Optimality of Log Hölder Continuity of the Integrated Density of States

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Task

We investigate optimality of the log Hölder continuity of the integrated density of states.

Ergodic Schrödinger Operator

Let (Ω, μ) be a probability space, $T : \Omega \to \Omega$ an invertible ergodic transformation, and $f : \Omega \to \mathbb{R}$ a bounded measurabe function.

The ergodic Schödinger operator $H_\omega : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is defined by

$$H_\omega u(n) = u(n+1) + u(n-1) + V_\omega(n)u(n)$$

and

$$V_\omega(n) = f(T^n\omega).$$

Integrated Density of States

The integrated density of states (IDS) k is defined by

$$k(E) = \lim_{N\to\infty} \int_\Omega \left(\frac{1}{N} \mathrm{tr}(P_{(-\infty,E)}(H_{\omega,[0,N-1]})) \right) d\mu(\omega),$$

where $H_{\omega,[0,N-1]}$ denotes the restriction of H_ω to $\ell^2([0,N-1])$.

Log Hölder Continuity

Theorem (Craig & Simon, 1983)

There exists a constant $C = C(\| f \|_\infty)$ *such that*

$$|k(E) - K(\tilde{E})| \leq \frac{C}{\log |E - \tilde{E}|^{-1}}$$

for $|E - \tilde{E}| \leq \frac{1}{2}$.

A natural question arises: can $\epsilon \to \frac{1}{\log(\epsilon^{-1})}$ be replaced by another function, which goes to zero faster?

Examples

- If f is periodic, that is, there is p such that $f(T^p\omega) = f(\omega)$ for every $\omega \in \Omega$, then $k(E)$ is $\frac{1}{2}$-Hölder continuous.
- For random Schrödinger operators, $k(E)$ is Hölder continuous.
- Craig (1983) gave examples such that the regularity of $k(E)$ cannot be improved to

$$\epsilon \to \frac{1}{\log(\epsilon^{-1})(\log(\log(\epsilon^{-1})))^{\beta}}$$

where $\beta > 1$.

Motivation

Our motivation in this topic comes from the importance of the Wegner estimate in multiscale analyis. If one could improve the continuity of $k(E)$ to $\epsilon \to \frac{1}{(\log(\epsilon^{-1}))^{\beta}}$ for some large enough $\beta > 1$, one would be able to use this for multiscale analysis.

Already Craig's result shows that this is impossible. However one could hope that a combination of an improved continuity result and an improvement of multiscale analysis might remove the Wegner estimate assumption.

We show that the continuity of IDS cannot be improved for all potentials beyond log Hölder continuity.

Limit-Periodic Model

We assume that the compact space Ω is a Cantor group that admits a minimal translation T.

Definition

Ω is called a Cantor group if it is an infinite totally disconnected compact Abelian topological group.

For any $f \in C(\Omega, \mathbb{R})$, $(f(T^n(\omega)))_{n\in\mathbb{Z}}$ is limit-periodic, that is, it can be uniformly approximated by periodic potentials. The set of all periodic f is dense in $C(\Omega, \mathbb{R})$.

Integrated Density of States

If V is limit-periodic, then $\mathrm{hull}(V)$ is a Cantor group that admits a minimal translation, and it has a unique Haar measure. In this context, the IDS can be replaced by

$$k_V(E) = \lim_{N\to\infty} \frac{1}{N} \mathrm{tr}(P_{(-\infty,E)}(H_{[0,N-1]})).$$

Our Result

Now we are ready to give

Theorem (to appear in Math. Nachr.)

Given any increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *with*

$$\lim_{x \to 0} \varphi(x) = 0$$

and a constant $C_0 > 0$*, there is a limit-periodic* V *satisfying* $\| V \|_\infty \leq C_0$ *such that its integrated density of states satisfies*

$$\limsup_{E \to E_0} \frac{|k_V(E) - k_V(E_0)| \log(|E - E_0|^{-1})}{\varphi(|E - E_0|)} = \infty,$$

for any $E_0 \in \sigma(\Delta + V)$.

This result tells us that with φ as in the previous theorem, we cannot have

$$|k_V(E) - k_V(E_0)| \leq C \cdot \frac{\varphi(|E - E_0|)}{\log(|E - E_0|^{-1})}$$

for any $C > 0$ and all V.

That is, log Hölder continuity cannot be improved for all V.

Idea

The proof idea comes from the well known result about periodic potentials.

Theorem

If V is a p-periodic potential, and $[E-, E+]$ is a band of $\sigma(\Delta + V)$, then we have

$$k(E+) - K(E-) = \frac{1}{p}.$$

Construct Desired Limit-periodic Potentials.

We construct a sequence V_j of p_j periodic potentials, with the following properties

1. V_j is p_j-periodic.
2. The Lebesgue measure $\epsilon_j = |\sigma(\Delta + V_j)|$ satisfies $\log(\epsilon_j^{-1}) \geq p_{j-1} \cdot p_j \cdot \varphi(2\epsilon_j)$.
3. We have that $\| V_j - V_{j-1} \| \leq \frac{\min(\epsilon, \epsilon_1, \ldots, \epsilon_{j-1})}{2^j}$.

How to get the statement 2 ?

We heavily use Lemma 3.1 and Lemma 3.2 in

A. Avila, "*On the spectrum and Lyapunov exponent of limit periodic Schrödinger operators*", Comm. Math. Phys. **288:3** (2009), 907–918.

Thank You !