2}m^{-\gamma},\alpha,\omega})\cap[E_m-\frac{\kappa}{m^{\gamma+\varepsilon}},E_m+\frac{\kappa}{m^{\gamma+\varepsilon}}]=\emptyset.$$

Next, since

$$\widehat{H}_{\frac{1}{2}m^{-\gamma},\alpha,\omega}=\widehat{H}^{Q}_{\frac{1}{2}m^{-\gamma},\alpha,\omega}\oplus\widehat{H}^{\mathbb{Z}\setminus Q}_{\frac{1}{2}m^{-\gamma},\alpha,\omega}+K$$

with K a rank 4 operator, we have

$$\#\left(\sigma(\widehat{H}^{Q}_{\frac{1}{2}m^{-\gamma},\alpha,\omega})\cap[E_m-\frac{\kappa}{m^{\gamma+\varepsilon}},E_m+\frac{\kappa}{m^{\gamma+\varepsilon}}]\right)\leq 4.$$

Now, the claim follows from Lemma 3.1. □

Introduce the set $\mathcal{E}_{1,k}$ as the set of energies

$$E\in[-2+\delta,-\delta]\cup[\delta,2-\delta] \tag{3.12}$$

such that for some s and t, we have

$$\operatorname{dist}(E,\sigma(H_{\gamma,\rho}^{Q_s^+}))\geq\frac{1}{2^{k(\gamma+\varepsilon)}}, \tag{3.13}$$

$$\operatorname{dist}(E,\sigma(H_{\gamma,\rho}^{Q_t^-}))\geq\frac{1}{2^{k(\gamma+\varepsilon)}}. \tag{3.14}$$

We have that

Lemma 3.4. *We have*

$$|\mathcal{E}_{1,k}|\leq\frac{1}{2}\frac{1}{k^2}. \tag{3.15}$$

Proof. Because of the properties of the set $I_s^{\pm}$, it suffices to eliminate a $\frac{1}{2^{(k+1)(\gamma+\frac{1}{2}\varepsilon)}}$ of the energies in (3.11). Hence, we have the bound

$$|\mathcal{E}_{1,k}|\leq 6S\frac{1}{2^{(k+1)(\gamma+\frac{1}{2}\varepsilon)}}\leq\frac{1}{2^{\frac{k\varepsilon}{3}}}\leq\frac{1}{2}\frac{1}{k^2},$$

for k large enough. This finishes the proof. □

Proof of Proposition 2.2. Let $E \in \mathcal{E}_{1,k}$. Then we may choose $I^+ = Q_s^+$ from (3.13) and $I^- = Q_t^-$ from (3.14). The claim follows. □

4. Proof of Corollary 2.3

Define the set $\mathcal{F}_k$ as the set of energies $E \in [-2,2]$ such that

$$\mathrm{dist}(E, \sigma(H_{\gamma,\rho}^{[a,b]})) \leq 2\mathrm{e}^{-(2^k)^{\frac{\varepsilon}{4}}} \tag{4.1}$$

for some $a \in \Lambda_k^-$ and $b \in \Lambda_k^+$.

Lemma 4.1. *For k large enough, we have that*

$$|\mathcal{F}_k| \leq \frac{1}{2}\frac{1}{k^2}. \tag{4.2}$$

Proof. There are less then 2^{2k} many possible choices for a, b. Each set $\sigma(H_{\gamma,\rho}^{[a,b]})$ contains less then 2^{k+1} elements. Hence, we are taking the $\mathrm{e}^{-2^{k\varepsilon}}$ neighborhood of a set containing less then 2^{3k+1} elements. The claimed estimate follows. □

We introduce

$$\mathcal{E}_{\delta,k}^2 = \mathcal{E}_{\delta,k}^1 \cup \mathcal{F}_k. \tag{4.3}$$

We see that $|\mathcal{E}_{\delta,k}^2| \leq \frac{1}{k^2}$.

Proof of Corollary 2.3. Let $E \in \mathcal{E}_{\delta,k}^2$ and let $I^{\pm}$ be the two intervals from Proposition 2.2 (ii.b). By the Combes–Thomas estimate (see [3] or Lemma 10.1 in [8]), we obtain from (3.11) that

$$|G_{\gamma,\rho}^{I^{\pm}}(E,x,y)| \leq \exp\left(-\frac{c|x-y|}{2^{(\gamma+\varepsilon)k}}\right)$$

for an universal constant $c > 0$ and $x, y \in I^{\pm}$.

Choose a in the center of I^+ and b in the one of I^-. By the second resolvent equation

$$\begin{aligned}|G_{\gamma,\rho}^{\Lambda_k}(E,x,y)| &\leq \mathrm{e}^{2^{\frac{k\varepsilon}{4}}}\left(|G_{\gamma,\rho}^{\Lambda_k}(E,x,a)| + |G_{\gamma,\rho}^{\Lambda_k}(E,x,b)|\right)\\ &\leq 4\mathrm{e}^{2\cdot 2^{\frac{k\varepsilon}{4}}}\exp\left(-\frac{c}{2^{(\gamma+\varepsilon)k}}2^{k(\gamma+2\varepsilon)}\right).\end{aligned}$$

The claim follows by choosing k large enough. □

Acknowledgements

I am thanking to Daniel Ueltschi and Robert Sims for their kind invitation to the second Arizona School of Analysis with Applications, where the idea for this project originated. Furthermore, I thank Milivoje Lukic for useful discussions.

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