Controlling Quantum Fluctuations in the Mean-Field Limit

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The quantum Bose gas

A quantum Bose gas of N particles is described by a normalized symmetric wave function

$$\Psi_N \in L^2(\mathbb{R}^{dN}, \mathrm{d}x_1 \cdots \mathrm{d}x_N)\,.$$

The time evolution is given by the N-body Schrödinger equation

$$\mathrm{i}\partial_t \Psi_N(t) \;=\; H_N \Psi_N(t)\,, \qquad \Psi_N(0) \;=\; \Psi_{N,0}\,.$$

The N-body Hamiltonian is

$$H_N \;:=\; \sum_{i=1}^N h_i + \lambda_N \sum_{1 \leqslant i < j \leqslant N} w(x_i - x_j)$$

The mean-field limit of the quantum Bose gas

Goal: understand behaviour of $\Psi_N(t)$. This is hard!

Simplifications:

- Take the limit $N \to \infty$ (macroscopic gas).
- Consider factorized initial data (condensate): $\Psi_{N,0} = \varphi_0^{\otimes N}$ for $\varphi_0 \in L^2(\mathbb{R}^d)$.

A nontrivial limit necessitates that both terms of H_N be of the same order in the limit $N \to \infty$. Set $\lambda_N = 1/N$: mean-field scaling.

$\Psi_N(t)$ is no longer factorized for $t > 0$. Heuristics: expect that

$$\Psi_N(t) \;\approx\; \varphi(t)^{\otimes N}\,,$$

where

$$\mathrm{i}\partial_t \varphi(t) \;=\; h\varphi(t) + (w * |\varphi(t)|^2)\varphi(t)\,,$$

the Hartree equation.

Reduced density matrices

We need a means of quantifying the error in $\Psi_N \approx \varphi^{\otimes N}$ (control the quantum fluctuations around φ). Define the reduced k-particle density matrix

$$\begin{aligned}\gamma_N^{(k)}&(x_1,\dots,x_k;y_1,\dots,y_k)\\ &:= \int \mathrm{d}x_{k+1}\cdots\mathrm{d}x_N\;\Psi_N(x_1,\dots,x_N)\overline{\Psi_N(y_1,\dots,y_k,x_{k+1},\dots,x_N)}\,.\end{aligned}$$

Good indicator: trace norm distance

$$R_N^{(k)} \;:=\; \mathrm{Tr}\left|\gamma_N^{(k)} - |\varphi\rangle\langle\varphi|^{\otimes k}\right|,$$

where $\mathrm{Tr}|A| = \sum_{\lambda\in\mathrm{Sp}(A)}|\lambda|$.

Previous results

- (Rodnianski & Schlein, 2007) If $h = -\Delta$ and $w(x) = |x|^{-1}$ then

$$R_N^{(k)}(t) \;\leqslant\; \frac{C(k)}{\sqrt{N}} \mathrm{e}^{K(k)t} \,.$$

- (Erdős & Schlein, 2008) If $\widehat{w} \in L^1$ then

$$R_N^{(k)}(t) \;\leqslant\; \frac{C(k)}{N} \mathrm{e}^{K(k)t} \,.$$

A different indicator of condensation

Define

$$E_N^{(k)} \;:=\; 1 - \left\langle \varphi^{\otimes k} , \gamma_N^{(k)} \varphi^{\otimes k} \right\rangle$$

Properties:

(i) $E_N^{(k)} \;\leqslant\; k E_N^{(1)}$

(ii) $E_N^{(k)} \leqslant R_N^{(k)} \leqslant \sqrt{E_N^{(k)}}$

Result I: nice w

Theorem

Assume that

(i) h and H_N are self-adjoint and bounded from below

(ii) $w \in L^{p_1} + L^{p_2}$ where $2 \leqslant p_1 \leqslant p_2 \leqslant \infty$

Then

$$E_N^{(1)}(t) \ \leqslant \ \left(E_N^{(1)}(0) + \frac{1}{N}\right)\mathrm{e}^{\Phi(t)}$$

where

$$\Phi(t) \ := \ 32\|w\|_{L^{p_1}+L^{p_2}} \int_0^t \mathrm{d}s \, \big(\|\varphi(s)\|_{q_1} + \|\varphi(s)\|_{q_2}\big)$$

and

$$\frac{1}{2} \ = \ \frac{1}{p_i} + \frac{1}{q_i}$$

- For $d = 3$, allowed singularities in w up to $|x|^{-3/2}$
- If $E_N^{(1)}(0) \leqslant C/N$ then

$$E_N^{(k)}(t) \;\leqslant\; C\frac{k}{N}\mathrm{e}^{\Phi(t)} \quad \text{and} \quad R_N^{(k)}(t) \;\leqslant\; C\sqrt{\frac{k}{N}}\mathrm{e}^{\Phi(t)/2}$$

$\Longrightarrow$ control the condensation of $k = o(N)$ particles

Examples

- The boson star: $h = \sqrt{\mathbb{1} - \Delta}$ and $w(x) = \lambda|x|^{-1}$ with $\lambda > -4/\pi$
- Scattering: If $\int \mathrm{d}t \, \|\varphi(t)\|_{q_i} < \infty$ for $i = 1, 2$ then $\Phi(t) \leqslant C$
  $\Longrightarrow$ uniformity in time

Result II: singular w

"Venerable physical folklore": $h = -\Delta$ and $w(x) = |x|^{-\zeta}$ for $\zeta < 2$ produces reasonable quantum dynamics for $d = 3$.

Mathematics:

- H_N is self-adjoint.
- Hartree equation is globally well-posed.

Does mean-field convergence hold?

Yes! Provided that $w \in L^{6/5} + L^{\infty}$ (in $d = 3$)

$\Longrightarrow$ allowed singularities up to $|x|^{-5/2}$.

Theorem

Assume that

(i) h and H_N are self-adjoint and bounded from below

(ii) $w \in L^p + L^\infty$ for $p_0 < p \leqslant 2$

(iii) $E_N^{(1)}(0) \to 0$ sufficiently fast in N

Then

$$E_N^{(1)}(t) \;\leqslant\; \frac{1}{N^\eta} \mathrm{e}^{\Phi(t)}$$

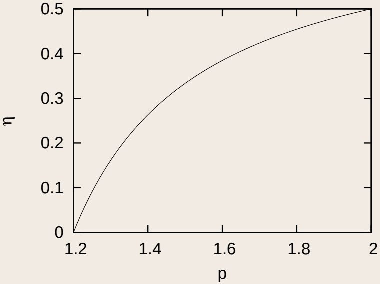
where

$$\eta \;:=\; \frac{p/p_0 - 1}{2p/p_0 - p/2 - 1}$$

and

$$\Phi(t) \;=\; C \int_0^t \mathrm{d}s \, \bigl(1 + \|\varphi(s)\|_X^3\bigr)$$

for $X \approx H^2 \cap L^\infty$.



Sketch of proofs

Control the quantity $\alpha := E_N^{(1)}(t)$. Set

$$p \;:=\; |\varphi\rangle\langle\varphi| \,, \qquad q \;:=\; \mathbb{1} - p \,,$$

and

$$W^\varphi \;:=\; w * |\varphi|^2 \,.$$

With the abbreviation $W_{ij} := w(x_i - x_j)$ we get

$$\dot\alpha \;=\; \frac{\mathrm{i}}{2N} \big\langle \Psi , \big[(N-1)W_{12} - NW_1^\varphi - NW_2^\varphi \,, q_1 + q_2\big] \Psi \big\rangle \,.$$

Main work: prove a bound of the form

$$\dot\alpha_N(t) \;\leqslant\; A_N(t) + B_N(t)\alpha_N(t) \tag{1}$$

with $\lim_N A_N(t) = 0$. Then by Grönwall we are done.

To get (1), insert $\mathbb{1} = (p_1 + q_1)(p_2 + q_2)$ in front of both Ψ's in

$$\frac{\mathrm{i}}{2N}\big\langle \Psi\,, \big[(N-1)W_{12} - NW_1^{\varphi} - NW_2^{\varphi}\,, q_1 + q_2\big]\Psi\big\rangle$$

and multiply everything out.

Only three types of terms survive:

$$p_1p_2(\cdot)q_1p_2\,, \qquad q_1p_2(\cdot)q_1q_2\,, \qquad p_1p_2(\cdot)q_1q_2\,.$$

Heuristics:

- p controls singularities in W_{12} (since φ is smooth).
- q is small: $\langle \Psi\,, q_1\Psi\rangle = \alpha$.

Use energy estimates to control singularities.