

# Universality of Wigner Random Matrices: a Survey of Recent Results

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## Abstract

In this survey we study the universality of spectral statistics of large random matrices. We consider  $N \times N$  symmetric, hermitian or quaternion self-dual random matrices with independent, identically distributed entries (Wigner matrices) where the probability distribution for each matrix element is given by a measure  $\nu$  with a subexponential decay. Our main result is that the correlation functions of the local eigenvalue statistics in the bulk of the spectrum coincide with those of the Gaussian Orthogonal Ensemble (GOE), the Gaussian Unitary Ensemble (GUE) and the Gaussian Symplectic Ensemble (GSE), respectively, in the limit  $N \rightarrow \infty$ . Our approach is based on the study of the Dyson Brownian motion via a related new dynamics, the local relaxation flow.

As a main input, we establish that the density of eigenvalues converges to the Wigner semicircle law and this holds even down to the smallest possible scale, and, moreover, we show that eigenvectors are fully delocalized. These results hold even without the condition that the matrix elements are identically distributed, only independence is used. We also prove a Wegner type estimate and that the eigenvalues repel each other on arbitrarily small scales.

*Keywords.* Wigner random matrix, Dyson Brownian Motion, semicircle law, sine-kernel.

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# 1 Introduction

This survey is based upon the lecture notes that the author has prepared for the participants at the Arizona School of Analysis and Applications, Tucson, AZ in 2010. The style of the presentation is closer to the informal style of a lecture than to a formal research article. For the details and sometimes even for the precise formulation we refer to the original papers.

In the first introductory section we give an overview about universality of random matrices, including previous results, history and motivations. We introduce some basic concepts such as Wigner matrix, Wigner semicircle law, Stieltjes transform, moment method, sine-kernel, gap distribution, level repulsion, bulk and edge universality, invariant ensembles, Green function comparison theorem, local relaxation flow and reverse heat flow. Several of these concepts will not be used for our main results, but we included them to help the orientation of the reader.

The selection of the material presented in the first section is admittedly reflects a personal bias of the author and it is not meant to be comprehensive. There are several very active research directions connected with random matrices that are not mentioned in this survey at all, e.g. supersymmetric methods or connection with free probability. Some other very rich topics, e.g. edge universality and orthogonal polynomial methods, are mentioned only very superficially. We refer the reader to more comprehensive surveys on random matrices, especially the classical book of Mehta [68], the survey of the Riemann-Hilbert approach of Deift [20], the recent book of Anderson, Guionnet and Zeitouni [4] and the forthcoming book of Forrester [50]. An excellent short summary about the latest developments is by Guionnet [57].

Starting from Section 2, we present the shortest and up to now the most powerful approach to the bulk universality for  $N \times N$  Wigner matrices. In this part we strive for mathematical rigor, but several details will be referred to the original papers. The argument has three distinct steps:

1. Local semicircle law (Section 2);
2. Universality for Gaussian convolutions via the local relaxation flow (Section 3);
3. Green function comparison theorem (Section 4).

Finally, in Section 5, we put together the proof from these ingredients. Some technical lemmas are collected in the Appendices that can be neglected at first reading.

*Convention:* Throughout the paper the letters  $C$  and  $c$  denote positive constants whose values may change from line to line and they are independent of the relevant parameters. Since we will always take the  $N \rightarrow \infty$  limit at the end, all estimates are understood for sufficiently large  $N$ . In informal explanations we will often neglect logarithmic factors, by introducing the notation  $\lesssim$  and  $\ll$  to indicate inequality “up to some  $\log N$  factor”. More precisely,  $A \lesssim B$  means  $A \leq (\log N)^C B$  with some non-negative constant  $C$ , and  $A \ll B$  means  $A \leq (\log N)^{-C} B$  with some positive constant  $C$ .

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## 1.1 Summary of the main results: an orientation for the reader

We will consider  $N$  by  $N$  matrices  $H = (h_{ij})_{i,j=1}^N$  whose entries are real or complex random variables (our results for matrices with quaternion entries will not be discussed here, see [44]). In most cases we assume that  $H$  is hermitian or symmetric, but our method applies to other ensembles as well. We assume that the entries are independent up to the symmetry constraint,  $h_{ij} = \bar{h}_{ji}$ , they are centered,  $\mathbb{E}h_{ij} = 0$ , and their tail probability has a subexponential decay. We do *not* assume that the matrix elements are identically distributed but we assume that the variances,  $\sigma_{ij}^2 := \mathbb{E}|h_{ij}|^2$  satisfy the normalization condition

$$\sum_{j=1}^N \sigma_{ij}^2 = 1, \quad i = 1, 2, \dots, N. \quad (1.1)$$

i.e., the deterministic  $N \times N$  matrix of variances,  $\Sigma = (\sigma_{ij}^2)$ , is symmetric and doubly stochastic. These conditions guarantee that  $-1 \leq \Sigma \leq 1$ . We will always assume that 1 is a simple eigenvalue of  $\Sigma$  and there is a positive number  $\delta_- > 0$  such that  $-1 + \delta_- \leq \Sigma$ . This assumption is satisfied for practically any random matrix ensembles. Sometimes we will need a uniform gap condition, i.e. that there exists a positive  $\delta_+ > 0$  such that

$$\text{Spec } \Sigma \subset [-1 + \delta_-, 1 - \delta_+] \cup \{1\}.$$

For example, for the standard Wigner matrix  $\sigma_{ij}^2 = N^{-1}$  and  $\delta_- = \delta_+ = 1$ . For random band matrices (see (1.18) for the precise definition) with band width  $W$  satisfying  $1 \ll W \ll N$ , the gap  $\delta_+$  goes to zero as the size of the matrix increases.

The normalization (1.1) ensures that the bulk of the spectrum of  $H$  lies in the interval  $[-2, 2]$  and the density of eigenvalues  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$  is given by the Wigner semicircle law as  $N \rightarrow \infty$ . Apart from the vicinity of the edges  $\pm 2$ , the typical spacing of neighboring eigenvalues is of order  $1/N$ . We are interested in the statistics of the eigenvalues in the  $N \rightarrow \infty$  limit.

There will be three groups of main result which we summarize below. The strongest results are found in our recent papers [46, 47] with H.T. Yau and J. Yin, which partially build upon methods and ideas of our previous results [37, 38, 39, 41, 42, 44] obtained in collaboration with B. Schlein and, in case of [41], also with J. Ramirez and in case of [42] with S. Péché and J. Ramirez.

### 1.1.1 Summary of Section 2: Main results on the local semicircle law

In Section 2 we prove that the density of eigenvalues follows the semicircle law *down to the smallest possible scale*, i.e., to scales only a slightly larger than  $1/N$ . We will call it *local semicircle law*.

The local semicircle law is identified via the Stieltjes transform of the empirical density of the eigenvalues,

$$m(z) = m_N(z) = \frac{1}{N} \sum_{j=1}^N \frac{1}{\lambda_j - z}, \quad z = E + i\eta, \quad E \in \mathbb{R}, \quad \eta > 0,$$

and we show that  $m_N(z)$  converges to the Stieltjes transform of the semicircle density

$$m_{sc}(z) = \int_{\mathbb{R}} \frac{\varrho_{sc}(x) dx}{x - z}, \quad \varrho_{sc}(x) := \frac{1}{2\pi} \sqrt{(4 - x^2)_+},$$

in the limit  $N \rightarrow \infty$ . The imaginary part  $\eta = \Im z$  may depend on  $N$  and it corresponds to the local scale on which the density is identified. The precision of our approximation is of order  $(N\eta)^{-1}$  for Wigner matrices, i.e., we show in Theorem 2.6 later that

$$|m(z) - m_{sc}(z)| \lesssim \frac{CN^\varepsilon}{N\eta\kappa}, \quad \kappa := ||E| - 2|, \quad (1.2)$$

with a very high probability and with an arbitrarily  $\varepsilon > 0$ . The factor  $N^\varepsilon$  can be removed but the power of  $N\eta$  is reduced. For example,

$$|m(z) - m_{sc}(z)| \lesssim \frac{C}{\sqrt{N\eta\kappa}} \quad (1.3)$$

was proved in Theorem 4.1 of [41] (under some additional assumptions). Both results (1.2), (1.3) also hold for Wigner matrices whose variances are not necessarily equal but only comparable (see (1.62) for precise definition); the key input is that in this case we have  $\delta_+ > 0$ . There are  $\kappa$ -independent bounds also available, for example in Proposition 8.1. [44] we proved that

$$|m(z) - m_{sc}(z)| \lesssim \frac{C}{(N\eta)^{1/4}} \quad (1.4)$$

(the proof was given for sample covariance matrices, but the same argument applies to the Wigner case as well).

For more general Wigner matrices the key quantity that measures the precision is the *spread of the matrix*, defined by

$$M := \frac{1}{\max_{ij} \sigma_{ij}^2}. \quad (1.5)$$

For typical random band matrices (see (1.18) for the precise definition),  $M$  is comparable with the band width  $W$ . If  $M \ll N$ , then the precision of our estimates is determined by  $M$  instead of  $N$ , for example, in [47] we obtain

$$|m(z) - m_{sc}(z)| \lesssim \frac{N^\varepsilon}{M\eta\kappa^2} \quad (1.6)$$

(see Theorem 2.6 later), or

$$|m(z) - m_{sc}(z)| \lesssim \frac{1}{\sqrt{M\eta} \kappa} \quad (1.7)$$

was proven in Theorem 2.1 of [46].

It is well known that the identification of the Stieltjes transform of a measure for the complex parameters  $z = E + i\eta$ ,  $E \in \mathbb{R}$ , is equivalent to knowing the density down to scales essentially  $O(\eta)$ , thus we obtain the control on the density down to scales essentially of order  $\eta \sim 1/M$ .

The Stieltjes transform  $m(z)$  can also be viewed as the normalized trace of the resolvent,

$$m(z) = \frac{1}{N} G(z) = \frac{1}{N} \sum_{i=1}^N G_{ii}(z), \quad G(z) = \frac{1}{H - z}.$$

In addition to (1.2), we are able to prove that not only the sum, but each diagonal element  $G_{ii}(z)$  is given by the semicircle law, but the precision is weaker:

$$\max_i |G_{ii}(z) - m_{sc}(z)| \lesssim \frac{C_\kappa}{\sqrt{N\eta}}, \quad z = E + i\eta. \quad (1.8)$$

Finally, we can also show that the off-diagonal resolvent elements are small:

$$\max_{ij} |G_{ij}(z)| \lesssim \frac{C_\kappa}{\sqrt{N\eta}}. \quad (1.9)$$

The explicit constant  $C_\kappa$  in (1.8) and (1.9) deteriorates near the spectral edge as an inverse power of  $\kappa$ ; the exponent depends on whether a positive uniform lower bound  $\delta_+ > 0$  is available or not. For more general Wigner matrices, e.g. for band matrices, we obtain the same estimates but  $M$  replaces  $N$  on the right hand sides of (1.8) and (1.9). The precise statements are given in Theorem 2.6.

The asymptotics of the Stieltjes transform can be translated into the asymptotics of the counting function (e.g. Theorem 2.7) or into a result on the location of the eigenvalues (Theorem 2.8). Moreover, the local semicircle law easily implies that the eigenvectors are fully *delocalized* (see Section 2.4).

### 1.1.2 Summary of Section 3: Main results on the bulk universality with Gaussian component

Bulk universality refers to the fact that local eigenvalue statistics, i.e., correlation functions of eigenvalues rescaled by a factor  $N$ , or the distribution of the gap between consecutive eigenvalues exhibit universal behavior which is solely determined by the symmetry class of the ensemble.

Bulk universality has first been proved for Gaussian Wigner ensembles, i.e., when the matrix elements  $h_{ij}$  are i.i.d. Gaussian random variables by Dyson [30] and Mehta [69]. The Gaussian

character makes explicit calculations easier that are needed to identify the limiting correlation functions (e.g. the celebrated *sine kernel* for the hermitian case). The key fact is that the joint distribution function for the eigenvalues of such ensembles is explicit and it contains a Vandermonde determinant structure from which the local universality can be deduced.

It is a natural idea to consider a broader class of matrices that still have some Gaussian character; the useful concept is the *Gaussian divisible* matrices, i.e., where the probability law of each matrix element contains a Gaussian component (Gaussian convolution).

The first approach is to push the explicit calculations further by finding a similar Vandermonde structure. For an algebraic reason, this method is available for the hermitian case only. Using our local semicircle law and a modified version of an explicit representation formula of Johansson [61] we were able to prove bulk universality for hermitian Wigner matrices with a tiny Gaussian component of variance  $O(N^{-1+\varepsilon})$  with an improved formula in [42].

The second approach is to embed the Gaussian divisible ensemble into a stochastic flow of matrices, and use the key observation of Dyson [29] that under this flow the eigenvalues perform a specific stochastic dynamics with a logarithmic interaction, the Dyson Brownian Motion. Eventually the dynamics relaxes to equilibrium, which is the well known Gaussian model (GUE, GOE or GSE). Our main idea is that the *local* relaxation is much faster, i.e., the local statistics of eigenvalues already reach their equilibrium within a very short  $t = N^{-\varepsilon}$  time (with an explicit  $\varepsilon > 0$ ). Our main result states that the local correlation functions for a Gaussian divisible matrices with a small Gaussian component coincide with the correlation functions of the purely Gaussian ensembles.

This result can be formulated in a general setup and viewed as a strong local ergodicity of the Dyson Brownian motion or, in fact, of any one dimensional stochastic particle dynamics with logarithmic interaction. This general formulation appeared first in [44] and it will be given in Theorem 3.3, but most of the key ideas were invented in [40]. For the application of this general principle to random matrices one needs certain apriori information about the location of the eigenvalues, which we obtain from the local semicircle law. In particular, using this idea, the bulk universality for *symmetric* Wigner matrices was first proved in [40]. The case of quaternion self-dual and sample covariance matrices were solved in [44].

### 1.1.3 Summary of Section 4: Main results on the removal of the Gaussian component

To prove the universality of any Wigner ensemble, we need to compare it with a Gaussian divisible ensemble for which universality is already proven. Such comparison principle is plausible if the Gaussian component is small and indeed a perturbation argument can be applied.

The first result of this type used a *reversed heat flow* argument [42], where we showed that any smooth distribution can be approximated with a very high precision by a Gaussian divisible distribution. Combining this method with the bulk universality for hermitian Wigner matrices with an  $O(N^{-1+\varepsilon})$  Gaussian component, we were able to prove bulk universality for any Wigner ensemble under the smoothness condition [42].

A more robust approach is the Green function comparison theorem from [46], which states that



for two matrix ensembles, the joint distribution of the Green functions coincides, provided that the first four moments of the probability law of the matrix elements are identical or very close. The spectral parameter  $z$  can have a very small imaginary part  $\Im z \sim N^{-1-\varepsilon}$ , i.e., these Green functions can detect individual eigenvalues. The precise statement is given in Theorem 4.1.

The combination of the results of Section 1.1.2 on the bulk universality of Gaussian divisible matrices and the Green function comparison theorem gives the bulk universality for any Wigner ensemble by a simple matching argument. The method applies even to matrices with comparable variances (1.62). The only condition in the approach is a subexponential decay for the tail of the probability law of the matrix elements [47].

The four moment condition was first observed by Tao and Vu [91] in the four moment theorem for eigenvalues (Theorem 1.5). They used this result to prove universality for hermitian Wigner matrices without smoothness condition but under some moment and support condition, that especially excluded the Bernoulli distribution. The bulk universality for hermitian Wigner matrices including the Bernoulli case was first proved in our joint paper [43] after combining the results of [42] and [91].

Currently the hydrodynamical method combined with the Green function comparison theorem gives the most general approach to bulk universality. This path not only proves bulk universality for general Wigner ensembles, but it also offers a conceptual understanding how this universality emerges from simple principles.

In Sections 1.2–1.7 we review several facts, results and methods in connection with Wigner random matrices and some related ensembles. These sections are meant to provide a general background information. In Section 1.6 we also explain the key new ideas listed above in more details and give a summary of various results on bulk universality. A reader wishing to focus only on the most recent developments can skip Sections 1.2–1.7 and jump to Section 2.

## 1.2 Wigner matrix ensemble

A central question in probability theory is the universality of cumulative statistics of a large set of independent data. Given an array of  $N$  independent random variables

$$(X_1, X_2, \dots, X_N) \tag{1.10}$$

one forms linear statistics like the mean

$$\overline{X}^{(N)} = \frac{1}{N} \sum_{j=1}^N X_j$$

or the fluctuation

$$S^{(N)} = \frac{1}{\sqrt{N}} \sum_{j=1}^N (X_j - \mathbb{E}X_j).$$

Under very general conditions, a universal pattern emerges as  $N \rightarrow \infty$ : the mean converges to its expectation, in particular, it becomes deterministic,

$$\overline{X}^{(N)} \rightarrow \lim_{N \rightarrow \infty} \mathbb{E} \overline{X}^{(N)},$$

assuming that the latter limit

$$\lim_{N \rightarrow \infty} \mathbb{E} \overline{X}^{(N)} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N \mathbb{E} X_j$$

exists (law of large numbers). Moreover, the fluctuation  $S^{(N)}$  converges to a centered normal Gaussian random variable  $\xi$

$$S^{(N)} \rightarrow \xi \quad (\text{in distribution}) \quad (1.11)$$

(central limit theorem), i.e., the density function of  $\xi$  is given by  $(\sigma\sqrt{2\pi})^{-1} \exp(-x^2/2\sigma^2)$ . The variance  $\sigma^2$  of  $\xi$  is given by the average of the variances of  $X_j$ ,

$$\sigma^2 := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N \sigma_j^2, \quad \sigma_j^2 := \mathbb{E}[X_j - \mathbb{E}X_j]^2.$$

In particular, for independent, identically distributed (i.i.d.) random variables, we have  $\mathbb{E} \overline{X}^{(N)} = \mathbb{E}X_1$  and  $\sigma^2 = \mathbb{E}[X_1 - \mathbb{E}X_1]^2$ , thus  $\overline{X}^{(N)}$  converges to the common expectation value of  $X_j$ 's, and  $S^{(N)}$  converges to the centered normal distribution with the same variance as  $X_j$ .

The emergence of a single universal distribution, the Gaussian, is a remarkable fact of Nature. It shows that large systems with many independent components in a certain sense behave similarly, irrespective of the details of the distributions of the components.

It is natural to generalize this question of universality from arrays (1.10) to double arrays, i.e., to matrices:

$$X^{(N,M)} = \begin{pmatrix} X_{11} & X_{12} & \dots & X_{1N} \\ X_{21} & X_{22} & \dots & X_{2N} \\ \vdots & \vdots & \dots & \vdots \\ X_{M1} & X_{M2} & \dots & X_{MN} \end{pmatrix} \quad (1.12)$$

with independent entries. The statistics in question should involve a quantity which reflects the matrix character and is influenced by all entries; for example the norm of  $X^{(N,M)}$ . Although the norm of each random realization of  $X^{(N,M)}$  may differ, it is known, for example, that in the limit as  $N, M \rightarrow \infty$ , such that  $N/M \rightarrow d$ ,  $0 < d \leq 1$  is fixed, it becomes deterministic, e.g. we have [67, 99]

$$\frac{1}{\sqrt{M}} \|X^{(N,M)}\| \rightarrow \sigma(1 + \sqrt{d}) \quad (1.13)$$

assuming that the matrix elements are centered,  $\mathbb{E}X_{ij} = 0$ , and their average variance is  $\sigma^2$ . Note that the typical size of  $X^{(N,M)}$  is only of order  $\sqrt{M}$  despite that the matrix has dimensions of order  $M \times N$ , filled with elements of size  $O(1)$ . If the matrix elements were strongly correlated then the norm could very well be of order  $M$ . For example, in the extreme case, if all elements were the same,  $X_{ij} = X$ , then  $\|X^{(N,M)}\| \sim M$ . Independence of the matrix elements prevents such conspiracy and it reduces the typical size of the matrix by a factor of  $\sqrt{M}$ , similarly to the central limit theorem (note the  $\sqrt{N}$  normalization in (1.11)).

Matrices offer much richer structure than studying only their norm. Assuming  $M = N$ , the most important characteristics of a square matrix are the eigenvalues and eigenvectors. As (1.13) suggests, it is convenient to assume zero expectation for the matrix elements and rescale the matrix by a factor  $N^{-1/2}$  to have norm of order 1. For most of this presentation, we will therefore consider large  $N \times N$  square matrices of the form

$$H = H^{(N)} = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1N} \\ h_{21} & h_{22} & \dots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{N1} & h_{N2} & \dots & h_{NN} \end{pmatrix} \quad (1.14)$$

with *centered* entries

$$\mathbb{E}h_{ij} = 0, \quad i, j = 1, 2, \dots, N. \quad (1.15)$$

As for normalization, we assume that the matrix of variances

$$\Sigma := \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 & \dots & \sigma_{1N}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 & \dots & \sigma_{2N}^2 \\ \vdots & \vdots & & \vdots \\ \sigma_{N1}^2 & \sigma_{N2}^2 & \dots & \sigma_{NN}^2 \end{pmatrix}, \quad \sigma_{ij}^2 = \mathbb{E}|h_{ij}|^2, \quad (1.16)$$

is doubly stochastic, i.e., for every  $i = 1, 2, \dots, N$  we have

$$\sum_j \sigma_{ij}^2 = \sum_j \sigma_{ji}^2 = 1.$$

The most natural example is the **mean-field model**, when

$$\sigma_{ij}^2 = \frac{1}{N} \quad i, j = 1, 2, \dots, N,$$

i.e., each matrix element is of size  $h_{ij} \sim N^{-1/2}$ . This corresponds to the standard Wigner matrix. For most of this presentation the reader can restrict the attention to this case.

Random matrices are typically subject to some symmetry restrictions, e.g. we will consider symmetric ( $h_{ij} = h_{ji} \in \mathbb{R}$ ) or hermitian ( $h_{ij} = \overline{h_{ji}} \in \mathbb{C}$ ) random matrices. We will mostly assume that the matrix elements are *independent* up to the symmetry requirement (i.e. in case of symmetric or hermitian matrices, the variables  $\{h_{ij} : i \leq j\}$  are independent). This leads us to the

**Definition 1.1** An  $N \times N$  symmetric or hermitian random matrix (1.14) is called **universal Wigner matrix (ensemble)** if the entries are centered (1.15), their variances  $\sigma_{ij}^2 = \mathbb{E}|h_{ij}|^2$  satisfy

$$\sum_j \sigma_{ij}^2 = 1, \quad i = 1, 2, \dots, N \quad (1.17)$$

and  $\{h_{ij} : i \leq j\}$  are independent. In the special case  $\sigma_{ij}^2 = 1/N$ , we recover the original definition of the **Wigner matrices** or **Wigner ensemble** [100].

The most prominent Wigner ensembles are the *Gaussian Orthogonal Ensemble* (GOE) and the *Gaussian Unitary Ensemble* (GUE); i.e., symmetric and hermitian Wigner matrices with rescaled matrix elements  $\sqrt{N}h_{ij}$  being standard Gaussian variables (in the hermitian case,  $\sqrt{N}h_{ij}$  is a standard complex Gaussian variable, i.e.  $\mathbb{E}|\sqrt{N}h_{ij}|^2 = 1$ ).

For simplicity of the presentation, in case of the Wigner ensembles, we will assume that  $h_{ij}$ ,  $i < j$ , are identically distributed (i.e. not only their variances are the same). In this case we fix a distribution  $\nu$  and we assume that the rescaled matrix elements  $\sqrt{N}h_{ij}$  are distributed according to  $\nu$ . Depending on the symmetry type, the diagonal elements may have a different distribution, but we will omit this subtlety from the discussion. The distribution  $\nu$  will be called the *single entry distribution* of  $H$ .

We will sometimes mention a special class of universal Wigner matrices that have a band structure; they will be called *random band matrices*. The variances are given by

$$\sigma_{ij}^2 = W^{-1} f\left(\frac{[i-j]_N}{W}\right), \quad (1.18)$$

where  $W \geq 1$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}_+$  is a bounded nonnegative symmetric function with  $\int f = 1$  and we defined  $[i-j]_N \in \{1, 2, \dots, N\}$  by the property that  $[i-j]_N \equiv i-j \pmod{N}$ . Note that the relation (1.17) holds only asymptotically as  $W \rightarrow \infty$  but this can be remedied by an irrelevant rescaling. One can even consider  $d$ -dimensional band matrices, where the rows and columns are labelled by a finite lattice  $\Lambda \subset \mathbb{Z}^d$  and  $\sigma_{ij}^2$  depends on the difference  $i-j$  for any  $i, j \in \Lambda$ .

Another class of random matrices, that even predate Wigner, are the *random covariance matrices*. These are matrices of the form

$$H = X^* X, \quad (1.19)$$

where  $X$  is a rectangular  $M \times N$  matrix of the form (1.12) with centered i.i.d. entries with variance  $\mathbb{E}|X_{ij}|^2 = M^{-1}$ . Note that the matrix elements of  $H$  are not independent, but they are generated from the independent matrix elements of  $X$  in a straightforward way. These matrices appear in statistical samples and were first considered by Wishart [101]. In the case when  $X_{ij}$  are centered Gaussian, the random covariance matrices are called *Wishart matrices or ensemble*.

### 1.3 Motivations: from Schrödinger operators to the $\zeta$ -function

We will primarily study the eigenvalue statistics of large random matrices and some results about eigenvectors will also be mentioned. The main physical motivation is that a random matrix can model the Hamilton operator of a disordered quantum system. The symmetry properties of  $H$  stem from this consideration: symmetric matrices represent Hamiltonians of systems with time reversal invariance (e.g. no magnetic field), hermitian matrices correspond to systems without time reversal symmetry. (There is a third class of matrices, the quaternion self-dual matrices, most prominently modelled by the *Gaussian Symplectic Ensemble (GSE)*, that describe systems with odd-spin and no rotational symmetry, but we will not discuss it in detail.)

E. Wigner has originally invented random matrices to mimic the eigenvalues of the unknown Hamiltonian of heavy nuclei; lacking any information, he assumed that the matrix elements are i.i.d. (independent, identically distributed) random variables subject to the hermitian condition. His very bold vision was that, although such a crude approximation cannot predict individual energy levels (eigenvalues) of the nucleus, their *statistical properties* may be characteristic to some global feature shared by any nucleus. By comparing measured data of energy levels of nuclei with numerical calculations of eigenvalues of certain random matrices, he found that the *level statistics*, i.e., the distribution of the energy gaps between neighboring energy levels (eigenvalues), show remarkable coincidence and robustness. In particular, he observed that energy levels tend to repel each other, a significant difference from the level statistics of fully uncorrelated random points (Poisson point process). Similar feature was found for random matrices: even Wigner matrices that are “as stochastic as possible” delivered plots of strongly correlated (repelling) eigenvalues. This correlation is due to the underlying fundamental symmetry of the matrix ensemble, in particular symmetric and hermitian matrices were found to have a different strength of level repulsion, but within a fixed symmetry class a universal pattern emerged. For more details on the history of this remarkable discovery, see [68].

Universality of local eigenvalue statistics is believed to hold for a much broader class of matrix ensembles than we have introduced. There is no reason to believe that the matrix elements of the Hamiltonian of the heavy nuclei are indeed i.i.d. random variables. Conceivably, the matrix elements need not be fully independent or identically distributed for universality. There is little known about matrices with correlated entries, apart from the unitary invariant ensembles (Section 1.5.3) that represent a very specific correlation. In case of a certain class of Wigner matrices with weakly correlated entries, the semicircle law and its Gaussian fluctuation have been proven [79, 80].

Much more studied are various classes of random matrices with independent but not identically distributed entries. The most prominent example is the tight binding Anderson model [6], i.e., a Schrödinger operator,  $-\Delta + \lambda V$ , on a regular square lattice with a random on-site potential  $V$  and disorder strength  $\lambda$ . This model describes electron propagation (conductance) in an ionic lattice with a disordered environment. Restricted to a finite box, it can be represented by a matrix whose diagonal elements are i.i.d. random variables; the deterministic off-diagonal elements are given by the Laplacian.

The general formulation of the *universality conjecture for random Schrödinger operators* states that there are two distinctive regimes depending on the energy and the disorder strength. In the strong disorder regime, the eigenfunctions are localized and the local spectral statistics is Poisson. In the weak disorder regime, the eigenfunctions are delocalized and the local statistics coincides with that of a Gaussian matrix ensemble. Separate conjectures, that will not be discussed here, relate these two regimes to chaotic vs. integrable behavior of the underlying classical dynamical system. According to the Berry-Tabor conjecture [10], Poisson statistics of eigenvalues should emerge from quantizations of integrable classical dynamics, while random matrix theory stems from quantization of chaotic classical dynamics (Bohigas, Giannoni, Schmit [13])

Returning to the more concrete Anderson model, in space dimensions three or higher and for weak randomness, the model is conjectured to exhibit metal-insulator transition, i.e., in  $d \geq 3$  dimensions the eigenfunctions of  $-\Delta + \lambda V$  are delocalized for small  $\lambda$ , while they are localized for large  $\lambda$ . It is a fundamental open question to establish this transition.

The localization regime at large disorder or near the spectral edges has been well understood by Fröhlich and Spencer with the multiscale technique [53, 54], and later by Aizenman and Molchanov by the fractional moment method [1]; many other works have since contributed to this field. In particular, it has been established that the local eigenvalue statistics are Poisson [71] and that the eigenfunctions are exponentially localized with an upper bound on the localization length that diverges as the energy parameter approaches the presumed phase transition point [87, 33].

The progress in the delocalization regime has been much slower. For the Bethe lattice, corresponding to the infinite-dimensional case, delocalization has been established in [64, 2, 51] (in an apparent controversy to the general conjectures, the eigenvalue statistics, however, are Poisson but for a well understood specific reason [3]). In finite dimensions only partial results are available. The existence of an absolutely continuous spectrum (i.e., extended states) has been shown for a rapidly decaying potential, corresponding to a scattering regime [76, 14, 23]. Diffusion has been established for a heavy quantum particle immersed in a phonon field in  $d \geq 4$  dimensions [52]. For the original Anderson Hamiltonian with a small coupling constant  $\lambda$ , the eigenfunctions have a localization length of at least  $\lambda^{-2}$  (see [17]). The time and space scale  $\lambda^{-2}$  corresponds to the kinetic regime where the quantum evolution can be modelled by a linear Boltzmann equation [89, 45]. Beyond this time scale the dynamics is diffusive. This has been established in the scaling limit  $\lambda \rightarrow 0$  up to time scales  $t \sim \lambda^{-2-\kappa}$  with an explicit  $\kappa > 0$  in [37, 38, 39].

There are no rigorous results on the local spectral statistics of the Anderson model in the disordered regime, but it is conjectured – and supported by numerous arguments in the physics literature, especially by supersymmetric methods (see [32]) – that the local correlation functions of the eigenvalues of the finite volume Anderson model follow the GOE statistics in the thermodynamic limit. GUE statistics is expected if an additional magnetic field breaks the time-reversal symmetry of the Anderson Hamiltonian. Based upon this conjecture, the local eigenvalue statistics are used to compute the phase diagram numerically. It is very remarkable that the random Schrödinger operator, represented by a very sparse random matrix, exhibits the same universality class as the

full Wigner matrix, at least in a certain energy range.

Due to their mean-field character, Wigner matrices are simpler to study than the Anderson model and they are always in the delocalization regime. In this survey we mainly focus on Wigner matrices, but we keep in mind the original motivation from general disordered systems. In particular, we will study not only eigenvalue statistics but also eigenvectors that are shown to be completely delocalized [40]. The local spectral statistics in the bulk is universal, i.e., it follows the statistics of the corresponding Gaussian ensemble (GOE, GUE, GSE), depending on the symmetry type of the matrix. This topic will be the main goal of this presentation.

To close this section, we mention some other possible research directions that we will not pursue here further. The list is incomplete.

A natural intermediate class of ensembles between the fully stochastic Wigner matrices and the Anderson model with diagonal randomness is the family of *random band matrices*. These are hermitian or symmetric random matrices  $H$  with independent but not identically distributed entries. The variance of  $h_{ij}$  depends only on  $|i-j|$  and it becomes negligible if  $|i-j|$  exceeds a given parameter  $W$ , the band-width; for example,  $\sigma_{ij}^2 = \mathbb{E}|h_{ij}|^2 \sim \exp(-|i-j|/W)$ . It is conjectured that the system is completely delocalized if  $W \gg \sqrt{N}$ , otherwise the localization length is  $W^2$ . Moreover, for narrow bands,  $W \ll \sqrt{N}$ , the local eigenvalue statistics are expected to be Poisson, while for broad bands,  $W \gg \sqrt{N}$  it should be given by GUE or GOE, depending on the symmetry class. Localization properties of  $H$  for  $W \ll N^{1/8}$  have been recently shown [78] but not local statistics. From the delocalization side, with A. Knowles we recently proved [34] diffusion up to time scale  $t \ll W^{1/3}$  which implies that the localization length is at least  $W^{1+1/6}$ .

We mention that the universality of local eigenvalue statistics is often investigated by supersymmetric techniques in the physics literature. These methods are extremely powerful to extract the results by saddle point computations, but the analysis justifying the saddle point approximation still lacks mathematical rigor. So far only the density of states has been investigated rigorously by using this technique [24].

Finally, we point out that we focused on the physical motivations coming from disordered quantum systems, but random matrices appear in many other branches in physics and mathematics. It is a fundamental object with extremely rich structure. The most remarkable connection is with the  $\zeta$ -function. It is conjectured that the zeros of the Riemann  $\zeta$ -function,  $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ , lying on the vertical line  $\Re s = \frac{1}{2}$ , have the same local statistics as the GUE (after appropriate rescaling). A review and further references to many numerical evidences is found in the classical book of Mehta [68].

## 1.4 Eigenvalue density and delocalization

For symmetric or hermitian matrix  $H$ , let  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$  denote the eigenvalues. They form a random point process on the real line with a distribution generated from the joint probability

law of the matrix elements. Since the functional relation between matrix elements and eigenvalues is highly nontrivial, the product measure on the entries turns out to generate a complicated and highly correlated measure for the eigenvalues. Our main goal is to understand this induced measure.

Under the chosen normalization (1.17), the typical size of the eigenvalues is of order one. We will prove a much more precise statement later, but it is instructive to have a rough feeling about the size via computing  $\text{Tr } H^2$  in two ways:

$$\sum_{i=1}^N \lambda_i^2 = \text{Tr } H^2 = \sum_{i,j=1}^N |h_{ij}|^2.$$

Taking expectation and using (1.17) we have

$$\frac{1}{N} \sum_i \mathbb{E} \lambda_i^2 = \frac{1}{N} \sum_{ij} \sigma_{ij}^2 = 1$$

i.e., in an average sense  $\mathbb{E} \lambda_i^2 = 1$ .

#### 1.4.1 Wigner Semicircle law and other canonical densities

The empirical distribution of eigenvalues follows a universal pattern, the **Wigner semicircle law**. To formulate it more precisely, note that the typical spacing between neighboring eigenvalues is of order  $1/N$ , so in a fixed interval  $[a, b] \subset \mathbb{R}$ , one expects macroscopically many (of order  $N$ ) eigenvalues. More precisely, it can be shown (first proof was given by Wigner, [100]) that for any fixed  $a \leq b$  real numbers,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{i : \lambda_i \in [a, b]\} = \int_a^b \varrho_{sc}(x) dx, \quad \varrho_{sc}(x) := \frac{1}{2\pi} \sqrt{(4-x^2)_+}, \quad (1.20)$$

where  $(a)_+ := \max\{a, 0\}$  denotes the positive part of the number  $a$ . Note the emergence of the universal density, the semicircle law, that is independent of the details of the distribution of the matrix elements.

The semicircle law is characteristic for the universal Wigner matrices (see Definition 1.1). For random square matrices with independent entries but *without symmetry* (i.e.,  $h_{ij}$  are independent for all  $i, j$ ) a similar universal pattern emerges, the **circular law**. For example, if  $h_{ij}$  are centered i.i.d. random variables with common variance  $\sigma_{ij}^2 = N^{-1}$ , then the empirical density of eigenvalues converges to the uniform measure on the unit disk in the complex plane [94]. If independence is dropped, one can get many different density profiles.

For example, in case of the random covariance matrices (1.19), the empirical density of eigenvalues  $\lambda_i$  of  $H$  converges to the *Marchenko-Pastur law* [67] in the limit when  $M, N \rightarrow \infty$  such that  $d = N/M$  is fixed  $0 \leq d \leq 1$ :

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{i : \lambda_i \in [a, b]\} = \int_a^b \varrho_{MP}(x) dx, \quad \varrho_{MP}(x) := \frac{1}{2\pi d} \sqrt{\frac{[(\lambda_+ - x)(x - \lambda_-)]_+}{x^2}} \quad (1.21)$$



with  $\lambda_{\pm} := (1 \pm \sqrt{d})^2$  being the spectral edges. Note that in case  $M \leq N$ , the matrix  $H$  has macroscopically many zero eigenvalues, otherwise the spectra of  $XX^*$  and  $X^*X$  coincide so the Marchenko-Pastur law can be applied to all nonzero eigenvalues with the role of  $M$  and  $N$  exchanged.

### 1.4.2 The moment method

The eigenvalue density is commonly approached via the fairly robust *moment method* (see [4] for an exposé) that was also the original approach of Wigner to prove the semicircle law [100]. For example, for hermitian Wigner matrices, it consists of computing  $\mathbb{E}\text{Tr} H^{2k}$  for large  $k$  powers by expanding it as

$$\mathbb{E} \sum_{i_1, i_2, \dots, i_{2k}} h_{i_1 i_2} h_{i_2 i_3} \dots h_{i_{2k} i_1}$$

and noticing that each factor  $h_{xy}$  must be at paired with at least another copy  $h_{yx} = \bar{h}_{xy}$ , otherwise the expectation value is zero. The possible index sequences that satisfy this pairing conditions can be classified according to their complexity, and it turns out that the main contribution comes from the so-called *backtracking paths*. These are index sequences  $i_1 i_2 i_3 \dots i_{2k} i_1$ , returning to the original index  $i_1$ , that can be successively generated by a substitution rule

$$a \rightarrow aba, \quad b \in \{1, 2, \dots, N\}, \quad b \neq a,$$

with an arbitrary index  $b$ . These index sequences satisfy the pairing condition in an obvious manner and it turns out that they involve the largest possible number ( $N^k$ ) independent indices. The number of backtracking paths is explicitly given by the Catalan numbers,  $C_k = \frac{1}{k+1} \binom{2k}{k}$ , so  $\mathbb{E}\text{Tr} H^{2k}$  can be computed fairly precisely for each finite  $k$ :

$$\frac{1}{N} \mathbb{E}\text{Tr} H^{2k} = \frac{1}{k+1} \binom{2k}{k} + O_k(N^{-2}). \quad (1.22)$$

Note that the number of independent labels,  $N^k$ , exactly cancels the size of the  $k$ -fold product of variances,  $(\mathbb{E}|h|^2)^k = N^{-k}$ . If the distribution of the matrix elements is symmetric, then the traces of odd powers all vanish since they can never satisfy the pairing condition. Without the symmetry condition the traces of odd powers are non-zero but negligible.

We will compute the trace of the resolvent, or the *Stieltjes transform* of the empirical density

$$\varrho_N(dx) = \frac{1}{N} \sum_1^N \delta(x - \lambda_j)$$

of the eigenvalues. For any  $z = E + i\eta$ ,  $E \in \mathbb{R}$ ,  $\eta > 0$ , we define

$$m_N(z) := \frac{1}{N} \text{Tr} \frac{1}{H - z} = \frac{1}{N} \sum_{j=1}^N \frac{1}{\lambda_j - z} = \int_{\mathbb{R}} \frac{d\varrho_N(x)}{x - z}. \quad (1.23)$$

For large  $z$  one can expand it as follows

$$m_N(z) = \frac{1}{N} \text{Tr} \frac{1}{H - z} = -\frac{1}{Nz} \sum_{m=0}^{\infty} \left(\frac{H}{z}\right)^m, \quad (1.24)$$

so after taking the expectation, using (1.22) and neglecting the error terms, we get

$$\mathbb{E}m_N(z) \approx -\sum_{m=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} z^{-(2k+1)}. \quad (1.25)$$

which, after some calculus, can be identified as the power series of  $\frac{1}{2}(-z + \sqrt{z^2 - 4})$ . The approximation becomes exact in the  $N \rightarrow \infty$  limit. Although the expansion (1.24) is valid only for large  $z$ , given that the limit is an analytic function of  $z$ , one can extend the relation

$$\lim_{N \rightarrow \infty} \mathbb{E}m_N(z) = \frac{1}{2}(-z + \sqrt{z^2 - 4})$$

by analytic continuation to the whole upper half plane  $z = E + i\eta$ ,  $\eta > 0$ . It is an easy exercise to see that this is exactly the Stieltjes transform of the semicircle density, i.e.,

$$m_{sc}(z) := \frac{1}{2}(-z + \sqrt{z^2 - 4}) = \int_{\mathbb{R}} \frac{\varrho_{sc}(x) dx}{x - z} \quad (1.26)$$

(the branch of the square root is chosen such that  $\Im m_{sc}(z) > 0$  for  $\Im z > 0$ ). Since Stieltjes transforms identify the measure uniquely, and pointwise convergence of Stieltjes transforms implies weak convergence of measures, we obtain that

$$\mathbb{E}d\varrho_N(x) \rightharpoonup \varrho_{sc}(x) dx. \quad (1.27)$$

With slightly more efforts one can show that the convergence holds also in probability not only in expectation,

$$\lim_{N \rightarrow \infty} m_N(z) = \frac{1}{2}(-z + \sqrt{z^2 - 4}) \quad (1.28)$$

holds with high probability. For more details, see [4].

### 1.4.3 The local semicircle law

Moment method can typically identify the resolvent for any fixed  $z$  and thus give the semicircle law as a weak limit, i.e., (1.20) will hold *for any fixed* interval  $I := [a, b]$  as  $N \rightarrow \infty$ . However, a fixed interval  $I$  typically contains of order  $N|I|$  eigenvalues. It is natural to ask if the semicircle law holds *locally* as well, i.e., for intervals whose length may shrink with  $N$ , but still  $N|I| \gg 1$ .

Eventually, the semicircle law is a type of law of large numbers that should require only that the number of random objects in consideration goes to infinity. Due to the formula

$$\Im m_N(z) = \frac{1}{N} \sum_{i=1}^N \frac{\eta}{(\lambda_i - E)^2 + \eta^2} \sim \frac{\pi}{N} \sum_{i=1}^N \delta_\eta(\lambda_i - E) \quad z = E + i\eta,$$

where  $\delta_\eta$  denotes an approximate delta function on scale  $\eta$ , we see that knowing the Stieltjes transform for some  $z \in \mathbb{C}$  with  $\Im z = \eta$  is essentially equivalent to knowing the local density on scale  $\eta$ , i.e., in an interval of length  $|I| \sim \eta$ .

In [37, 38, 39] we proved that the *local* semicircle law holds on the smallest possible scale of  $\eta \gg 1/N$ , i.e., the limit (1.20) holds even if the length of  $I = [a, b]$  is essentially of order  $1/N$ , hence it typically contains only large but finite number of eigenvalues. This will be the key technical input for further investigations on local spectral statistics. There are several versions of the local semicircle law; we will give two precise statements: Theorem 1.8 (from [39]) and Theorem 2.6 (from [47]).

The method of the proof is different from the moment method, but we still work with the resolvent, or Stieltjes transform. The key observation (see also several previous works, e.g. [7, 67]) is that the Stieltjes transform  $m_{sc}(z)$  of the semicircle density  $\varrho_{sc}$  (1.26) satisfies the following simple quadratic equation:

$$m_{sc}(z) + \frac{1}{z + m_{sc}(z)} = 0, \quad (1.29)$$

and among the two possible solutions,  $m_{sc}(z)$  is identified by the requirement that  $\Im m_{sc}(z) > 0$  for  $\Im z > 0$ . The strategy is that expanding the empirical Stieltjes transform  $m_N(z)$  (1.23) according to minors of  $H$ , we prove that  $m_N$  satisfies the self-consistent equation (1.29) approximately and with a high probability:

$$m_N(z) + \frac{1}{z + m_N(z)} \approx 0.$$

Then we conclude the proof of  $m_N \approx m_{sc}$  by invoking the stability of the equation (1.29). Since the stability deteriorates near the edges,  $E = \Re z \approx \pm 2$ , the estimate will be weaker there, indicating that the eigenvalue fluctuation is larger at the edge.

Our best results in this direction are obtained in [47] (which is partly a streamlined version of [46]), where not only the trace of the Green function (1.23) but also individual diagonal elements were shown to be given by the semicircle law. The results were already listed informally in Section 1.1.1 and we pointed out that they hold also for universal Wigner matrices, see Theorem 2.6.

For these ensembles Guionnet [56] and Anderson-Zeitouni [5] already proved that the density of the eigenvalues converges to the Wigner semi-circle law on a large scale, our result improves this to small scales. For example, for band matrices (for Definition see (1.18)) with band width  $W$  we obtain that the semicircle law holds down to energy scales  $1/W$ . The delocalization length is shown to be at least as large as the band width  $W$ . We note that a certain three dimensional version of Gaussian band matrices was also considered by Disertori, Pinson and Spencer [24] using the

supersymmetric method. They proved that the expectation of the density of eigenvalues is smooth and it coincides with the Wigner semicircle law up to a precision of the spread of the matrix.

#### 1.4.4 Density for invariant ensembles

There is another natural way to define a probability distributions on symmetric or hermitian matrices apart from directly imposing a given probability law  $\nu$  on its entries. They are obtained by defining a density function directly on the set of matrices, e.g. of the form

$$\mathcal{P}(H)dH := Z^{-1} \exp(-N\text{Tr} V(H))dH. \quad (1.30)$$

Here  $dH = \prod_{i \leq j} dH_{ij}$  is the flat Lebesgue measure (in case of hermitian matrices and  $i < j$ ,  $dH_{ij}$  is the Lebesgue measure on the complex plane  $\mathbb{C}$ ). The function  $V : \mathbb{R} \rightarrow \mathbb{R}$  is assumed to grow mildly at infinity (some logarithmic growth would suffice) to ensure that the measure defined in (1.30) is finite, and  $Z$  is the normalization factor. Probability distributions of the form (1.30) are called **invariant ensembles** since they are invariant under the orthogonal or unitary conjugation (in case of symmetric or hermitian matrices, respectively). E.g. in the hermitian case, for any fixed unitary matrix  $U$ , the transformation

$$H \rightarrow U^* H U$$

leaves the distribution (1.30) invariant thanks to  $\text{Tr} V(U^* H U) = \text{Tr} V(H)$ .

Wigner matrices and invariant ensembles form two different universes with quite different mathematical tools available for their studies. In fact, these two classes are almost disjoint, the Gaussian ensembles being the only invariant Wigner matrices. The following lemma is a relatively easy exercise (see [20] or Theorem 2.6.3 [68]).

**Lemma 1.1** *Suppose that the symmetric or hermitian matrix ensemble given in (1.30) have independent entries  $h_{ij}$ ,  $i \leq j$ . Then  $V(x)$  is a quadratic polynomial,  $V(x) = ax^2 + bx + c$  with  $a > 0$ . This means that apart from a trivial shift and normalization, the ensemble is GOE or GUE.*

The density of eigenvalues of the invariant ensemble (1.30) is determined by a variational problem [20]. It is given by the equilibrium density of a gas with a logarithmic self-interaction and external potential  $V$ , i.e., as the solution of

$$\inf_{\varrho} \left\{ \int_{\mathbb{R}} \int_{\mathbb{R}} \log |s - t|^{-1} \varrho(ds) \varrho(dt) + \int V(t) \varrho(dt) \right\}$$

where the infimum is over all probability measures  $\varrho$ . Under some mild conditions on  $V$ , the equilibrium measure is absolutely continuous,  $\varrho_{eq}(dt) = \varrho_{eq}(t)dt$  and it has compact support. If  $V$  is a polynomial, then the support consist of finitely many intervals. The empirical density of eigenvalues converges to  $\varrho_{eq}$  in the sense of (1.20) where  $\varrho_{sc}$  is replaced with the function  $\varrho_{eq}$ . It is an amusing calculus exercise to check that the solution of this variational problem for the Gaussian case,  $V(x) = x^2/2$ , is indeed  $\varrho_{sc}$ .

### 1.4.5 Eigenvector delocalization

Apart from the statistics of the eigenvalues, one may also study the eigenvectors of a random matrix. In light of the universality conjecture about disordered systems explained in Section 1.3, it is a challenging question to test this hypothesis on the level of eigenvectors as well. Wigner matrices are mean-field models and from the physics intuition they are always in the delocalized regime. Of course they are still finite, so they cannot have absolutely continuous spectrum, a standard signature for delocalization that people working in random Schrödinger operators are often looking for. But the delocalization of eigenvectors is a perfectly meaningful question for large but finite matrices as well. Surprisingly, this question was largely neglected both by the random matrix community and the random Schrödinger operator community within mathematics until Tom Spencer has raised it recently. He pointed out in a lecture that in the case of the Gaussian ensembles, a simple invariance argument proves that the eigenvectors  $\mathbf{v}$  are fully delocalized in the sense that their  $\ell^4$ -norm is  $\|\mathbf{v}\|_4 \sim N^{-1/4}$  (assuming  $\|\mathbf{v}\|_2 = 1$ ). This is a signature of strong delocalization, since, on the one hand, by Schwarz inequality

$$N^{-1/2}\|\mathbf{v}\|_2 = \left(\frac{1}{N} \sum_{i=1}^N |v_i|^2\right)^{1/2} \leq \left(\frac{1}{N} \sum_{i=1}^N |v_i|^4\right)^{1/4} = N^{-1/4}\|\mathbf{v}\|_4,$$

i.e.,  $\|\mathbf{v}\|_4 \geq N^{-1/4}$  always holds, on the other hand this inequality is essentially saturated if all coordinates of the eigenvector are of approximately the same size,  $|v_i| \sim N^{-1/2}$ .

The simple invariance argument works only for the Gaussian case, where the unitary invariance is present, but it is a natural question to ask whether eigenvectors of Wigner ensembles are also delocalized and the answer is affirmative. We have proved [39, Corollary 3.2] that if  $\mathbf{v}$  is an  $\ell^2$ -normalized eigenvector of a Wigner matrix  $H$  with eigenvalue  $\lambda$  away from the edge

$$H\mathbf{v} = \lambda\mathbf{v}, \quad \lambda \in [-2 + \kappa, 2 - \kappa]$$

for some  $\kappa > 0$ , then the  $\ell^p$  norm of  $\mathbf{v}$ , for any  $2 < p < \infty$  is bounded by

$$\|\mathbf{v}\|_p \leq QN^{-\frac{1}{2} + \frac{1}{p}} \tag{1.31}$$

with a very high probability, the exceptional set being subexponentially small in  $Q$  for large  $Q$ . A similar bound with a logarithmic correction holds for  $p = \infty$  as well. The precise statement will be given in Theorems 2.20 and 2.21. It is essentially a straightforward corollary of the local semicircle law, Theorem 2.6, that was informally outlined in Section 1.4.3.

Note that if  $\mathbf{v}$  is an  $\ell^2$ -normalized eigenvector, then the size of the  $\ell^p$ -norm of  $\mathbf{v}$ , for  $p > 2$ , gives information about delocalization. Complete delocalization occurs when  $\|\mathbf{v}\|_p \lesssim N^{-1/2+1/p}$  since this corresponds to the  $\ell^p$ -norm of the fully delocalized vector  $\mathbf{v} = (N^{-1/2}, N^{-1/2}, \dots, N^{-1/2})$ . In contrast, a fully localized eigenvector,  $\mathbf{v} = (0, 0, \dots, 0, 1, 0, \dots, 0)$  has  $\ell^p$  norm one.

## 1.5 Local statistics of eigenvalues: previous results.

A central question concerning random matrices is the universality conjecture which states that *local statistics* of eigenvalues of large  $N \times N$  square matrices  $H$  are determined by the symmetry type of the ensembles but are otherwise independent of the details of the distributions. It turns out that local statistics exhibit even stronger universality features than the eigenvalue densities.

Local statistics refers to observables that can distinguish among individual eigenvalues. For all ensembles we presented so far, we used a normalization of the matrix such that the typical eigenvalues remain in a compact set as  $N \rightarrow \infty$ , in other words, the limiting density function  $\varrho$  was compactly supported. In this case, the typical spacing between neighboring eigenvalues is of order  $N^{-1}$ . This holds *in the bulk of the spectrum*, i.e., at a positive distance away the *spectral edges*. The spectral edges are characterized by the points where  $\varrho$  goes to zero. For example, for the Wigner semicircle distribution,  $\varrho_{sc}$ , they are at  $\pm 2$ , for the Marchenko-Pastur distribution (1.21) they are at  $\lambda_{\pm}$ , and for certain invariant ensembles the support of the eigenvalue density might consist of several intervals i.e., it can have more than two spectral edges.

### 1.5.1 Bulk universality: the sine kernel and gap distribution

To see individual eigenvalues and their joint distribution in the bulk spectrum, one needs to “zoom” the point process of the eigenvalues by magnifying it by a factor of  $N$ . We fix two real numbers,  $\alpha_1, \alpha_2$  and an energy  $E$  with  $\varrho(E) > 0$ , and we ask the probability that there is an eigenvalue at  $E + \alpha_1/[N\varrho(E)]$  and simultaneously there is an eigenvalue at  $E + \alpha_2/[N\varrho(E)]$  (the normalization is chosen such that the typical number of eigenvalues between these two points is independent of  $E$ ). It turns out that the answer is independent of the details of the ensemble and of the energy  $E$ , it depends only on the symmetry type. For example, for the hermitian case, it is given by

$$\mathbb{P}\left\{\text{there are eigenvalues } \lambda \in E + \frac{\alpha_1 + d\alpha_2}{N\varrho(E)} \text{ and } \lambda' \in E + \frac{\alpha_2 + d\alpha_1}{N\varrho(E)}\right\} \approx \left[1 - \left(\frac{\sin \pi(\alpha_1 - \alpha_2)}{\pi(\alpha_1 - \alpha_2)}\right)^2\right] d\alpha_1 d\alpha_2. \quad (1.32)$$

The function on the r.h.s. is obtained from the celebrated **sine kernel** and it should be viewed as a two by two determinant of the form

$$\det \left( K(\alpha_i - \alpha_j) \right)_{i,j=1}^2, \quad K(x) = \frac{\sin \pi x}{\pi x}. \quad (1.33)$$

The explicit formula for the  $K$  kernel in the symmetric case is more complicated (see [20]), but it is universal and the correlation function has the same *determinantal structure*.

Note that (1.32) contains a much more delicate information about the eigenvalues than the semicircle law (1.20). First, it is a local information after a magnification to a scale where individual eigenvalues matter. Second, it expresses a correlation among two eigenvalues. For example, due to  $\frac{\sin y}{y} \rightarrow 1$  as  $y \rightarrow 0$ , we see that the eigenvalues repel each other.

In general, the  $k$ -th correlation functions (or  $k$ -point marginals) give information about the joint behavior of a  $k$ -tuple of eigenvalues. Their definition is as follows:

**Definition 1.2** Let  $p_N(\lambda_1, \lambda_2, \dots, \lambda_N)$  be the joint symmetrized probability distribution of the eigenvalues. For any  $k \geq 1$ , the  $k$ -point correlation function is defined by

$$p_N^{(k)}(\lambda_1, \lambda_2, \dots, \lambda_k) := \int_{\mathbb{R}^{N-k}} p_N(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \dots, \lambda_N) d\lambda_{k+1} \dots d\lambda_N.$$

*Remark.* We usually label the eigenvalues in increasing order. For the purpose of this definition, however, we dropped this restriction and we consider  $p_N(\lambda_1, \lambda_2, \dots, \lambda_N)$  to be a symmetric function of  $N$  variables,  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$  on  $\mathbb{R}^N$ . Alternatively, one could consider the density  $\tilde{p}_N(\boldsymbol{\lambda}) = N! p_N(\boldsymbol{\lambda}) \cdot \mathbf{1}(\boldsymbol{\lambda} \in \Xi^{(N)})$  where

$$\Xi^{(N)} := \{\lambda_1 < \lambda_2 < \dots < \lambda_N\} \subset \mathbb{R}^N.$$

The significance of the  $k$ -point correlation functions is that they give the expectation value of symmetrized observables (functions)  $O$  depending on  $k$  eigenvalues via the formula

$$\frac{(N-k)!}{N!} \mathbb{E} \sum_{i_1, i_2, \dots, i_k=1}^N O(\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_k}) = \int_{\mathbb{R}^k} O(x_1, x_2, \dots, x_k) p_N^{(k)}(x_1, x_2, \dots, x_k) dx_1 \dots dx_k,$$

where the summation is over all distinct indices  $i_1, i_2, \dots, i_k$  and the prefactor is just a normalization of the sum.

For example, the one-point function  $p_N^{(1)}$  expresses the density, in particular, by choosing the observable  $O(x) = \mathbf{1}(x \in [a, b])$  to be the characteristic function of  $[a, b]$ , we have

$$\frac{1}{N} \#\{i : \lambda_i \in [a, b]\} = \frac{1}{N} \sum_{i=1}^N O(\lambda_i) = \int O(x) p_N^{(1)}(x) dx = \int_a^b p_N^{(1)}(x) dx.$$

Therefore, the Wigner semicircle law (1.20) states that  $p_N^{(1)}$  converges weakly to  $\varrho_{sc}$  as  $N \rightarrow \infty$ .

The sine kernel universality in the hermitian case expresses that the (weak) limit of the rescaled  $k$ -point correlation function, as  $N \rightarrow \infty$ , is given by the determinant of  $K(x)$  from (1.33), i.e.,

$$\frac{1}{[\varrho(E)]^k} p_N^{(k)}\left(E + \frac{\alpha_1}{N\varrho(E)}, E + \frac{\alpha_2}{N\varrho(E)}, \dots, E + \frac{\alpha_k}{N\varrho(E)}\right) \rightarrow \det(K(\alpha_i - \alpha_j))_{i,j=1}^k \quad (1.34)$$

for any fixed  $E$ , as a weak convergence of functions in the variables  $(\alpha_1, \dots, \alpha_k)$ .

Once the  $k$ -point correlation functions are identified, it is easy to derive limit theorems for other quantities related to individual eigenvalues. The most interesting one is the **gap distribution**, i.e., the distribution of the difference of subsequent eigenvalues,  $\lambda_{j+1} - \lambda_j$ . Note that it apparently involves only two eigenvalues, but it is not expressible solely by the two point correlation function,

since the two eigenvalues must be consecutive. Nevertheless, the gap distribution can be expressed in terms of all correlation functions as follows.

Fix an energy  $E$  with  $|E| < 2$  and for any  $s > 0$  and for some  $N$ -dependent parameter  $t$  with  $1/N \ll t \ll 1$  let

$$\Lambda(s) = \Lambda_N(s) := \frac{1}{2Nt\varrho_{sc}(E)} \#\left\{1 \leq j \leq N-1 : \lambda_{j+1} - \lambda_j \leq \frac{s}{N\varrho_{sc}(E)}, |\lambda_j - E| \leq t\right\},$$

i.e., the proportion of rescaled eigenvalue differences below a threshold  $s$  in a large but still microscopic vicinity of an energy  $E$ . Let  $\mathcal{K}_\alpha$  be the operator acting on  $L^2((0, \alpha))$  with integral kernel  $K(x, y) := \frac{\sin \pi(x-y)}{\pi(x-y)}$ . Then for any  $E$  with  $|E| < 2$  and for any  $s > 0$  we have

$$\lim_{N \rightarrow \infty} \mathbb{E} \Lambda_N(s) = \int_0^s p(\alpha) d\alpha, \quad p(\alpha) = \frac{d^2}{d\alpha^2} \det(1 - \mathcal{K}_\alpha), \quad (1.35)$$

where  $\det$  denotes the Fredholm determinant of the operator  $1 - \mathcal{K}_\alpha$  (note that  $\mathcal{K}_\alpha$  is a compact operator). The density function  $p(s)$  of the nearest neighbor eigenvalue spacing behaves, with a very good but not exact approximation (called the *Wigner surmise*), as  $p(s) \approx \frac{\pi s}{2} e^{-\pi s^2/4}$  for the symmetric case and  $p(s) \approx 32\pi^{-2}s^2 e^{-4s^2/\pi}$  for the hermitian case [68].

Note that this behavior is in sharp contrast to the level spacing statistics of the *Poisson point process*, where the corresponding density is  $p(s) = e^{-s}$  (after rescaling the process so that the mean distance is one). In particular, random matrices exhibit **level repulsion** whose strength depends on the symmetry class (note the different behavior of  $p(s)$  near  $s \approx 0$ ).

For the proof of (1.35), we can use the exclusion-inclusion formula to express

$$\begin{aligned} \mathbb{E} \Lambda(s) &= \frac{1}{2Nt\varrho} \sum_{m=2}^{\infty} (-1)^m \binom{N}{m} \int_{-t}^t dv_1 \dots \int_{-t}^t dv_m \mathbf{1}\left\{\max |v_i - v_j| \leq \frac{s}{N\varrho}\right\} \\ &\quad \times p_N^{(m)}(E + v_1, E + v_2, \dots, E + v_m), \end{aligned} \quad (1.36)$$

where  $\varrho = \varrho_{sc}(E)$ . After a change of variables,

$$\begin{aligned} \mathbb{E} \Lambda(s) &= \frac{1}{2Nt\varrho} \sum_{m=2}^{\infty} (-1)^m \int_{-N\varrho t}^{N\varrho t} dz_1 \dots \int_{-N\varrho t}^{N\varrho t} dz_m \\ &\quad \times \binom{N}{m} \frac{1}{(N\varrho)^m} p_N^{(m)}\left(u + \frac{z_1}{N\varrho}, \dots, u + \frac{z_m}{N\varrho}\right) \mathbf{1}\left\{\max |z_i - z_j| \leq s\right\} \\ &= \frac{1}{2Nt\varrho} \sum_{m=2}^{\infty} (-1)^m m \int_{-N\varrho t}^{N\varrho t} dz_1 \int_0^s da_2 \dots \int_0^s da_m \\ &\quad \times \binom{N}{m} \frac{1}{(N\varrho)^m} p_N^{(m)}\left(u + \frac{z_1}{N\varrho}, u + \frac{z_1 + a_2}{N\varrho}, \dots, u + \frac{z_1 + a_m}{N\varrho}\right), \end{aligned} \quad (1.37)$$



where the factor  $m$  comes from considering the integration sector  $z_1 \leq z_j$ ,  $j \geq 2$ . Taking  $N \rightarrow \infty$  and using (1.34), we get

$$\lim_{N \rightarrow \infty} \mathbb{E} \Lambda(s) = \sum_{m=2}^{\infty} \frac{(-1)^m}{(m-1)!} \int_0^s da_2 \cdots \int_0^s da_m \det \left( \frac{\sin \pi(a_i - a_j)}{\pi(a_i - a_j)} \right)_{i,j=1}^m, \quad (1.38)$$

where in the last determinant term we set  $a_1 = 0$ . The interchange of the limit and the summation can be easily justified by an alternating series argument. We note that the left hand side of (1.38) is  $\int_0^s p(\alpha) d\alpha$ , where  $p(\alpha)$  is the second derivative of the Fredholm determinant  $\det(1 - \mathcal{K}_\alpha)$  given in (1.35) (see [83] or [4] for more details). We thus have

$$\lim_{N \rightarrow \infty} \mathbb{E} \Lambda(s) = \int_0^s p(\alpha) d\alpha. \quad (1.39)$$

### 1.5.2 Edge universality: the Airy kernel

Near the spectral edges and under a different scaling another type of universality emerges. It also has a determinantal form, but the kernel is given by the Airy kernel,

$$A(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y}$$

where  $\text{Ai}(x)$  is the Airy function, i.e.,

$$\text{Ai}(x) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{1}{3}t^3 + xt\right) dt$$

which is the solution to the second order ODE,  $y'' - xy = 0$ , with vanishing boundary condition at  $x = \infty$ . The result, that is analogous to (1.34), at the upper spectral edge  $E = 2$  of the hermitian Wigner matrices, is the following weak limit as  $N \rightarrow \infty$

$$p_N^{(k)}\left(2 + \frac{\alpha_1}{N^{2/3}}, 2 + \frac{\alpha_2}{N^{2/3}}, \dots, 2 + \frac{\alpha_k}{N^{2/3}}\right) \rightharpoonup \det(A(\alpha_i, \alpha_j))_{i,j=1}^k. \quad (1.40)$$

Similar statement holds at the lower spectral edge,  $E = -2$ . For Wigner matrices this was first proved by Soshnikov [86] following the work of Sinai and Soshnikov [84] and recently a different proof was given by Tao and Vu [92]. Note the different magnification factor  $N^{2/3}$  that expresses the fact that near the edge the typical eigenvalue spacing is  $N^{-2/3}$ . Intuitively, this spacing is consistent with the semicircle law, since

$$\#\{\lambda_j \geq 2 - \varepsilon\} \approx \frac{N}{2\pi} \int_{2-\varepsilon}^2 \sqrt{4 - x^2} dx = \frac{2}{3\pi} \varepsilon^{3/2} N,$$

so we expect finitely many eigenvalues at a distance  $\varepsilon \sim N^{-2/3}$  away from the edge. Note however, that this argument is not rigorous, since the semicircle law (1.20) requires the test interval  $[a, b]$  to

be fixed, independent of  $N$ . Even our local semicircle result does not prove this result directly, since the estimates (1.3)–(1.7) deteriorate near the edge.

The largest eigenvalue  $\lambda_N$  may extend above 2, but not more than by  $O(N^{-2/3})$ . More precisely, the distribution function of the largest eigenvalue is given by another universal function, the *Tracy-Widom distribution* [95]

$$\lim_{N \rightarrow \infty} \mathbb{P}\left(\lambda_N \leq 2 + \frac{s}{N^{2/3}}\right) = F_{2,1}(s) := \exp\left(-\int_s^\infty (x-s) \cdot q^2(x) dx\right)$$

where  $q(s)$  is the solution to the Painlevé II differential equation  $q''(s) = sq(s) + 2q^3(s)$  with asymptotics  $q(s) \sim \text{Ai}(s)$  at  $s = +\infty$  as a boundary condition. One can prove that

$$F_{2,1}(s) \sim 1 - \frac{1}{16\pi s^{3/2}} e^{-\frac{4}{3}s^{3/2}}$$

as  $s \rightarrow \infty$ , i.e., eigenvalues beyond the  $O(N^{-2/3})$  scale are superexponentially damped. Similar formula holds for symmetric matrices as well. Note that, in particular, this result precisely identifies the limiting distribution of the norm of a large Wigner matrix.

The edge universality is commonly approached via the moment method presented in Section 1.4. The error term in (1.22) deteriorates as  $k$  increases, but with a more careful classification and evaluation of the possible pairing structure, it is possible to determine the moments up to order  $k = O(N^{2/3})$ , see [86]. We just mention the simpler result

$$\frac{1}{N} \mathbb{E} \text{Tr} H^{2k} = \frac{2^{2k}}{\sqrt{\pi k^3}} (1 + o(1)) \quad (1.41)$$

as long as  $k = o(N^{2/3})$ . Such precision is sufficient to identify the upper spectral edge of  $H$  with a precision almost  $N^{-2/3}$  since

$$P(\lambda_N \geq 2 + \varepsilon) \leq \frac{\mathbb{E} \text{Tr} H^{2k}}{(2 + \varepsilon)^{2k}} \leq \frac{CN}{k^{3/2}(1 + \frac{\varepsilon}{2})^{2k}} = o(1)$$

if  $\varepsilon \geq (\log N)/k \gg N^{-2/3} \log N$ . The computation (1.41) can be refined to include powers of order  $k \sim N^{2/3}$  and identify the common distribution of the largest eigenvalues precisely [86]. We remark that the original work of Soshnikov assumed that the single entry distribution is symmetric and all its moments are finite; this condition has been subsequently relaxed [77, 73].

The moment method does not seem to be applicable beyond Soshnikov's scale, i.e., for  $k$  much larger than  $N^{2/3}$ . On the other hand, bulk universality would require to compute moments up to  $k \sim O(N)$  since  $1/k$  is essentially the resolution scale for which knowing the moments of order  $k$  precisely still gives some information. The proof of the bulk universality requires completely new methods.

We mention a useful rule of thumb. There is a strong relation among the control of  $e^{-itH}$ ,  $H^k$  and  $(H - z)^{-1}$ . Modulo some technicalities and logarithmic factors, the following three statements are roughly equivalent for any  $0 < \varepsilon \ll 1$ :

$e^{-itH}$  can be controlled up to times  $|t| \leq \varepsilon^{-1}$

$H^k$  can be controlled up to powers  $k \leq \varepsilon^{-1}$

$(H - z)^{-1}$  can be controlled down to  $\Im z = \eta \geq \varepsilon$ .

These relations follow from the standard identities

$$\frac{1}{H - z} = i \int_0^\infty e^{-it(H-z)} dt, \quad z = E + i\eta, \quad \eta > 0$$

$$e^{-itH} = \sum_{k=0}^{\infty} \frac{(-itH)^k}{k!} = \frac{1}{2\pi i} \int_\gamma \frac{e^{-itz}}{H - z} dz$$

(where the contour  $\gamma$  encircles the spectrum of  $H$ ).

### 1.5.3 Invariant ensembles

For ensembles that remain invariant under the transformations  $H \rightarrow U^* H U$  for any unitary matrix  $U$  (or, in case of symmetric matrices  $H$ , the  $U$  is any orthogonal matrix), the joint probability density function of all the  $N$  eigenvalues can be explicitly computed. These ensembles are typically given by the probability density (1.30). The eigenvalues are strongly correlated and they are distributed according to a Gibbs measure with a long range logarithmic interaction potential (this connection was exploited first in [28]). The joint probability density of the eigenvalues of  $H$  can be computed explicitly:

$$p_N(\lambda_1, \lambda_2, \dots, \lambda_N) = \text{const.} \prod_{i < j} (\lambda_i - \lambda_j)^\beta \prod_{j=1}^N e^{-N \sum_{j=1}^N V(\lambda_j)}, \quad (1.42)$$

where  $\beta = 1$  for symmetric and  $\beta = 2$  for hermitian ensembles. In particular, for the Gaussian case,  $V$  is quadratic and thus the joint distribution of the GOE ( $\beta = 1$ ) and GUE ( $\beta = 2$ ) eigenvalues is given by

$$p_N(\lambda_1, \lambda_2, \dots, \lambda_N) = \text{const.} \prod_{i < j} (\lambda_i - \lambda_j)^\beta \prod_{j=1}^N e^{-\frac{1}{4}\beta N \sum_{j=1}^N \lambda_j^2}. \quad (1.43)$$

It is often useful to think of this measure as a Gibbs measure of the form

$$\mu_N(d\boldsymbol{\lambda}) = p_N(\boldsymbol{\lambda}) d\boldsymbol{\lambda} = \frac{e^{-N\mathcal{H}(\boldsymbol{\lambda})}}{Z}, \quad \mathcal{H}(\boldsymbol{\lambda}) = \sum_{i=1}^N V(\lambda_i) + \frac{\beta}{N} \sum_{i < j} \log |\lambda_j - \lambda_i|^{-1} \quad (1.44)$$

with confining potential  $V(\lambda) = \frac{\beta}{4}\lambda^2$ . The proof of (1.42) is a direct (but involved) calculation; it is based upon a change of variable. We sketch it for the hermitian (unitary invariant) case. The key observation is that invariance of the measure under conjugation implies that the eigenvalues, organized in a diagonal matrix  $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ , are independent of the eigenvectors, organized in a unitary matrix  $V$ . Writing  $H = VDV^*$ , one obtains that  $\mathcal{P}(H)dH$  factorizes as

$$\mathcal{P}(H)dH = e^{-N\text{Tr}V(H)}dH = \left[ p_N(\lambda_1, \lambda_2, \dots, \lambda_N)d\lambda_1 \dots d\lambda_N \right] dV,$$

where  $dV$  denotes the uniform (Haar) measure on the unitary group  $U(N)$  and  $p_N$  is the induced density function on the diagonal matrices (or its entries). Thus the computation of the function  $p_N$  amounts to computing the Jacobian of the change of variables from the matrix elements of  $H$  to the parametrization coordinates in terms of eigenvalues and eigenvectors. The result is

$$dH = (\text{const.}) [\Delta_N(\boldsymbol{\lambda})]^\beta d\boldsymbol{\lambda} dV, \quad \Delta_N(\boldsymbol{\lambda}) := \prod_{i < j} (\lambda_i - \lambda_j), \quad (1.45)$$

where  $\beta = 1$  is the symmetric and  $\beta = 2$  is the hermitian case, see [4] or Section 3.1–3.3 of [68] for details.

Especially remarkable is the emerging Vandermonde determinant in (1.42) which directly comes from integrating out the Haar measure. Note that the symmetry type of the ensemble appears through the exponent  $\beta$ . Only  $\beta = 1, 2$  or  $4$  cases correspond to matrix ensembles of the form (1.30), namely, to the symmetric, hermitian and quaternion self-dual matrices. We will not give the precise definition of the latter (see, e.g. Chapter 7 of [68] or [44]), just mention that this is the natural generalization of symmetric or hermitian matrices to quaternion entries and it has real eigenvalues (similarly as hermitian matrices have real eigenvalues despite their complex entries).

Irrespective of any underlying matrix ensemble, one can nevertheless study the distribution (1.42) for any  $\beta > 0$ ; these are called the *general  $\beta$ -ensembles*. In fact, for the Gaussian case,  $V(\lambda) = \lambda^2/2$ , there are corresponding tridiagonal matrix ensembles for any  $\beta > 0$ , obtained from successive Householder transformations, whose eigenvalue distribution is described by (1.42), see [27] for an overview. Using the tridiagonal structure, methods from the theory of Jacobi matrices can be applied. For example, the universality at the edge eigenvalues is understood in a sense that they are shown to converge to the lowest eigenvalues for a one dimensional Schrödinger operator with a white noise drift and, in particular, the  $\beta$ -analogue of the Tracy-Widom distribution has been identified in [75, 74] following the conjectures of [31]. A different method, the *Brownian carousel representation* [96], has been used to generalize the tail distribution of large eigenvalue gaps (“Wigner surmise”) for Gaussian  $\beta$ -ensembles [97]. More precisely, it has been shown that the probability under the distribution (1.42) with  $V(\lambda) = \lambda^2/2$  that there is no point falling into a fixed interval of length  $s$  (after locally rescaling the spectrum so that the typical distance is  $2\pi$ ) is given by

$$q_s = (\kappa_\beta + o(1)) s^{\gamma_\beta} \exp\left(-\frac{\beta}{64}s^2 + \left(\frac{\beta}{8} - \frac{1}{4}\right)s\right), \quad \gamma_\beta := \frac{1}{4}\left(\frac{\beta}{2} + \frac{2}{\beta} - 3\right), \quad \kappa_\beta > 0,$$

as  $s \rightarrow \infty$  (after  $N \rightarrow \infty$  limit).

The bulk universality, i.e., the analogue of the sine-kernel behavior (1.34) for general  $\beta$ -ensembles is unknown, even for the Gaussian case. The main difficulty is that (1.42) represents an  $N$ -particle system with long range interaction. We can write the joint density as Gibbs measure (1.44), we have  $N$  particles in a confining potential  $V$  that repel each other with potential that has locally a logarithmic repulsion, but also a large (in fact increasing) long range component. Standard methods from statistical physics to construct and analyse Gibbs measures do not seem to apply. Although here we do not attempt to construct the analogue of an infinite volume Gibbs measure, we only want to compute correlation functions, but even this is a daunting task with standard methods unless an extra structure is found.

#### 1.5.4 Universality of classical invariant ensembles via orthogonal polynomials

Much more is known about the classical invariant ensembles, i.e., the  $\beta = 1, 2, 4$  cases, with a general potential  $V$ . For these specific values an extra mathematical structure emerges, namely the orthogonal polynomials with respect to the weight function  $e^{-NV(\lambda)}$  on the real line. This approach was originally applied by Mehta and Gaudin [68, 70] to compute the gap distribution for the Gaussian case that involved classical Hermite orthonormal polynomials. Dyson [30] computed the local correlation functions for a related ensemble (circular ensemble) that was extended to the standard Gaussian ensembles by Mehta [69]. Later a general method using orthogonal polynomials has been developed to tackle a very general class of unitary ensembles (see, e.g. [11, 20, 21, 22, 49, 68, 72] and references therein).

For simplicity, to illustrate the connection, we will consider the hermitian case  $\beta = 2$  with a Gaussian potential  $V(\lambda) = \lambda^2/2$  (which, by Lemma 1.1, is also a Wigner matrix ensemble, namely the GUE). To simplify the presentation further, for the purpose of this argument only, we rescale the eigenvalues  $\lambda \rightarrow \sqrt{N}\lambda$ , which effectively removes the factor  $N$  from the exponent in (1.42). (This pure scaling works only in the Gaussian case, but it is only a technical convenience to simplify formulas.)

Let  $P_k(x)$  be the  $k$ -th orthogonal polynomial with respect to the weight function  $e^{-x^2/2}$  with leading coefficient 1. Let

$$\psi_k(x) := \frac{e^{-x^2/4} P_k(x)}{\|e^{-x^2/4} P_k\|}$$

be the corresponding orthonormal function, i.e.,

$$\int \psi_k(x) \psi_\ell(x) dx = \delta_{k,\ell}. \tag{1.46}$$

In the particular case of the Gaussian weight function,  $P_k$  is given by the Hermite polynomials

$$P_k(x) := (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}, \quad \text{and} \quad \psi_k(x) = \frac{P_k(x)}{(2\pi)^{1/4} (k!)^{1/2}} e^{-x^2/4},$$

but for the following discussion we will not need these explicit formulae.

The key observation is that by simple properties of the Vandermonde determinant, we have

$$\Delta_N(\mathbf{x}) = \prod_{1 \leq i < j \leq N} (x_j - x_i) = \det (x_i^{j-1})_{i,j=1}^N = \det (P_{j-1}(x_i))_{i,j=1}^N, \quad (1.47)$$

exploiting that  $P_j(x) = x^j \pm \dots$  is a polynomial of degree  $j$  with leading coefficient equal one. Define the kernel

$$K_N(x, y) := \sum_{k=0}^{N-1} \psi_k(x) \psi_k(y),$$

i.e., the projection kernel onto the subspace spanned by the first  $K$  orthonormal functions. Then (1.47) immediately implies

$$\begin{aligned} p_N(x_1, \dots, x_N) &= C_N \left[ \det (P_{j-1}(x_i))_{i,j=1}^N \right]^2 \prod_{i=1}^N e^{-x_i^2/2} \\ &= C'_N \left[ \det (\psi_{j-1}(x_i))_{i,j=1}^N \right]^2 = C'_N \det (K_N(x_i, x_j))_{i,j=1}^N, \end{aligned}$$

where in the last step we used that the square of the matrix  $(\psi_{j-1}(x_i))_{i,j=1}^N$  is exactly  $(K_N(x_i, x_j))_{i,j=1}^N$  and we did not follow the precise constants for simplicity.

To compute the correlation functions, we expand the determinant:

$$\begin{aligned} p_N^{(k)}(x_1, \dots, x_k) &= C_{k,N} \int_{\mathbb{R}^{N-k}} \det (K_N(x_i, x_j))_{i,j=1}^N \prod_{i=k+1}^N dx_i \\ &= C_{k,N} \sum_{\sigma, \tau \in S_N} (-1)^{\tau+\sigma} \int_{\mathbb{R}^{N-k}} \prod_{j=1}^N \psi_{\sigma(j)-1}(x_j) \psi_{\tau(j)-1}(x_j) \prod_{i=k+1}^N dx_i \\ &= C_{k,N} \sum_{\alpha_1 < \alpha_2 < \dots < \alpha_k} \sum_{\sigma, \tau \in S_k} (-1)^{\tau+\sigma} \prod_{j=1}^k \psi_{\alpha_{\sigma(j)}-1}(x_j) \psi_{\alpha_{\tau(j)}-1}(x_j) \\ &= C_{k,N} \sum_{\alpha_1 < \alpha_2 < \dots < \alpha_k} \left[ \det (\psi_{\alpha_j-1}(x_j))_{i,j=1}^k \right]^2, \end{aligned}$$

where  $S_N$  is the permutation group on  $N$  elements and  $(-1)^\tau$  is the parity character of the permutation. In the third line we used (1.46) to perform the integrations that have set  $\sigma(j) = \tau(j)$  for all  $j \geq k+1$  and we denoted by  $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$  the ordering of the set  $\{\sigma(1), \sigma(2), \dots, \sigma(k)\} = \{\tau(1), \tau(2), \dots, \tau(k)\}$ . Finally, using that the matrix  $[K_N(x_i, x_j)]_{i,j=1}^k$  can be written as  $A^t A$  with  $A_{ij} = \psi_{i-1}(x_j)$  and using the Cauchy-Binet expansion formula for the determinant of a product matrix, we get

$$\det [K_N(x_i, x_j)]_{i,j=1}^k = \sum_{\alpha_1 < \alpha_2 < \dots < \alpha_k} \left[ \det (\psi_{\alpha_j-1}(x_j))_{i,j=1}^k \right]^2.$$

Apart from the constant, that can be computed, we thus proved that

$$p_N^{(k)}(x_1, \dots, x_k) = \frac{(N-k)!}{N!} \det[K_N(x_i, x_j)]_{i,j=1}^k,$$

i.e., the correlation functions have a determinantal structure.

In order to see the sine-kernel (1.33) emerging, we need a basic algebraic property of the orthogonal polynomials, the Christoffel–Darboux formula:

$$K_N(x, y) = \sum_{j=0}^{N-1} \psi_j(x)\psi_j(y) = \sqrt{N} \left[ \frac{\psi_N(x)\psi_{N-1}(y) - \psi_N(y)\psi_{N-1}(x)}{x-y} \right].$$

Furthermore, orthogonal polynomials of high degree have asymptotic behavior as  $N \rightarrow \infty$

$$\psi_{2m}(x) \approx \frac{(-1)^m}{N^{1/4}\sqrt{\pi}} \cos(\sqrt{N}x) + o(N^{-1/4}), \quad \psi_{2m+1}(x) \approx \frac{(-1)^m}{N^{1/4}\sqrt{\pi}} \sin(\sqrt{N}x) + o(N^{-1/4}), \quad (1.48)$$

for any  $m$  such that  $|2m - N| \leq C$ . The approximation is uniform for  $|x| \leq CN^{-1/2}$ . These formulas will be useful if we set  $E = 0$  in (1.34), since we rescaled the eigenvalues by a factor of  $\sqrt{N}$ , so the relation between the notation of (1.34) (for  $k = 2$ ) and  $x, y$  is

$$x = \sqrt{N} \left( E + \frac{\alpha_1}{N\rho(E)} \right), \quad y = \sqrt{N} \left( E + \frac{\alpha_2}{N\rho(E)} \right). \quad (1.49)$$

For different values of  $E$  one needs somewhat different asymptotic formulae for the orthogonal polynomials. We can thus compute that

$$K_N(x, y) \approx \frac{1}{\pi} \frac{\sin(\sqrt{N}x) \cos(\sqrt{N}y) - \sin(\sqrt{N}y) \cos(\sqrt{N}x)}{x-y} = \frac{\sin \sqrt{N}(x-y)}{\pi(x-y)}. \quad (1.50)$$

Using (1.49) and that  $\rho(0) = \pi^{-1}$ , we have

$$\frac{1}{\rho(0)\sqrt{N}} K_N(x, y) \approx \frac{\sin \pi(\alpha_1 - \alpha_2)}{\pi(\alpha_1 - \alpha_2)}$$

which gives (1.34) for  $E = 0$  after undoing the  $\lambda \rightarrow \sqrt{N}\lambda$  magnification.

The main technical input is the refined asymptotic formulae (1.48) for orthogonal polynomials. In case of the classical orthogonal polynomials (appearing in standard Gaussian Wigner and Wishart ensembles) they are usually obtained by a Laplace asymptotics from their integral representation. For a general potential  $V$  the corresponding analysis is quite involved and depends on the regularity properties of  $V$ . One successful approach was initiated by Fokas, Its and Kitaev [49] and by P. Deift and collaborators via the Riemann–Hilbert method, see [20] and references therein. An alternative method was presented in [66, 65] using more direct methods from orthogonal polynomials.

There have been many refinements and improvements in this very active research area related to invariant ensembles as it reveals fruitful connections between random matrices, orthogonal polynomials, complex analysis and even combinatorics (see [20]). One common input, however, is the explicit formula (1.42) for the joint probability density that allows one to bring in orthogonal polynomials. We now depart from this topic and we will focus on ensembles when such explicit formula is not available; the most prominent example is the Wigner matrix. Apart from the Gaussian case, no explicit formula is available for the joint eigenvalue distribution. Thus the basic algebraic connection between eigenvalue ensembles and orthogonal polynomials is missing and completely new methods needed to be developed. In the next section we summarize recent results in this direction.

## 1.6 Local statistics of eigenvalues: new results

### 1.6.1 Hermitian matrices with Gaussian convolutions

The first rigorous partial result for bulk universality in the non-unitary case was given by Johansson [61], see also Ben Arous and P ech e [9] for extending [61] to the full bulk spectrum and the recent improvement [62] on weakening moment conditions. The main result states that the bulk universality holds for Gaussian divisible *hermitian* ensembles, i.e., hermitian ensembles of the form

$$H = \sqrt{1 - \varepsilon} \widehat{H} + \sqrt{\varepsilon} V, \quad (1.51)$$

where  $\widehat{H}$  is a hermitian Wigner matrix,  $V$  is an independent standard GUE matrix and  $\varepsilon$  is a positive constant of order one, independent of  $N$ .

We will often use the parametrization

$$H = H_t = e^{-t/2} \widehat{H} + (1 - e^{-t})^{1/2} V, \quad (1.52)$$

where  $t$  can be interpreted as time of an *Ornstein-Uhlenbeck (OU) process*. This formalism incorporates the idea that matrices with Gaussian convolutions can be obtained as a matrix valued stochastic process, namely as the solution of the following stochastic differential equation:

$$dH_t = \frac{1}{\sqrt{N}} d\beta_t - \frac{1}{2} H_t dt, \quad H_0 = \widehat{H},$$

where  $\beta_t$  is a hermitian matrix valued process whose diagonal matrix elements are standard real Brownian motions and whose off-diagonal matrix elements are standard complex Brownian motions. Note that infinite time,  $t = \infty$ , corresponds to the GUE ensemble, so the process (1.52) *interpolates* between the Wigner matrix  $\widehat{H}$  and the GUE. This point of view will be extremely useful in the sequel as it allows us to compare Wigner matrices with Gaussian ones if the effect of the time evolution is under control.

Alternatively, one can consider the density function  $u_t$  of the real and imaginary parts of the matrix elements as being evolved by the *generator* of the OU process:

$$\partial_t u_t = \mathcal{L} u_t, \quad \mathcal{L} = \frac{1}{4} \frac{\partial^2}{\partial x^2} - \frac{x}{2} \frac{\partial}{\partial x}, \quad (1.53)$$



where the initial density  $u_0$  is the distribution  $\nu(dx)$  of the matrix elements of  $\sqrt{N}\widehat{H}$ . For the diagonal elements, an OU process with a slightly different normalization is used. The OU process (1.53) keeps the expectation zero and variance  $\frac{1}{2}$  if  $u_0$  had these properties.

The joint distribution of the eigenvalues of Gaussian divisible hermitian random matrices of the form (1.52) still has a certain determinantal structure. The formula is somewhat simpler if we write

$$H = \sqrt{1 - \varepsilon} \left( \widehat{H} + aV \right)$$

with  $a = \sqrt{\varepsilon/(1 - \varepsilon)}$ , i.e., we use the standard Gaussian convolution

$$\widetilde{H} = \widehat{H} + aV \tag{1.54}$$

and then rescale at the end. Note that (1.54) can be generated by evolving the matrix elements by standard Brownian motions  $\beta$  up to time  $t = a^2$ , i.e.

$$d\widetilde{H}_t = \frac{1}{\sqrt{N}} d\beta_t, \quad \widetilde{H}_0 = \widehat{H}.$$

Moreover, to be in line with the normalization convention of [42] that follows [61], we assume that the matrix elements of the Wigner matrix  $\widehat{H}$  have variance  $\frac{1}{4N}$  instead of  $1/N$  as in Definition 1.1. This means that the eigenvalues are scaled by a factor  $\frac{1}{2}$  compared with the convention in the previous sections, and the semicircle law (1.20) is modified to  $2\pi^{-1}\sqrt{(1 - x^2)_+}$ . This convention applies only up to the end of this section.

Let  $\mathbf{y} = (y_1, \dots, y_N)$  denote the eigenvalues of  $\widehat{H}$  and  $\mathbf{x} = (x_1, \dots, x_N)$  denote the eigenvalues of  $\widetilde{H}$ . Then we have the following representation (a variant of these formulae was given and used by Johansson in [61] and they were motivated by similar formulae by Brézin and Hikami [15]):

**Lemma 1.2** [42, Proposition 3.2] *Let  $V$  be a standard GUE matrix. For any fixed hermitian matrix  $\widehat{H}$  with eigenvalues  $\mathbf{y}$ , the density function of the eigenvalues  $\mathbf{x}$  of  $\widetilde{H} = \widehat{H} + aV$  is given by*

$$q_S(\mathbf{x}; \mathbf{y}) := \frac{1}{(2\pi S)^{N/2}} \frac{\Delta_N(\mathbf{x})}{\Delta_N(\mathbf{y})} \det \left( e^{-(x_j - y_k)^2/2S} \right)_{j,k=1}^N, \tag{1.55}$$

with  $S = a^2/N$  and we recall that  $\Delta_N$  denotes the Vandermonde determinant (1.47). The  $m$ -point correlation functions of the eigenvalues of  $\widetilde{H} = \widehat{H} + aV$ ,

$$p_{N,\mathbf{y}}^{(m)}(x_1, \dots, x_m) := \int_{\mathbb{R}^{N-m}} q_S(x_1, x_2, \dots, x_N; \mathbf{y}) dx_{m+1} \dots dx_N,$$

are given by the following formula

$$p_{N,\mathbf{y}}^{(m)}(x_1, \dots, x_m) = \frac{(N - m)!}{N!} \det \left( \mathcal{K}_N^S(x_i, x_j; \mathbf{y}) \right)_{i,j=1}^m, \quad S = a^2/N. \tag{1.56}$$

Here we define

$$\begin{aligned} \mathcal{K}_N^S(u, v; \mathbf{y}) := & \frac{1}{(2\pi i)^2 (v-u)S} \int_{\gamma} dz \int_{\Gamma} dw (e^{-(v-u)(w-r)/S} - 1) \prod_{j=1}^N \frac{w - y_j}{z - y_j} \\ & \times \frac{1}{w-r} \left( w - r + z - u - S \sum_j \frac{y_j - r}{(w - y_j)(z - y_j)} \right) e^{(w^2 - 2uw - z^2 + 2uz)/2S}, \end{aligned} \quad (1.57)$$

where  $r \in \mathbb{R}$  is an arbitrary constant. The integration curves  $\gamma$  and  $\Gamma$  in the complex plane are given by  $\gamma = \gamma_+ \cup \gamma_-$  as the union of two lines  $\gamma_+ : \tau \rightarrow -\tau + i\omega$  and  $\gamma_- : \tau \rightarrow \tau - i\omega$  ( $\tau \in \mathbb{R}$ ) for any fixed  $\omega > 0$  and  $\Gamma$  is  $\tau \rightarrow i\tau$ ,  $\tau \in \mathbb{R}$ .

We note that  $\Gamma$  can be shifted to any vertical line since the integrand is an entire function in  $w$  and has a Gaussian decay as  $|Im w| \rightarrow \infty$ . The constants  $r \in \mathbb{R}$  and  $\omega > 0$  (appearing in the definition of the contour  $\gamma$ ) can be arbitrary and can be appropriately chosen in the estimates.

The key step behind the proof of (1.55) is the Harish-Chandra-Itzykson-Zuber integral [60]

$$\int_{U(N)} e^{\text{Tr}(UAU^*B)} dU = (\text{const.}) \frac{\det(e^{a_i b_j})_{i,j=1}^N}{\Delta_N(\mathbf{a}) \Delta_N(\mathbf{b})},$$

where  $A, B$  are two hermitian matrices with eigenvalues  $\mathbf{a} = (a_1, \dots, a_N)$  and  $\mathbf{b} = (b_1, \dots, b_N)$  and the integration is over the unitary group  $U(N)$  with respect to the Haar measure. We note that this is the step where the unitary invariance (or the hermitian character of  $H$ ) is crucially used; analogous simple formula does not exist for other symmetry groups, see [16].

For any testfunction  $f$ , and for a fixed matrix  $\tilde{H}$ , we have

$$\int f(\mathbf{x}) q_S(\mathbf{x}; \mathbf{y}) d\mathbf{x} = (\text{const.}) \int f(\mathbf{x}) e^{-\frac{1}{2} N \text{Tr}(\tilde{H} - \hat{H})^2} d\tilde{H},$$

where we used that  $V = a^{-1}(\tilde{H} - \hat{H})$  is a GUE matrix with distribution

$$\mathcal{P}(V) dV = e^{-\frac{1}{2} N \text{Tr} V^2} dV$$

We set  $\tilde{H} = UXU^*$  to be the diagonalization of  $\tilde{H}$  with  $X = \text{diag}(\mathbf{x})$ , then we have, using (1.45),

$$\begin{aligned} \int f(\mathbf{x}) q_S(\mathbf{x}; \mathbf{y}) d\mathbf{x} &= (\text{const.}) \int_{\mathbb{R}^N} \int_{U(N)} f(\mathbf{x}) e^{-\frac{1}{2a^2} N \text{Tr}(UXU^* - \hat{H})^2} dU \Delta_N^2(\mathbf{x}) d\mathbf{x} \\ &= (\text{const.}) \int_{\mathbb{R}^N} \left[ \int_{U(N)} e^{\frac{N}{a^2} \text{Tr} UXU^* \hat{H}} dU \right] f(\mathbf{x}) e^{-\frac{N}{2S} \sum_i (x_i^2 + y_i^2)} \Delta_N^2(\mathbf{x}) d\mathbf{x} \\ &= (\text{const.}) \int_{\mathbb{R}^N} f(\mathbf{x}) \frac{\det(e^{x_i y_j / S})_{i,j=1}^N}{\Delta_N(\mathbf{x}) \Delta_N(\mathbf{y})} e^{-\frac{N}{2S} \sum_i (x_i^2 + y_i^2)} \Delta_N^2(\mathbf{x}) d\mathbf{x} \\ &= (\text{const.}) \int_{\mathbb{R}^N} f(\mathbf{x}) \frac{\Delta_N(\mathbf{x})}{\Delta_N(\mathbf{y})} \det(e^{-\frac{1}{2S} (x_i - y_j)^2})_{i,j=1}^N d\mathbf{x}, \end{aligned}$$

which proves (1.55) (apart from the constant). This shows how the Vandermonde determinant structure emerges for Gaussian convolutions. The proof of the contour integral representation (1.57) from (1.55) is a bit more involved, see Proposition 3.2 of [42] (or Proposition 2.3 [61]) for the details.

Once (1.57) is given, the key idea is to view it as a complex integral suited for Laplace asymptotics or saddle point calculation. More precisely, after some straightforward algebraic steps, it can be brought into the following form (see Section 3.1 of [42]), where we already changed variables in the argument to detect the microscopic structure. For any fixed  $|u| < 1$  and  $t = a^2$  we find from (1.57) that

$$\frac{1}{N\varrho(u)}\mathcal{K}_N^{t/N}\left(u, u + \frac{\tau}{N\varrho(u)}; \mathbf{y}\right) = N \int_{\gamma} \frac{dz}{2\pi i} \int_{\Gamma} \frac{dw}{2\pi i} h_N(w) g_N(z, w) e^{N(f_N(w) - f_N(z))} \quad (1.58)$$

with

$$\begin{aligned} f_N(z) &:= \frac{1}{2t}(z^2 - 2uz) + \frac{1}{N} \sum_j \log(z - y_j) \\ g_N(z, w) &:= \frac{1}{t(w-r)}[w - r + z - u] - \frac{1}{N(w-r)} \sum_j \frac{y_j - r}{(w - y_j)(z - y_j)} \\ h_N(w) &:= \frac{1}{\tau} \left( e^{-\tau(w-r)/t\varrho(u)} - 1 \right). \end{aligned}$$

Notice the  $N$  factor in front of the exponent in (1.58), indicating that the main contribution to the integral comes from the saddle points, i.e., from  $z$  and  $w$  values where  $f'_N(z) = f'_N(w) = 0$ . Note that

$$f'_N(z) = \frac{z - u}{t} + \frac{1}{N} \sum_j \frac{1}{z - y_j}, \quad (1.59)$$

i.e., it is essentially given by the empirical Stieltjes transform (1.23) of the eigenvalues of  $\widehat{H}$ . Suppose that the Wigner semicircle law or, equivalently, (1.28) holds, then the saddle point  $z_N$ ,  $f'_N(z_N) = 0$  can be well approximated by the solution to

$$\frac{z - u}{t} + 2(z - \sqrt{z^2 - 1}) = 0 \quad (1.60)$$

(note that the formula for the Stieltjes transform slightly differs from (1.28) because of the different normalization of  $\widehat{H}$ ). It is easy to check that there are two solutions,  $z^{\pm}$ , with imaginary part given by  $\Im z_{\pm} = \pm 2t\sqrt{1 - u^2} + O(t^2)$  for small  $t$ .

Once the saddle points are identified, the integration contours  $\gamma$  and  $\Gamma$  in (1.58) can be shifted to pass through the saddle points from a direction where  $f_N$  is real and its second derivative has the “good sign”, so that usual saddle point approximation holds. There are altogether four pairs of

saddles  $(z_N^\pm, w_N^\pm)$ , but only two give contributions to leading order. Their explicit evaluation gives the sine kernel (1.33) for  $\mathcal{K}_N$  in the  $N \rightarrow \infty$  limit.

The key technical input is to justify the semicircle approximation used in passing from (1.59) to (1.60) near the saddle point (some estimate is needed away from the saddle as well, but those are typically easier to obtain). The standard argument for the semicircle law presented in Section 1.4.2 holds for any *fixed*  $z$  with  $\Im z > 0$ , independently of  $N$ , especially important is that  $\Im z > 0$  uniformly in  $N$ . Therefore this argument can be used to justify (1.60) only for a fixed  $t > 0$ . Recall that  $t = a^2$  is the variance of the Gaussian convolution. This was the path essentially followed by Johansson who proved [61] that sine-kernel universality (1.34) holds if the Wigner matrix has a Gaussian component comparable with its total size.

One implication of the local semicircle law explained in Section 1.1.1 is that the approximation from (1.59) to (1.60) could be justified even for very short times  $t$ . Essentially any time of order  $\gg 1/N$  is allowed, but for technical reasons we carried out the estimates only for  $t = N^{-1+\varepsilon}$  for any  $\varepsilon > 0$  and we showed that the contour integral (1.58) is given by the sine-kernel even for such short times [42]. This proved the sine-kernel universality in the form of (1.34) for any fixed  $E$  in the bulk spectrum for hermitian matrices with a Gaussian component of variance  $N^{-1+\varepsilon}$ .

### 1.6.2 Gaussian convolutions for arbitrary matrices: the local relaxation flow

The method presented in Section 1.6.1 heavily relies on the Brézin-Hikami type formula (1.57) that is available only for hermitian matrices. For ensembles with other symmetries, in particular for symmetric Wigner matrices, a new method was necessary. In a series of papers [41, 40, 44, 46, 47] we developed an approach based upon hydrodynamical ideas from interacting particle systems. We will present it in more details in Section 3, here we summarize the key points.

The starting point is a key observation of Dyson [29] from 1962. Let  $\widehat{H}$  be an arbitrary fixed matrix and let  $V$  be a standard GUE matrix. Dyson noticed that if the parameter  $s$  in the matrix  $\widehat{H} + sV$  is varied and  $t = s^2$  is interpreted as time, then the evolution of the eigenvalues is given by a coupled system of stochastic differential equations, commonly called the Dyson Brownian motion (DBM). For convenience, we will replace the Brownian motions by Ornstein-Uhlenbeck processes to keep the variance constant, i.e., we use (1.52) to generate the matrix flow. The Dyson Brownian motion we will use (and we will still call it DBM) is given by the following system of stochastic differential equations for the eigenvalues  $\boldsymbol{\lambda}(t) = (\lambda_1(t), \dots, \lambda_N(t))$ , see, e.g. Section 4.3.1 of [4],

$$d\lambda_i = \frac{dB_i}{\sqrt{N}} + \left[ -\frac{\beta}{4}\lambda_i + \frac{\beta}{2N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right] dt, \quad 1 \leq i \leq N, \quad (1.61)$$

where  $\{B_i : 1 \leq i \leq N\}$  is a collection of independent Brownian motions. The initial condition  $\boldsymbol{\lambda}(0)$  is given by the eigenvalues of  $\widehat{H}$ . The choice of the parameter  $\beta = 2$  corresponds to the hermitian case, but the process (1.61) is well defined for any  $\beta \geq 1$  and the eigenvalues do not cross due to the strong repulsion among them. The threshold  $\beta = 1$  is critical for the non-crossing

property. As  $t \rightarrow \infty$ , the distribution of  $\lambda(t)$  converges to the invariant Gaussian  $\beta$ -ensemble distribution (1.43) as the global invariant measure. For example, for  $\beta = 2$ , it converges to the GUE as  $t \rightarrow \infty$ , in accordance with the fact that (1.52) converges to GUE.

Thus the result of Johansson [61] can be interpreted as stating that the local statistics of GUE is reached via DBM for time of order one. This is in fact the correct time scale to approach the *global* equilibrium. Our result from [42], explained in the previous section, indicates that the decay to the *local* equilibrium is much faster, it occurs already in time  $t \sim N^{-1+\varepsilon}$ . Although the proof was valid only for hermitian matrices, the fast local equilibration is a general feature of DBM. The key point is that in the hydrodynamical approach there is no restriction on the symmetry type; the argument works equally for symmetric, hermitian or quaternion self-dual ensembles, moreover, with some obvious modifications, it also works for random covariance matrices.

Our first paper that used hydrodynamical ideas is [41] and it predates [42]. In [41] we extended Johansson's result [61] to  $s^2 \gg N^{-3/4}$  by capitalizing on the fact that the local statistics of eigenvalues depend exclusively on the approach to *local* equilibrium which in general is faster than reaching global equilibrium. Unfortunately, the identification of local equilibria in [41] still used explicit representations of correlation functions by orthogonal polynomials (following e.g. [72]), and the extension to other ensembles in this way is not a simple task (see [81] for extension of [72] to symmetric matrices to prove edge universality). With this method we also needed to take a small averaging in the energy parameter  $E$  in (1.34).

To depart from using orthogonal polynomials, we introduced new hydrodynamical tools in [40] which entirely eliminated explicit formulas and it gave a unified proof for the universality of *symmetric* and hermitian Wigner matrices with a small Gaussian convolution. The time needed to reach the local equilibrium in a sufficiently strong sense has increased from  $N^{-3/4}$  to  $N^{-\xi}$  with a small positive  $\xi$  but the method has become general. The result was further generalized in [44] to quaternion self-dual Wigner matrices and sample covariance matrices and even to *generalized Wigner matrices* in [46, 47]. A universal Wigner matrix (see Definition 1.1) is called *generalized Wigner matrix* if

$$0 < C_{inf} \leq N\sigma_{ij}^2 \leq C_{sup} < \infty, \quad i, j = 1, 2, \dots, N, \quad (1.62)$$

holds with some fixed positive constants  $C, c$ . More importantly, the hydrodynamical method not only applies to all these specific ensembles, but it also gives a conceptual interpretation that the occurrence of the universality is due to the relaxation to local equilibrium of the DBM.

The hydrodynamical approach consists of two parts. First, we have a general theorem stating that under certain mild convexity conditions on the Hamiltonian  $\mathcal{H}$  of the equilibrium measure of the DBM (see (1.44) for a special case) and under a fairly strong control on the local density of eigenvalues, the local equilibrium is reached within a short time  $t \sim N^{-\xi}$ ,  $\xi > 0$ , in the sense that the local correlation functions rescaled in the form of (1.34) coincide with the same correlation functions in equilibrium. By the general Bakry-Emery [8] criterion, the speed of convergence to global equilibrium for DBM depends on the lower bound on the Hessian of the Hamiltonian  $\mathcal{H}$ ,

which in our case is of order one. The key idea is to speed up this convergence by modifying the Hamiltonian. We add to  $\mathcal{H}$  an auxiliary potential of the form

$$W(\boldsymbol{\lambda}) = \frac{1}{2R^2} \sum_j (\lambda_j - \gamma_j)^2,$$

where  $R$  is a small parameter, depending on  $N$ , and  $\gamma_j$ 's are the classical location of the eigenvalues, defined by

$$N \int_{-\infty}^{\gamma_j} \varrho(x) dx = j. \quad (1.63)$$

Here  $\varrho(x)$  is the limiting density, e.g.,  $\varrho(x) = \varrho_{sc}(x)$  for Wigner matrices. The Hamiltonian  $\widehat{\mathcal{H}} = \mathcal{H} + W$  generates a new stochastic flow of the eigenvalues, called the *local relaxation flow*. The convergence to equilibrium for this flow is faster, it occurs in a time scale  $R^2$ . In fact, due to the strong repulsion between eigenvalues (reflected by a singular repulsive force (1.61)), in some sense the convergence is even faster for observables that depend only on *eigenvalue differences*. Then we show that the modified dynamics is, in fact, not far from the original one, using that the typical size of the auxiliary potential  $W$  is small. More precisely, we will need to prove that the eigenvalues  $\lambda_j$  lie near  $\gamma_j$  with a precision  $N^{-1/2-\varepsilon}$ , i.e., that

$$\mathbb{E} \frac{1}{N} \sum_{i=1}^N (\lambda_i - \gamma_i)^2 = N^{-1-2\varepsilon} \quad (1.64)$$

holds with some  $\varepsilon > 0$ . This is the key input condition to our general theorem and it will be proved by a strong control on the local density. The exponent  $\xi$  in the time scale  $t \sim N^{-\xi}$  is essentially  $2\varepsilon$  appearing in the estimate (1.64).

The second part of the hydrodynamical approach is to prove the necessary input conditions on the local density for the general theorem, especially (1.64). This is the step where specific properties of the matrix ensemble come into the game. For example, in the Wigner case, we need to prove that the semicircle law holds with precision almost  $N^{-1}$ , similarly to [42]. The technical difference is that we need a fairly good control on the local density near the spectral edges as well, since (1.64) involves *all* eigenvalues. Although we are interested only in the bulk universality, i.e., local behavior away from the edges, we still need the global speed of convergence for the modified dynamics that is influenced by the eigenvalues at the edge as well. Recall that the control on the density near the edges becomes weaker since eigenvalues near the edge tend to fluctuate more.

A good local density control had been developed in our previous work on Wigner matrices [37, 38, 39], but the edge behavior was not optimal. Nevertheless, in [44] we succeeded in proving (1.64) in a somewhat complicated way, relying on some improvements of estimates from [37, 38, 39]. Recently, in [46], we found a more direct way to control the local density and prove (1.64) more efficiently. The most streamlined version is given in [47] which we will sketch in Section 2.

We mention that this proof also applies to generalized Wigner matrices where the variances satisfy (1.62). In this case, we prove the local semicircle law down to essentially the smallest possible energy scale  $N^{-1}$  (modulo  $\log N$  factors). This is sufficient to prove (1.64) and thus we can apply our general theorem and prove the bulk universality of local statistics for these matrices. A much more difficult case is the *Wigner band matrices* (1.18) where, roughly speaking,  $\sigma_{ij}^2 = 0$  if  $|i - j| > W$  for some  $W \ll N$ . In this case, we obtain the local semicircle law to the energy scale  $W^{-1}$  which is not strong enough to prove (1.64) if  $W$  is much smaller than  $N$  (the case  $W \geq N^{1-\delta}$  with some small  $\delta$  still works).

### 1.6.3 Removing the Gaussian convolution I. The reverse heat flow

In the previous two sections we discussed how to prove bulk universality for matrices with a small Gaussian convolution. The method of [42] (Section 1.6.1) required only a very small Gaussian component (variance  $N^{-1+\varepsilon}$ ) but it was restricted to the hermitian case. The hydrodynamical method (Section 1.6.2) assumes a larger Gaussian component (variance  $\sim N^{-2\varepsilon}$ ) but it works in general. Both methods, however, need to be complemented by a perturbative step to remove this small Gaussian component.

There have been two independent approaches developed to remove the restriction on Gaussian divisibility. The first method is our *reverse heat flow* argument that appeared first in [40] and was streamlined in Section 6 of [44]. The advantage of this method is that it can prove universality for a fixed energy  $E$  as formulated in (1.34), moreover, it is also very simple. The disadvantage is that it requires some smoothness of the distribution of the entries of the Wigner matrix, in addition to a natural decay condition.

The second method, appeared slightly after the first, is via *perturbation with four moment condition*. The advantage of this approach is that it holds for any distribution with the only condition of subexponential decay (1.69). The disadvantage is that it proves (1.34) only after some averaging over  $E$ . The four moment condition was introduced by Tao and Vu [91] and used later in [92, 93] in their study of eigenvalue perturbation. In Section 4 we will present our approach [46] based on resolvent perturbation whose proof is much simpler than in [91].

In this section we sketch the method of the reverse heat flow and in the next section we explain the four moment comparison principle.

For simplicity of the presentation, we consider the hermitian case and we assume that the real and imaginary parts of the matrix elements are i.i.d. Let

$$\partial_t u_t = B u_t, \quad B = \frac{1}{4} \frac{\partial^2}{\partial x^2} - \frac{x}{2} \frac{\partial}{\partial x} \tag{1.65}$$

be the OU process that keeps variance  $\frac{1}{2}$  fixed and let  $\gamma(dx) := \pi^{-1/2} e^{-x^2} dx$  denote the reversible measure for this process. Let  $\nu(dx) = u(x)\gamma(dx)$  be the initial measure of the real and imaginary

parts of the entries of the Wigner matrix  $\widehat{H}$ . We let the OU process act on the matrix elements, i.e., we consider the matrix

$$H_t = e^{-t/2}\widehat{H} + (1 - e^{-t})^{1/2}V, \quad (1.66)$$

where  $V$  is a GUE matrix, independent of  $\widehat{H}$ . The distribution of the real and imaginary parts of the matrix elements of  $H_t$  is then given by  $u_t(x)\gamma(dx)$ , the solution to (1.65) with initial data  $u_0 = u$  (strictly speaking, these formulas hold for the offdiagonal elements, the diagonal element has twice bigger variance and it is subject to a slightly different OU flow).

The main observation is that the arguments in Section 1.6.1 or Section 1.6.2 guarantee that sine kernel holds for *any* hermitian matrix that has a Gaussian component of variance  $N^{-1+\varepsilon}$  or  $N^{-\xi}$ , respectively. Given a Wigner matrix  $\widehat{H}$ , we do not necessary have to compare  $\widehat{H}$  with its Gaussian convolution,  $H_t$  (1.66); it is sufficient to find another Wigner matrix  $\widetilde{H}$  such that

$$\widehat{H} \approx e^{-t/2}\widetilde{H} + (1 - e^{-t})^{1/2}V \quad (1.67)$$

with a very high precision. In fact,  $\widetilde{H}$  can even be chosen  $t$ -dependent. The following lemma shows that *any* Wigner matrix  $\widehat{H}$  with a sufficiently smooth distribution can be arbitrary well approximated by Gaussian divisible matrices of the form (1.67) if  $t \sim N^{-\delta}$  for some  $\delta > 0$ .

We assume that  $u(x) > 0$  and it can be written as

$$u(x) = e^{-V(x)}, \quad \text{with} \quad \sum_{j=1}^{2K} |V^{(j)}(x)| \leq C_K(1 + x^2)^{C_K} \quad (1.68)$$

with any  $K \in \mathbb{N}$  and with sufficiently large constants  $C_K$ . Moreover, we assume that the initial single entry distribution  $d\nu = u d\gamma$  has a subexponential decay, i.e., there is constants  $C, \alpha > 0$  such that for any  $t$

$$\int \mathbf{1}(|x| \geq t^\alpha) d\nu(x) \leq C e^{-t}. \quad (1.69)$$

The key technical lemma is the following approximation statement.

**Lemma 1.3** *Suppose that for some  $K > 0$ , the measure  $\nu = u\gamma$  satisfies (1.68) and (1.69). Then there is a small constant  $\alpha_K$  depending on  $K$  such that for any  $t \leq \alpha_K$  there exists a probability density  $g_t$  with mean zero and variance one such that*

$$\int |e^{tB}g_t - u| d\gamma \leq C t^K \quad (1.70)$$

for some  $C > 0$  depending on  $K$ .

Furthermore, let  $\mathcal{B} = B^{\otimes n}$ ,  $F = u^{\otimes n}$  with some  $n \leq CN^2$ . Denote by  $G_t = g_t^{\otimes n}$ . Then we also have

$$\int |e^{t\mathcal{B}}G_t - F| d\gamma^{\otimes n} \leq C N^2 t^K \quad (1.71)$$

for some  $C > 0$  depending on  $K$ .



*Sketch of the proof.* Given  $u$ , we want to solve the equation

$$e^{Bt}g_t = u,$$

i.e., formally  $g_t = e^{-Bt}u$ . However, the operator  $e^{-Bt}$  is like running a heat flow (with an OU drift) in reverse time which is typically undefined unless  $u$  is analytic. But we can define an approximate solution to the backward heat equation, i.e., we set

$$g_t := \left( I - Bt + \frac{t^2 B^2}{2!} - \dots + \frac{(-tB)^{K-1}}{(K-1)!} \right) u.$$

Since  $B$  is a second order differential operator and  $u$  is sufficiently smooth, this expression is well defined, moreover

$$e^{Bt}g_t = O\left(t^K B^K u\right) = O(t^K).$$

This proves (1.70) and (1.71) directly follows from it.  $\square$

Armed with Lemma 1.3, we can prove the sine kernel universality in the form of (1.34) for any hermitian Wigner matrix satisfying (1.68) and (1.69) and for any fixed  $|E| < 2$ . We choose  $n \sim N^2$  to be the number of independent OU processes needed to generate the flow of the matrix elements. By choosing  $K$  large enough, we can compare the two measures  $e^{tB}G_t$  and  $F$  in the total variational norm; for any observable  $J : \mathbb{R}^n \rightarrow \mathbb{R}$  of the matrix elements, we have

$$\left| \int J(e^{tB}G_t - F) d\gamma^{\otimes n} \right| \leq \|J\|_\infty C N^2 t^K.$$

In order to prove (1.34), appropriate observables  $J$  need to be chosen that depend on the matrix elements via the eigenvalues and they express local correlation functions. It is easy to see that for such  $J$ , its norm  $\|J\|_\infty$  may grow at most polynomially in  $N$ . But we can always choose  $K$  large enough to compensate for it with the choice  $t = N^{-\delta}$ . Since the sine kernel holds for the distribution  $e^{tB}G_t$  with  $t = N^{-1+\varepsilon}$  (Section 1.6.1) or  $t = N^{-\xi}$  (Section 1.6.2) it will also hold for the Wigner measure  $F$ .

#### 1.6.4 Removing the Gaussian convolution II. The Green function comparison theorem

Let  $H$  and  $H'$  be two Wigner matrix ensembles such that the first four moments of the single entry distribution,  $\nu$  and  $\nu'$ , coincide:

$$m_j = m'_j, \quad j = 1, 2, 3, 4 \tag{1.72}$$

where

$$m_j = \int x^j d\nu(x) \quad \text{and} \quad m'_j = \int x^j d\nu'(x).$$

For complex entries one has to take the collection of all  $j$ -moments, i.e.,  $m_j$  represents the collection of all  $\int x^a \bar{x}^b d\nu(x)$  with  $a + b = j$ . Recall that  $\nu$  is the distribution of  $\sqrt{N}h_{ij}$ . By our normalization of Wigner matrices, the first moment is always zero and the second moment is one, so (1.72) is really a condition on the third and fourth moments.

Our main result is the following comparison theorem for joint distribution of Green functions. Here we only state the result, a more detailed presentation will be given in Section 4.

**Theorem 1.4 (Green function comparison theorem. Thm. 2.3 [46])** *Let  $H$  and  $H'$  be two Wigner matrices with single entry distributions  $\nu$  and  $\nu'$ . Assume that (1.69) and (1.72) hold for  $\nu$  and  $\nu'$ . Let  $G(z) = (H - z)^{-1}$  and  $G'(z) = (H' - z)^{-1}$  denote the resolvents. Fix  $k$  and suppose that the function  $F : \mathbb{R}^k \rightarrow \mathbb{R}$  satisfies*

$$\sup_{x \in \mathbb{R}^k} |\nabla^j F(x)| \leq N^{\varepsilon'}, \quad 0 \leq j \leq 5. \quad (1.73)$$

*Let  $\kappa$  and  $\varepsilon$  be small parameters. Then, for sufficiently small  $\varepsilon'$  and  $\varepsilon$  there is  $c_0 > 0$  such that for any integers  $\ell_1, \dots, \ell_k$  and spectral parameters  $z_j^m = E_j^m \pm i\eta$ ,  $1 \leq j \leq \ell_m$ ,  $m = 1, 2, \dots, k$  with  $E_j^m \in [-2 + \kappa, 2 - \kappa]$  and  $\eta \geq N^{-1-\varepsilon}$ , we have*

$$\left| \mathbb{E} F \left( \frac{1}{N} \text{Tr} \left[ \prod_{j=1}^{\ell_1} G(z_j^1) \right], \dots, \frac{1}{N} \text{Tr} \left[ \prod_{j=1}^{\ell_k} G(z_j^k) \right] \right) - \mathbb{E}' F(G \rightarrow G') \right| \leq N^{-c_0}. \quad (1.74)$$

In fact, the condition (1.72) can be weakened to require that the third and fourth moment be only close;

$$m_j = m'_j, \quad j = 1, 2, \quad \text{and} \quad |m_3 - m'_3| \leq N^{-\frac{1}{2}-\delta}, \quad |m_4 - m'_4| \leq N^{-\delta} \quad (1.75)$$

with some  $\delta > 0$ . Then (1.74) still holds, but  $\varepsilon$  and  $\varepsilon'$  has to be sufficiently small, depending on  $\delta$  and  $c_0$  will also depend on  $\delta$ .

In other words, under the four moment matching condition for the two Wigner ensembles, the expectation of traces of any combination of resolvent products coincides if the spectral parameters in the resolvents are not closer than  $\eta = N^{-1-\varepsilon}$  to the real axis. Such a small distance corresponds to spectral resolution on scale  $N^{-1-\varepsilon}$ , i.e., it can identify local correlation functions of individual eigenvalues. It is an easy algebraic identity to express correlation functions from traces of resolvents, for example the one point correlation function (density) on scale  $\eta$  is approximated by

$$p_N^{(1)}(E) \sim \frac{1}{\pi N} \Im \text{Tr} G(E + i\eta) = \frac{1}{2\pi} \left[ \frac{1}{N} \text{Tr} G(E + i\eta) - \frac{1}{N} \text{Tr} G(E - i\eta) \right]$$

and higher point correlation functions involve higher order polynomials of resolvents. Thus Theorem 1.4 directly compares correlation functions (for the precise statement, see Theorem 4.2). We remark that taking traces is not essential in (1.74), a similar comparison principle works for matrix elements of the resolvents as well (see [46] for the precise formulation).

A related theorem for eigenvalues was first proven by Tao and Vu [91]. Let  $\lambda_1 < \lambda_2 < \dots < \lambda_N$  and  $\lambda'_1 < \lambda'_2 < \dots < \lambda'_N$  denote the eigenvalues of  $H$  and  $H'$ , respectively. The following theorem states that the joint distribution of any  $k$ -tuple of eigenvalues on scale  $1/N$  is very close to each other.

**Theorem 1.5 (Four moment theorem for eigenvalues. Thm. 15 of [91])** *Let  $H, H'$  be two Wigner matrices and assume that (1.72) and (1.69) hold for their single entry distributions  $\nu$  and  $\nu'$ . For any sufficiently small positive  $\varepsilon$  and  $\varepsilon'$  and for any function  $F : \mathbb{R}^k \rightarrow \mathbb{R}$  satisfying (1.73), and for any selection of  $k$ -tuple of indices  $i_1, i_2, \dots, i_k \in [\varepsilon N, (1 - \varepsilon)N]$  away from the edge, we have*

$$\left| \mathbb{E}F\left(N\lambda_{i_1}, N\lambda_{i_2}, \dots, N\lambda_{i_k}\right) - \mathbb{E}'F\left(N\lambda'_{i_1}, N\lambda'_{i_2}, \dots, N\lambda'_{i_k}\right) \right| \leq N^{-c_0} \quad (1.76)$$

with some  $c_0 > 0$ . The condition (1.72) can be relaxed to (1.75), but  $c_0$  will depend on  $\delta$ .

After the proof of Theorem 1.4 in Section 4 we will comment on the differences between the proofs of these two theorems.

Note that the arguments in (1.76) are magnified by a factor  $N$  and  $F$  is allowed to be concentrated on a scale  $N^{-\varepsilon'/5}$ , so the result is sufficiently precise to detect eigenvalue correlations on scale  $N^{-1-\varepsilon'/5}$ , i.e., even somewhat smaller than the eigenvalue spacing. Therefore Theorem 1.4 or 1.5 can prove bulk universality for a Wigner matrix  $H$  if another  $H'$  is found with matching four moments for which universality is already proved. In the *hermitian* case, the GUE matrices, or more generally the Gaussian divisible matrices (1.52) provide a good reference ensemble. Matching with a GUE matrix requires that the third and the fourth moments match,  $m_3 = 0$ ,  $m_4 = 3$ . Since the *location* of the eigenvalues for GUE is known very precisely [58], (1.76) can be translated into the limit of correlation functions (as (1.34)) even at a fixed energy  $E$ . If one aims only at the limiting gap distribution (1.35) instead of (1.34), then one can directly use the Gaussian divisible matrix (1.52) for matching. It is easy to check [18] that for any probability distribution  $\nu$  with  $m_1 = 0$  and  $m_2 = 1$  that is supported on at least three points, there is a distribution with an order one Gaussian component so that the first four moments match. Therefore  $H$  can be matched with a Gaussian divisible matrix for which Johansson has proved universality. Using the result of [42] on the universality of hermitian Wigner matrices with a tiny Gaussian convolution (conclusion of Section 1.6.1), and using that exact moment matching (1.72) can be relaxed to (1.75), one can compare any Wigner matrix  $H$  with its very short time  $t \sim N^{-1+\varepsilon}$  Ornstein-Uhlenbeck convolution (1.52). This removes the requirements  $m_3 = 0$  and that the support has at least three points and proves universality of correlation functions for any hermitian Wigner matrix in the sense of (1.34) after a little averaging in  $E$  [43]. The only technical condition is the subexponential decay of  $\nu$  (1.69).

In the symmetric case, the analogue of Johansson's result is not available (unless one uses [40]), and the only reference ensemble is GOE. Theorem 1.5 thus proves universality for symmetric

Wigner matrices whose single entry distribution has first four moments matching with GOE in the sense of (1.75).

The careful reader may notice a subtle difference between the observable in (1.76) and the local correlation functions (1.34). While both detect the structure on scale  $1/N$ , in (1.76) the *indices* of the eigenvalues are fixed, while in (1.34) their *location*. Roughly speaking, (1.76) can answer the question, say, “where are the  $N/2$ -th and the  $(N/2 - 1)$ -th eigenvalues”. The local correlation function asks “what is the probability of the simultaneous event that there is an eigenvalue at  $E$  and another one at  $E' = E + \alpha/N$ ”. These questions can be related only if some a-priori information is known about the location of the eigenvalues.

Prior to [91], apart from the GUE case [58], for no other ensembles could the eigenvalues be located with a precision  $N^{-1+c_0}$  for small  $c_0$ , and such precision is needed to translate (1.76) into (1.34) for a fixed  $E$ . Using a three-moment matching version of Theorem 1.5, one can locate the eigenvalues of any Wigner ensembles with such precision, provided that the third moment vanishes (i.e. matches with GUE). Given this information, one can proceed to match the fourth moment by choosing an appropriate Gaussian divisible matrix. This is possible if the original distribution is supported on at least three points. This is why eventually (1.34) was proven in [91] under the condition that the third moment vanishes and the support contains at least three points. If one accepts that (1.34) will be proven after some averaging in  $E$ , then the necessary information on the location of the eigenvalues is much weaker and it can typically be obtained from the local semicircle law.

In fact, tracking individual eigenvalues can be a more challenging task than computing correlation functions (although Theorem 1.5 in itself does not directly imply convergence of correlation functions; one still needs some information about the location of the  $i$ -th eigenvalue). The main reason is that correlation functions can be identified from observables involving traces of resolvents  $(H - z)^{-1}$  with  $\Im z \sim N^{-1-\varepsilon}$  and these resolvents have an a-priori bound of order  $|\Im z|^{-1} \leq N^{1+\varepsilon}$ , so perturbation formulas involving resolvents do not blow up. The eigenvalues however may produce resonances which could render some terms even potentially infinite. While level repulsion is a general feature of Wigner ensembles and it strongly suppresses resonances, the direct proof of the level repulsion is not an easy task. In fact, the most complicated technical estimate in [91] is the lower tail estimate on the gap distribution (Theorem 17 of [91]). It states that for any  $c_0 > 0$  there is a  $c_1$  such that

$$\mathbb{P}(\lambda_{i+1} - \lambda_i \leq N^{-1-c_0}) \leq N^{-c_1} \tag{1.77}$$

if the index  $i$  lies in the bulk ( $\varepsilon N \leq i \leq (1 - \varepsilon)N$ ).

### 1.6.5 Summary of results on bulk universality

Even the expert reader may find the recent developments confusing since there have been many papers on bulk universality of Wigner matrices under various conditions and with different methods.

Their interrelation was not always optimally presented in the research publications since it was, and still is, a fast developing story. In this section we try to give some orientation to the reader about the recent literature.

As mentioned in the introduction, the main guiding principle behind our proofs of universality of the local eigenvalue statistics is to compare the local statistics of a Wigner matrix with another matrix with some Gaussian component. More precisely, our approach consists of three main ingredients:

- (1) the local semicircle law;
- (2) universality for Gaussian divisible ensembles, i.e., the probability law of matrix elements containing a small Gaussian component.
- (3) universality for general ensembles; removal of the small Gaussian component.

It was clear to us from the very beginning that a good local semicircle law must be the first step in any proof of universality. In fact, all proofs of universality rely heavily on the details of the estimates one can obtain for the local semicircle law. We now summarize the existing results according to these three components.

Although the proof of the local semicircle law to all scales down to  $1/N$  is rather simple now, it was only gradually achieved. In our first paper, [37], we gave an *upper bound* on the local density essentially down to the optimal energy scale  $\eta \sim (\log N)/N$ , but the local semicircle law itself was proven only on scale  $\eta \gg N^{-2/3}$ . In the second paper, [38], we proved the local semicircle law down to the scale  $\eta \geq (\log N)^8/N$ , almost optimal but still off by a logarithmic factor. The tail probability to violate the local semicircle law was also far from optimal. Both defects were remedied in [39] (see Theorem 1.8 below) where, additionally, an optimal delocalization result for eigenvectors was also proven. In the first paper [37] we assumed a strong (Gaussian) decay condition and some convexity property of the single entry distribution that implies concentration (either via Brascamp-Lieb or logarithmic Sobolev inequalities). These technical conditions were subsequently removed and the Gaussian decay condition was replaced by a subexponential decay. Finally, in [46] and in its improved and streamlined version in [47], we obtained a much stronger error estimate to the local semicircle law which we called the strong local semicircle law, see (1.6)–(1.9). As a by-product of our approach, we obtained eigenfunction delocalization estimate in a very strong sense.

As for the second step, the main point is that the Gaussian component enables one to exhibit the universal behavior. There are two ways to implement this idea:

- (i) the contour integral representation following Johansson [61] and Ben-Arous, Pécché [9] but this option is available only for the hermitian case (or for the complex sample covariance case, [9]);

- (ii) the hydrodynamical approach, where a small Gaussian component (equivalently, a small time evolution of the OU process) already drives the system to local equilibrium. This approach applies to all ensembles, including symmetric, hermitian, symplectic and sample covariance ensembles, and it also gives the conceptual interpretation that the universality arises from the Dyson Brownian motion.

Both approaches require a good local semicircle law. Additionally, the earlier papers on the hydrodynamical methods, [40] and [44], also assumed the logarithmic Sobolev inequality (LSI) for the single entry distribution  $\nu$ , essentially in order to verify (1.64) from the local semicircle law. Finally in [47] we removed this last condition by using the strong local semicircle law (1.2) which gave a simpler and more powerful proof of (1.64) (Theorem 2.8).

Summarizing the first two steps, we thus obtained bulk universality for generalized Wigner matrices (1.62) with a small Gaussian convolution under the sole condition of subexponential decay. This condition can be relaxed to a high order polynomial decay, but we did not pursue this direction. Furthermore, although the extension of the strong local semicircle law for sample covariance matrices is straightforward, but we did not carry out the details (the earlier detailed proof [44] required LSI).

The first two steps provide a large class of matrix ensembles satisfying the universality of local statistics. It remains to approximate arbitrary matrix ensembles by these matrices so that the local statistics are preserved. The approximation step can be done in two ways

- (i) via the reverse heat flow;
- (ii) via the Green function comparison theorem.

The reverse heat flow argument is very simple, but it requires smoothness on the single entry distribution. This approach was used in [42], [40] and [44] and this leads to universality for all ensembles mentioned under the smoothness condition. This smoothness condition was then removed in [46] where the Green function comparison theorem was first proved. Unfortunately, we still needed the LSI and universality was established for matrices whose distribution  $\nu$  is supported on at least three points.

Finally, the strong local semicircle law was proved in [47] and all smoothness and support conditions on the distributions were removed. In summary, in [47] we obtained bulk universality of correlation functions (1.34) and gap distribution (1.35) for all classical Wigner ensembles (including the generalized Wigner matrices, (1.62)). The universality in (1.34) is understood after a small averaging in the energy parameter  $E$ . The only condition on the single entry distribution  $\nu$  is that it has a subexponential decay (1.69).

The approach of Tao and Vu [91] uses a similar strategy of the three steps mentioned at the beginning of this section. For Step 2, the universality for *hermitian* Wigner matrices and *complex* sample covariance matrices were previously proved by Johansson [61] and Ben-Arous and P  ch   [9].

Step 3 follows from Tao-Vu's four moment theorem (Theorem 1.5). This leads to the universality for the *hermitian* Wigner matrices [91] and *complex* sample covariance matrices [93] satisfying the condition that the support of the distribution contains at least three points (and, if one aims at a fixed energy result in (1.34), then the third moment has also to vanish). For the symmetric case, the matching of the first four moments was required. The Tao-Vu's approach can also be applied to prove edge universality [92]. In the last paper [93] the subexponential decay was replaced with a condition of a sufficiently strong polynomial decay. The support and the third moment condition can be removed by combining [91] with the result from our approach [42] and this has led to the universality of hermitian matrices [43] for any distribution including the Bernoulli measure. On the other hand, even for hermitian matrices, the variances of the matrix elements are required to be constant in this approach.

Historically, Tao-Vu's first paper on the universality [91] appeared shortly after the paper [42] on the universality of hermitian matrices. A common ingredient for both [42] and [91] is the local semicircle law and the eigenfunction estimates that were essentially available from [38, 39], but due to certain technical conditions, they were reproved in [91]. The local semicircle law for sample covariance matrices was first proved in [44] and a slightly different version was given in [93] with some change in the technical assumptions tailored to the application.

The four moment condition first appeared in the four moment theorem by Tao-Vu [91] (Theorem 1.5) and it was used in the Green function comparison theorem [46] (Theorem 1.4). The four moment theorem concerns individual eigenvalues and thus it contains information about the eigenvalue gap distribution directly. In order to translate this information into correlation functions, the locations of individual eigenvalues of the comparison ensemble are required. The Green function comparison theorem, on the other hand, can be used to compare the correlation functions directly, but the information on the individual eigenvalues is weaker. Nevertheless, a standard exclusion-inclusion principle argument (like the one presented in (1.37)) concludes the universality of the gap distribution as well. Since individual eigenvalues tend to fluctuate and Green functions are more stable, this explains why the proof of the four moment theorem for eigenvalues is quite involved but the Green function comparison theorem is very simple. Furthermore, the Green function comparison theorem yields not only spectral information, but information on matrix elements as well.

## 1.7 Level repulsion and Wegner estimate on very short scales

The local semicircle law (1.3) can be turned into a direct estimate on the empirical density in the form

$$|\varrho_\eta(E) - \varrho_{sc}(E)| \lesssim \frac{C}{\sqrt{N\eta\kappa}}, \quad \kappa = ||E| - 2|.$$

Here  $\varrho_\eta$  denotes the empirical density  $\varrho(x) = \frac{1}{N} \sum_i \delta(\lambda_i - x)$  smoothed out on a scale  $\eta$ . This result asserts that the empirical density on scales  $\eta \gg O(1/N)$  is close to the semicircle density. On even smaller scales  $\eta \leq O(1/N)$ , the empirical density fluctuates, but its average,  $\mathbb{E} \varrho_\eta(E)$ ,



remains bounded uniformly in  $\eta$ . This is a type of *Wegner estimate* that plays a central role in the localization theory of random Schrödinger operators. In particular, it says that the probability of finding at least one eigenvalue in an interval  $I$  of size  $\eta = \varepsilon/N$  is bounded by  $C\varepsilon$  uniformly in  $N$  and  $\varepsilon \leq 1$ , i.e., no eigenvalue can stick to any point in the real line. Note that if the eigenvalues were independent (Poisson process), then the probability of finding  $n = 1, 2, 3, \dots$  eigenvalues in  $I$  were proportional with  $\varepsilon^n$ . For random matrices in the bulk of the spectrum this probability is much smaller. This phenomenon is known as *level repulsion*. The precise statement is the following:

**Theorem 1.6** [39, Theorem 3.4 and 3.5] *Consider symmetric or hermitian Wigner matrices with a single entry distribution  $\nu$  that has a Gaussian decay. Suppose  $\nu$  is absolutely continuous with a strictly positive and smooth density. Let  $|E| < 2$  and  $I = [E - \eta/2, E + \eta/2]$  with  $\eta = \varepsilon/N$  and let  $\mathcal{N}_I$  denote the number of eigenvalues in  $I$ . Then for any fixed  $n$ ,*

$$\mathbb{P}(\mathcal{N}_I \geq n) \leq \begin{cases} C_n \varepsilon^{n^2} & [\text{hermitian case}] \\ C_n \varepsilon^{n(n+1)/2} & [\text{symmetric case}] \end{cases} \quad (1.78)$$

uniformly in  $\varepsilon \leq 1$  and for all sufficiently large  $N$ .

The exponents are optimal as one can easily see from the Vandermonde determinant in the joint probability density (1.42) for unitary ensembles. The sine kernel behavior (1.33) implies level repulsion and even a lower bound on  $\mathbb{P}(\mathcal{N}_I \geq n)$ , but usually not on arbitrarily small scales since sine kernel is typically proven only as a weak limit (see (1.34)).

We also mention that (1.77) (Theorem 17 from [91]) is a certain type of level repulsion bound, but the exponents are not optimal and it does not hold on arbitrary small scales. However, the virtue of (1.77) is that it assumes no smoothness of the distribution, in particular it holds for discrete distributions as well. Clearly, say, for Bernoulli distribution, even the Wegner estimate, (1.78) for  $n = 1$ , cannot hold on superexponentially small scales  $\varepsilon \sim 2^{-N^2}$ .

*Sketch of the proof.* The first step of the proof is to provide an upper bound on  $\mathcal{N}_I$ . Let  $B^{(k)}$  denote the  $(N-1) \times (N-1)$  minor of  $H$  after removing the  $k$ -th row and  $k$ -th column. Let  $\lambda_\alpha^{(k)}$ ,  $\alpha = 1, 2, \dots, N-1$  denote the eigenvalues of  $B^{(k)}$  and  $\mathbf{u}_\alpha^{(k)}$  denote its eigenvectors. Computing the  $(k, k)$  diagonal element of the resolvent  $(H - z)^{-1}$  we easily obtain the following expression for  $m(z) = m_N(z)$

$$m(z) = \frac{1}{N} \sum_{k=1}^N \frac{1}{H - z}(k, k) = \frac{1}{N} \sum_{k=1}^N \left[ h_{kk} - z - \frac{1}{N} \sum_{\alpha=1}^{N-1} \frac{\xi_\alpha^{(k)}}{\lambda_\alpha^{(k)} - z} \right]^{-1}, \quad (1.79)$$

where

$$\xi_\alpha^{(k)} = N |\mathbf{a}^{(k)} \cdot \mathbf{u}_\alpha^{(k)}|^2, \quad (1.80)$$



and  $\mathbf{a}^{(k)}$  is the  $k$ -th column of  $H$  without the diagonal element  $h_{kk}$ . Taking the imaginary part, and using  $\mathcal{N}_I \leq C \Im m(z)$ , we have

$$\mathcal{N}_I \leq CN\eta^2 \sum_{k=1}^N \left| \sum_{\alpha: \lambda_\alpha^{(k)} \in I} \xi_\alpha^{(k)} \right|^{-1}. \quad (1.81)$$

It is an elementary fact that the eigenvalues of  $H$  and  $B^{(k)}$ , for each fixed  $k$ , are interlaced, i.e., the number of  $\lambda_\alpha^{(k)}$  in  $I$  is at least  $\mathcal{N}_I - 1$ . For each fixed  $k$  the random variables  $\{\xi_\alpha^{(k)} : \alpha = 1, 2, \dots, N-1\}$  are almost independent and have expectation value one, thus the probability of the event

$$\Omega_k := \left\{ \sum_{\alpha: \lambda_\alpha^{(k)} \in I} \xi_\alpha^{(k)} \leq \delta(\mathcal{N}_I - 1) \right\}$$

is negligible for small  $\delta$  [39, Lemma 4.7]. On the complement of all  $\Omega_k$  we thus have from (1.81) that

$$\mathcal{N}_I \leq \frac{CN^2\eta^2}{\delta(\mathcal{N}_I - 1)},$$

from which it follows that  $\mathcal{N}_I \leq CN\eta$  with very high probability. One of the precise results of this type is

**Lemma 1.7 (Theorem 4.6 of [39])** *Assuming the single entry distribution  $\nu$  has a Gaussian decay, then for any interval  $I$  with  $|I| \geq (\log N)/N$  we have*

$$\mathbb{P}(\mathcal{N}_I \geq KN|I|) \leq Ce^{-c\sqrt{KN|I|}}.$$

We remark that the Gaussian decay condition can be weakened and a somewhat weaker result holds also for shorter intervals  $|I| \geq 1/N$  (see Theorem 5.1 [39]).

The starting point for the proof of Theorem 1.6 is formula (1.79) together with

$$\mathcal{N}_I \leq CN\eta \Im m(E + i\eta).$$

This implies

$$\mathcal{N}_I \leq C\eta \sum_{k=1}^N \frac{1}{(a_k^2 + b_k^2)^{1/2}} \quad (1.82)$$

with

$$a_k := \eta + \frac{1}{N} \sum_{\alpha=1}^{N-1} \frac{\eta \xi_\alpha^{(k)}}{(\lambda_\alpha^{(k)} - E)^2 + \eta^2}, \quad b_k := h_{kk} - E - \frac{1}{N} \sum_{\alpha=1}^{N-1} \frac{(\lambda_\alpha^{(k)} - E) \xi_\alpha^{(k)}}{(\lambda_\alpha^{(k)} - E)^2 + \eta^2},$$

where  $a_k$  and  $b_k$  are the imaginary and real part, respectively, of the reciprocal of the summands in (1.79) and  $\xi_\alpha^{(k)}$  was defined in (1.80). The proof of Lemma 1.7 relied only on the imaginary part, i.e.,  $b_k$  in (1.82) was neglected in the estimate (1.81). In the proof of Theorem 1.6, however, we make an essential use of  $b_k$  as well. Since typically  $1/N \lesssim |\lambda_\alpha^{(k)} - E|$ , we note that  $a_k^2$  is much smaller than  $b_k^2$  if  $\eta \ll 1/N$  and this is the relevant regime for the Wegner estimate and for the level repulsion.

Assuming a certain smoothness on the distribution  $d\nu$ , the distribution of the variables  $\xi_\alpha^{(k)}$  will also be smooth even if we fix an index  $k$  and we condition on the minor  $B^{(k)}$ , i.e., if we fix the eigenvalues  $\lambda_\alpha^{(k)}$  and the eigenvectors  $\mathbf{u}_\alpha^{(k)}$ . Although the random variables  $\xi_\alpha^{(k)} = N|\mathbf{a}^{(k)} \cdot \mathbf{u}_\alpha^{(k)}|^2$  are not independent for different  $\alpha$ 's, they are sufficiently decorrelated so that the distribution of  $b_k$  inherits some smoothness from  $\mathbf{a}^{(k)}$ . Sufficient smoothness on the distribution of  $b_k$  makes the expectation value  $(a_k^2 + b_k^2)^{-p/2}$  finite for any  $p > 0$ . This will give a bound on the  $p$ -th moment on  $\mathcal{N}_I$  which will imply (1.78).

We present this idea for hermitian matrices and for the simplest case  $n = 1$ . From (1.82) we have

$$\mathbb{P}(\mathcal{N}_I \geq 1) \leq \mathbb{E} \mathcal{N}_I^2 \leq C(N\eta)^2 \mathbb{E} \frac{1}{a_1^2 + b_1^2}.$$

Dropping the superscript  $k = 1$  and introducing the notation

$$d_\alpha = \frac{N(\lambda_\alpha - E)}{N^2(\lambda_\alpha - E)^2 + \varepsilon^2}, \quad c_\alpha = \frac{\varepsilon}{N^2(\lambda_\alpha - E)^2 + \varepsilon^2},$$

we have

$$\mathbb{P}(\mathcal{N}_I \geq 1) \leq C\varepsilon^2 \mathbb{E} \left[ \left( \sum_{\alpha=1}^{N-1} c_\alpha \xi_\alpha \right)^2 + \left( h - E - \sum_{\alpha=1}^{N-1} d_\alpha \xi_\alpha \right)^2 \right]^{-1}. \quad (1.83)$$

From the local semicircle law (see Theorem 1.8 below), we know that with a very high probability, there are several eigenvalues  $\lambda_\alpha$  within a distance of  $O(1/N)$  of  $E$ . Choosing four such eigenvalues, we can guarantee that for some index  $\gamma$

$$c_\gamma, c_{\gamma+1} \geq C\varepsilon, \quad d_{\gamma+2}, d_{\gamma+3} \geq C \quad (1.84)$$

for some positive constant  $C$ . If  $\xi_\alpha$ 's were indeed independent and distributed according to the square of a complex random variable  $z_\alpha$  with a smooth and decaying density  $d\mu(z)$  on the complex plane, then the expectation in (1.83) would be bounded by

$$\sup_E \int \frac{1}{(c_\gamma |z_\gamma|^2 + c_{\gamma+1} |z_{\gamma+1}|^2)^2 + (E - d_{\gamma+2} |z_{\gamma+2}|^2 - d_{\gamma+3} |z_{\gamma+3}|^2)^2} \prod_{j=0}^3 d\mu(z_{\gamma+j}). \quad (1.85)$$

Simple calculation shows that this integral is bounded by  $C\varepsilon^{-1}$  assuming the lower bounds (1.84). Combining this bound with (1.83), we obtain (1.78) for  $n = 1$ . The proof for the general  $n$  goes

by induction. The difference between the hermitian and the symmetric cases manifests itself in the fact that  $\xi_\alpha$ 's are squares of complex or real variables, respectively. This gives different estimates for integrals of the type (1.85), resulting in different exponents in (1.78).  $\square$

In this proof we used the following version of the local semicircle law:

**Theorem 1.8** [Theorem 3.1 [39]] *Let  $H$  be an  $N \times N$  hermitian or symmetric Wigner matrix with a single entry distribution satisfying Gaussian decay. Let  $\kappa > 0$  and fix an energy  $E \in [-2+\kappa, 2-\kappa]$ . Then there exist positive constants  $C, c$ , depending only on  $\kappa$ , and a universal constant  $c_1 > 0$  such that the following hold:*

(i) *For any  $\delta \leq c_1\kappa$  and  $N \geq 2$  we have*

$$\mathbb{P}(|m(E+i\eta) - m_{sc}(E+i\eta)| \geq \delta) \leq C e^{-c\delta\sqrt{N\eta}} \quad (1.86)$$

*for any  $K/(N\sqrt{E}) \leq \eta \leq 1$ , where  $K$  is a large universal constant.*

(ii) *Let  $\mathcal{N}_{\eta^*}(E) = \mathcal{N}_{I^*}$  denote the number of eigenvalues in the interval  $I^* := [E-\eta^*/2, E+\eta^*/2]$ . Then for any  $\delta \leq c_1\kappa$  there is a constant  $K_\delta$ , depending only on  $\delta$ , such that*

$$\mathbb{P}\left\{\left|\frac{\mathcal{N}_{\eta^*}(E)}{N\eta^*} - \varrho_{sc}(E)\right| \geq \delta\right\} \leq C e^{-c\delta^2\sqrt{N\eta^*}} \quad (1.87)$$

*holds for all  $\eta^*$  satisfying  $K_\delta/N \leq \eta^* \leq 1$  and for all  $N \geq 2$ .*

## 2 Local semicircle law and delocalization

Each approach that proves bulk universality for generalized Wigner matrices requires first to analyze the local density of eigenvalues. The Wigner semicircle law [100] (and its analogue for Wishart matrices, the Marchenko-Pastur law [67]) has traditionally been among the first results established on random matrices. Typically, however, the empirical density is shown to converge weakly on macroscopic scales, i.e., on intervals that contain  $O(N)$  eigenvalues. Based upon our results [37, 38, 39, 41, 46, 47], here we show that the semicircle law holds on much smaller scales as well. In Section 2.2 we follow the formalism of [39], while in Section 2.3 we use [46, 47]. The former formalism directly aims at the Stieltjes transform, or the trace of the resolvent; the latter formalism is designed to establish the semicircle law for individual diagonal elements of the resolvent. Finally, in Section 2.4, we indicate how to prove delocalization of eigenvectors from local semicircle law.

### 2.1 Resolvent formulas

For definiteness, we do the proof for the hermitian case, but all formulas below carry over to the other symmetry classes with obvious modifications. We first collect a few useful formulas about resolvents. Their proofs are elementary results from linear algebra.

**Lemma 2.1** *Let  $A, B, C$  be  $n \times n, n \times m$  and  $m \times m$  matrices. We define  $(m + n) \times (m + n)$  matrix  $D$  as*

$$D = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \quad (2.1)$$

and  $n \times n$  matrix  $\widehat{D}$  as

$$\widehat{D} = A - B^*C^{-1}B. \quad (2.2)$$

Then for any  $1 \leq i, j \leq n$ , we have

$$(D^{-1})_{ij} = (\widehat{D}^{-1})_{ij} \quad (2.3)$$

for the corresponding matrix elements. □

Recall that  $G_{ij} = G_{ij}(z)$  denotes the matrix element of the resolvent

$$G_{ij} = \left( \frac{1}{H - z} \right)_{ij}.$$

Let  $G^{(i)}$  denote the resolvent of  $H^{(i)}$ , which is the  $(N-1) \times (N-1)$  minor of  $H$  obtained by removing the  $i$ -th row and column. Let  $\mathbf{a}^i = (h_{1i}, h_{2i}, \dots, h_{Ni})^t$  be the  $i$ -th column of  $H$ , sometimes after removing one or more elements. We always keep the original labelling of the rows and columns, so there will be no confusion: if  $\mathbf{a}^i$  is multiplied by a matrix whose  $j$ -th column and row are removed, then we remove the  $j$ -th entry from  $\mathbf{a}^i$  as well. With similar conventions, we can define  $G^{(ij)}$  etc. The superscript in parenthesis for resolvents always means “after removing the corresponding row and column”, in particular, by independence of matrix elements, this means that the matrix  $G^{(ij)}$  is independent of the  $i$ -th and  $j$ -th row and column of  $H$ .

Using Lemma 2.1 for  $n = 1, m = N - 1$ , we have

$$G_{ii} = \frac{1}{h_{ii} - z - \mathbf{a}^i \cdot G^{(i)} \mathbf{a}^i}. \quad (2.4)$$

For the  $i = 1$  case we can easily visualize it by

$$H = \begin{pmatrix} h_{11} & [\mathbf{a}^1]^* \\ \mathbf{a}^1 & H^{(1)} \end{pmatrix}$$

and noting that the first column of the resolvent,  $(G_{11}, G_{21}, \dots, G_{N1})^t = (x, \mathbf{w})^t$ ,  $x = G_{11}$ , solves the equation

$$\begin{aligned} (h_{11} - z)x + \mathbf{a}^1 \cdot \mathbf{w} &= 1 \\ x\mathbf{a}^1 + (H^{(1)} - z)\mathbf{w} &= 0. \end{aligned}$$

From the second equation  $\mathbf{w} = -xG^{(1)}\mathbf{a}^1$ , and then we express  $x$  from the first equation to get (2.4) for  $i = 1$ .

For the offdiagonal elements, one has to do a two-row expansion. Consider

$$H = \begin{pmatrix} h_{11} & h_{12} & \dots & [\mathbf{a}^1]^* & \dots \\ h_{21} & h_{22} & \dots & [\mathbf{a}^2]^* & \dots \\ \vdots & \vdots & & & \\ \mathbf{a}^1 & \mathbf{a}^2 & & H^{(12)} & \\ \vdots & \vdots & & & \end{pmatrix}.$$

With the notation  $D = H - z$  in Lemma 2.1 for  $n = 2$ ,  $m = N - 2$ , we can compute the  $\widehat{D}$  matrix which in this case we will call  $K^{(12)}$ :

$$\widehat{D} = \begin{pmatrix} h_{11} - z - \mathbf{a}^1 \cdot G^{(12)}\mathbf{a}^1 & h_{12} - \mathbf{a}^1 \cdot G^{(12)}\mathbf{a}^2 \\ h_{21} - \mathbf{a}^2 \cdot G^{(12)}\mathbf{a}^1 & h_{22} - z - \mathbf{a}^2 \cdot G^{(12)}\mathbf{a}^2 \end{pmatrix} =: \begin{pmatrix} K_{11}^{(12)} & K_{12}^{(12)} \\ K_{21}^{(12)} & K_{22}^{(12)} \end{pmatrix} \quad (2.5)$$

where we conveniently introduced

$$K_{ij}^{(12)} = h_{ij} - z\delta_{ij} - \mathbf{a}^i \cdot G^{(12)}\mathbf{a}^j, \quad i, j = 1, 2. \quad (2.6)$$

Thus, from Lemma 2.1, we have, e.g.

$$G_{11} = \frac{K_{22}^{(12)}}{K_{22}^{(12)}K_{11}^{(12)} - K_{12}^{(12)}K_{21}^{(12)}}, \quad (2.7)$$

and

$$G_{12} = -\frac{K_{12}^{(12)}}{K_{22}^{(12)}K_{11}^{(12)} - K_{12}^{(12)}K_{21}^{(12)}} = -G_{22}\frac{K_{12}^{(12)}}{K_{11}^{(12)}} = -G_{22}G_{11}^{(2)}K_{12}^{(12)}. \quad (2.8)$$

In the last step we used

$$G_{11}^{(2)} = \frac{1}{K_{11}^{(12)}} \quad (2.9)$$

which is exactly the one-row expansion (2.4) applied to the  $H^{(2)}$  minor of  $H$  after removing the second row/column.

There is another set of formulas that express how to compare resolvents of  $H$  and  $H^{(1)}$ , for example, for any  $i \neq j$ .

$$G_{ii} = G_{ii}^{(j)} + \frac{G_{ij}G_{ji}}{G_{jj}}. \quad (2.10)$$

This can be easily checked on a two by two matrix and its inverse:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad M^{-1} = \frac{1}{\Delta} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}, \quad \text{with} \quad \Delta = ad - bc,$$

so checking (2.10), e.g. for  $i = 1, j = 2$  boils down to the identity

$$\frac{d}{\Delta} = \frac{1}{a} + \frac{\frac{c}{\Delta} \frac{b}{\Delta}}{\frac{a}{\Delta}}.$$

For larger matrices, one just uses (2.3). Note that in formulas (2.7), (2.8) and (2.9) we already expressed all resolvents appearing in (2.10) in terms of the matrix elements of the two by two  $K^{(12)}$  matrix (2.5) which can play the role of  $M$  above.

Similarly one has for any three different indices  $i, j, k$  that

$$G_{ij} = G_{ij}^{(k)} + \frac{G_{ik}G_{kj}}{G_{kk}}. \quad (2.11)$$

This identity can be checked on 3 by 3 matrices and we leave it to the reader.

## 2.2 Semicircle via resolvents: Sketch of a crude method

In this section we sketch the proof of

**Theorem 2.2** *Let  $z = E + i\eta$ ,  $1/N \ll \eta \ll 1$  and  $\kappa := ||E| - 2|$ . Let  $H$  be a Wigner matrix and  $G(z) = (H - z)^{-1}$  its resolvent. We assume that the single entry distribution  $\nu$  has Gaussian decay ( $\alpha = 1/2$  in (1.69)). Then we have the following approximation for  $m(z) = \frac{1}{N} \text{Tr} G(z)$*

$$|m(z) - m_{sc}(z)| \leq \min \left\{ \frac{(\log N)^C}{\sqrt{N\eta\kappa}}, \frac{(\log N)^C}{(N\eta)^{1/4}} \right\} \quad (2.12)$$

*with a very high probability.*

We proved the local semicircle law in this form in Proposition 8.1. of [44] for sample covariance matrices (replacing semicircle with the Marchenko-Pastur distribution), but the same (or even easier) proof applies to Wigner matrices. The original proof was presented with a Gaussian decay condition, but it can easily be relaxed to subexponential decay (1.69), this affects only the tail estimate of the probability that the event (2.12) is violated. [For technical experts: in our previous papers, up to [44], we typically used the Hanson-Wright theorem [59] to estimate large deviation probabilities of quadratic forms. This gives a very good control for the tail, but requires Gaussian decay. In our more recent papers we use Lemma 2.13 based upon martingale inequalities, which requires only subexponential decay, and in fact can be relaxed to polynomial decay as well, but the tail probability estimate is weaker.]

For the proof, we start with the identity (2.4) and express  $G_{ii}$

$$G_{ii} = \frac{1}{h_{ii} - z - \mathbb{E}_i \mathbf{a}^i \cdot G^{(i)} \mathbf{a}^i - Z_i}, \quad (2.13)$$

where we split

$$\mathbf{a}^i \cdot G^{(i)} \mathbf{a}^i = \mathbb{E}_i \mathbf{a}^i \cdot G^{(i)} \mathbf{a}^i + Z_i, \quad Z_i := \mathbf{a}^i \cdot G^{(i)} \mathbf{a}^i - \mathbb{E}_i \mathbf{a}^i \cdot G^{(i)} \mathbf{a}^i$$

into its expectation and fluctuation, where  $\mathbb{E}_i$  denotes the expectation with respect to the variables in the  $i$ -th column/row. In particular,  $G^{(i)}$  is independent of  $\mathbf{a}^i$ , so we need to compute expectations and fluctuations of quadratic functions.

The expectation is easy

$$\mathbb{E}_i \mathbf{a}^i \cdot G^{(i)} \mathbf{a}^i = \mathbb{E}_i \sum_{kl \neq i} \overline{\mathbf{a}_k^i} G_{kl}^{(i)} \mathbf{a}_l^i = \sum_{kl \neq i} \mathbb{E}_i \overline{h_{ik}} G_{kl}^{(i)} h_{il} = \frac{1}{N} \sum_{k \neq i} G_{kk}^{(i)},$$

where in the last step we used that different matrix elements are independent, i.e.  $\mathbb{E}_i \overline{h_{ik}} h_{il} = \frac{1}{N} \delta_{kl}$ . The summations always run over all indices from 1 to  $N$ , unless those that are explicitly excluded.

Recall that

$$m(z) = \frac{1}{N} \text{Tr} G(z) = \frac{1}{N} \sum_{k=1}^N G_{kk}(z).$$

and we can similarly define

$$m^{(i)}(z) = \frac{1}{N-1} \text{Tr} G^{(i)}(z) = \frac{1}{N-1} \sum_{k \neq i} G_{kk}^{(i)}(z).$$

We have the following lemma to compare the trace of  $G$  and  $G^{(i)}$ :

**Lemma 2.3** *We have for any  $i$*

$$|m(z) - m^{(i)}(z)| \leq \frac{C}{N\eta}, \quad \eta = \Im z > 0. \quad (2.14)$$

*Proof.* Every self-adjoint matrix  $H$  and its minor  $H^{(i)}$  have *interlacing eigenvalues*, i.e., if  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$  are the eigenvalues of  $H$  and  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_{N-1}$  are the eigenvalues of  $H^{(i)}$ , then we have

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \lambda_3 \leq \dots \leq \lambda_{N-1} \leq \mu_{N-1} \leq \lambda_N.$$

The proof is an amusing exercise. One way to see it is first to derive the following equation for the eigenvalues  $\lambda$  in terms of the eigenvalues  $\mu$ 's

$$\lambda - h_{ii} = \sum_{\alpha=1}^{N-1} \frac{|\mathbf{a}^i \cdot \mathbf{u}_\alpha|^2}{\lambda - \mu_\alpha}.$$

Here  $\mathbf{u}_\alpha$  is the normalized eigenvector of the minor  $H^{(i)}$  belonging to  $\mu_\alpha$ . Then analyzing this equation how the left and right hand sides behave as a function of  $\lambda$ , one sees that there is exactly one solution  $\lambda$  in each interval  $[\mu_\alpha, \mu_{\alpha+1}]$ .

Once the interlacing property is established, let

$$F(x) = \frac{1}{N} \#\{\lambda_j \leq x\}, \quad F^{(i)}(x) = \frac{1}{N-1} \#\{\mu_j \leq x\},$$

denote the normalized counting functions of the eigenvalues. The interlacing property in terms of these functions means that

$$\sup_x |NF(x) - (N-1)F^{(i)}(x)| \leq 1.$$

Then, after integrating by parts,

$$\begin{aligned} \left| m(z) - \left(1 - \frac{1}{N}\right) m^{(i)}(z) \right| &= \left| \int \frac{dF(x)}{x-z} - \left(1 - \frac{1}{N}\right) \int \frac{dF^{(i)}(x)}{x-z} \right| \\ &= \frac{1}{N} \left| \int \frac{NF(x) - (N-1)F^{(i)}(x)}{(x-z)^2} dx \right| \\ &\leq \frac{1}{N} \int \frac{dx}{|x-z|^2} \leq \frac{C}{N\eta}, \end{aligned} \tag{2.15}$$

and this, together with the trivial bound  $|m^{(i)}| \leq \eta^{-1}$ , proves (2.14).  $\square$

Returning to (2.13), we have

$$G_{ii} = \frac{1}{-z - m(z) + \Omega_i} \tag{2.16}$$

where

$$\Omega_i = h_{ii} - Z_i + O\left(\frac{1}{N\eta}\right). \tag{2.17}$$

Summing up (2.16), we get

$$m = \frac{1}{N} \sum_i \frac{1}{-z - m(z) + \Omega_i}. \tag{2.18}$$

Suppose that  $\Omega := \max_i |\Omega_i|$  is small and  $|z + m| \geq C > 0$ , thus an expansion is possible, i.e.,

$$\frac{1}{-z - m(z) + \Omega_i} = \frac{1}{-z - m(z)} + O(\Omega).$$

Then we have the following *self-consistent equation* for  $m$

$$m + \frac{1}{z + m} = O(\Omega). \tag{2.19}$$



We recall that the Stieltjes transform of the semicircle law

$$m_{sc}(z) = \int_{\mathbb{R}} \frac{\varrho_{sc}(x) dx}{x - z}$$

can also be characterized as the only solution to the quadratic equation

$$m_{sc}(z) + \frac{1}{z + m_{sc}(z)} = 0 \quad (2.20)$$

with  $\Im m_{sc}(z) > 0$  for  $\Im z > 0$ . We can thus use the stability of the equation (2.20) to identify  $m$ , the solution to (2.19). The stability deteriorates near the spectral edges  $z \sim \pm 2$  and we have the following precise result that can be proved by elementary calculus (Lemma 8.4 of [44]):

**Lemma 2.4** *Fix  $z = E + i\eta$ ,  $\eta > 0$ , and set  $\kappa := ||E| - 2|$ . Suppose that  $m = m(z)$  has positive imaginary part. Then, if*

$$\left| m + \frac{1}{z + m} \right| \leq \delta,$$

then

$$|m - m_{sc}(z)| \leq \frac{C\delta}{\sqrt{\kappa + \delta}}.$$

□

Applying this result, we obtain

$$|m(z) - m_{sc}(z)| \leq \frac{C\Omega}{\sqrt{\kappa + \Omega}}. \quad (2.21)$$

We now give a rough bound on the size of  $\Omega$ . Clearly  $|h_{ii}| \lesssim N^{-1/2}$ . If the single entry distribution has a subexponential decay, then we can guarantee that *all* diagonal elements simultaneously satisfy essentially this bound with a very high probability. Recall that (1.69) implies

$$\mathbb{P}\left(|h_{ii}| \geq M^\alpha N^{-1/2}\right) \leq Ce^{-M}$$

for each  $i$ . Choosing  $M = (\log N)^{1+\varepsilon}$ , we have

$$\mathbb{P}\left(\exists i : |h_{ii}| \geq (\log N)^{(1+\varepsilon)\alpha} N^{-1/2}\right) \leq CN e^{-(\log N)^{1+\varepsilon}} \leq Ce^{-(\log N)^{1+\varepsilon}}, \quad (2.22)$$

which is faster than any polynomial decay.

To estimate  $Z_i$ , we compute its second moment

$$\mathbb{E}|Z_i|^2 = \sum_{kl \neq i} \sum_{k'l' \neq i} \mathbb{E}_i \left( \left[ \bar{h}_{ik} G_{kl}^{(i)} h_{il} - \mathbb{E}_i \bar{h}_{ik} G_{kl}^{(i)} h_{il} \right] \left[ h_{ik'} \bar{G}_{k'l'}^{(i)} \bar{h}_{il'} - \mathbb{E}_i h_{ik'} \bar{G}_{k'l'}^{(i)} \bar{h}_{il'} \right] \right). \quad (2.23)$$

Since  $\mathbb{E}h = 0$ , the non-zero contributions to this sum come from index combinations when all  $h$  and  $\bar{h}$  are paired. For pedagogical simplicity, assume that  $\mathbb{E}h^2 = 0$ , this can be achieved, for example, if the distribution of the real and imaginary parts are the same. Then the  $h$  factors in the above expression have to be paired in such a way that  $h_{ik} = h_{ik'}$  and  $h_{il} = h_{il'}$ , i.e.,  $k = k'$ ,  $l = l'$ . Note that pairing  $h_{ik} = h_{il}$  would give zero because we subtracted the expectation. The result is

$$\mathbb{E}|Z_i|^2 = \frac{1}{N^2} \sum_{kl \neq i} |G_{kl}^{(i)}|^2 + \frac{m_4 - 1}{N^2} \sum_{k \neq i} |G_{kk}^{(i)}|^2, \quad (2.24)$$

where  $m_4 = \mathbb{E}|\sqrt{N}h|^4$  is the fourth moment of the single entry distribution. The first term can be computed

$$\frac{1}{N^2} \sum_{kl \neq i} |G_{kl}^{(i)}|^2 = \frac{1}{N^2} \sum_{k \neq i} (|G^{(i)}|^2)_{kk} = \frac{1}{N\eta} \frac{1}{N} \sum_k \Im G_{kk}^{(i)} = \frac{1}{N\eta} \Im m^{(i)}, \quad (2.25)$$

where  $|G|^2 = GG^*$  and we used the identity

$$|G|^2 = \frac{1}{|H - E|^2 + \eta^2} = \frac{1}{\eta} \Im G.$$

To estimate this quantity, we need an upper bound on the local density on scale  $\eta \gg 1/N$ . For any interval  $I \subset \mathbb{R}$ , let

$$\mathcal{N}_I = \#\{\lambda_j \in I\}$$

denote the number of eigenvalues in  $I$ . In Section 1.7 we already sketched the proof of:

**Lemma 2.5 (Theorem 4.6 of [39])** *Assuming the single entry distribution  $\nu$  has a Gaussian decay, then for any interval  $I$  with  $|I| \geq (\log N)/N$  we have*

$$\mathbb{P}(\mathcal{N}_I \geq KN|I|) \leq Ce^{-c\sqrt{KN|I|}}.$$

We remark that the Gaussian decay condition can be weakened and a somewhat weaker result holds also for shorter intervals  $|I| \geq 1/N$  (see Theorem 5.1 [39]).

Using this lemma for the matrix  $H^{(i)}$  with eigenvalues  $\mu_1, \mu_2, \dots, \mu_{N-1}$ , we have

$$\Im m^{(i)} = \frac{1}{N-1} \sum_{\alpha} \frac{\eta}{(E - \mu_{\alpha})^2 + \eta^2} \lesssim \int \frac{\eta}{(x - E)^2 + \eta^2} \varrho_{sc}(x) dx \lesssim O(1). \quad (2.26)$$

Recall that the sign  $\lesssim$  here means “up to logarithmic factors”.

The second term in (2.24) we can estimate by using the trivial bound  $|G_{kk}^{(i)}| \leq \eta^{-1}$  and thus

$$\frac{1}{N^2} \sum_{k \neq i} |G_{kk}^{(i)}|^2 \leq \frac{1}{N^2\eta} \sum_{k \neq i} |G_{kk}^{(i)}| = \frac{1}{N^2\eta} \sum_{k \neq i} \sum_{\alpha=1}^{N-1} \frac{|\mathbf{u}_{\alpha}(k)|^2}{|\lambda_{\alpha} - z|} \leq \frac{1}{N\eta} \frac{1}{N} \sum_{\alpha=1}^{N-1} \frac{1}{|\lambda_{\alpha} - z|} \lesssim \frac{1}{N\eta}, \quad (2.27)$$

where  $\mathbf{u}_\alpha$  is a (normalized) eigenvector to  $\mu_\alpha$ . In the last step we used an estimate similar to (2.26).

The estimates (2.25) and (2.27) confirm that the size of  $Z_i$  is roughly

$$|Z_i| \lesssim \frac{1}{\sqrt{N\eta}} \quad (2.28)$$

at least in second moment sense. One can compute higher moments or use even stronger concentration results (if, for example, logarithmic Sobolev inequality is available), to strengthen (2.28) so that it holds in the sense of probability.

Altogether (2.17), (2.22) and (2.28) give

$$\Omega \lesssim \frac{C}{\sqrt{N}} + \frac{C}{\sqrt{N\eta}} + \frac{C}{N\eta}.$$

Since we are interested in  $1/N \ll \eta \leq 1$ , we get that  $\Omega \lesssim (N\eta)^{-1/2}$ . Combining this with the stability bound, (2.21), we have

$$|m(z) - m_{sc}(z)| \lesssim \min \left\{ \frac{1}{\sqrt{N\eta\kappa}}, \frac{1}{(N\eta)^{1/4}} \right\},$$

which proves Lemma 2.2, modulo several technical issues.  $\square$

The remarkable feature is that the method works down to scales  $\eta \sim 1/N$ . The factor  $\kappa$  expresses the fact that the estimate deteriorates near the edge.

The exponents are not optimal; in fact  $m - m_{sc}$  can be compared with a precision of order  $(N\eta)^{-1}$ . This will be presented in the next section. The gain will come from the fact that the main error term  $Z_i$  in (2.17) is fluctuating and it is possible to show [46] that its contributions to  $m - m_{sc}$  cancel to leading order and it is smaller than the size of  $Z_i$  predicted by the variance calculation (this effect was first exploited in Theorem 4.1 of [41] and substantially improved in [46] and [47]).

## 2.3 Semicircle via resolvents: refined method

### 2.3.1 Statement of the theorem and consequences

In this section we present a more refined method that can estimate matrix elements of the resolvent. This is also a key ingredient for the improved precision on the semicircle law both in terms of  $(N\eta)$ -power and edge behavior. The main ingredient is to analyze a self-consistent equation for the vector of the diagonal elements of the resolvent,  $(G_{11}, G_{22}, \dots, G_{NN})$ , instead of their sum which has led to (2.18). For definiteness, we formulate the result for hermitian matrices only, but the same result holds for other symmetry classes as well.

A key quantity will be the matrix of variances,  $\Sigma$ , introduced in (1.16). Recall that  $\Sigma$  is symmetric, doubly stochastic by (1.17), and in particular it satisfies  $-1 \leq \Sigma \leq 1$ . Let the spectrum of  $\Sigma$  be supported in

$$\text{Spec}(\Sigma) \subset [-1 + \delta_-, 1 - \delta_+] \cup \{1\} \quad (2.29)$$

with some nonnegative constants  $\delta_{\pm}$ . We will always have the following spectral assumption

$$1 \text{ is a simple eigenvalue of } \Sigma \text{ and } \delta_- \text{ is a positive constant, independent of } N. \quad (2.30)$$

For Wigner matrices, all entries of  $\Sigma$  are identical and  $\delta_{\pm} = 1$ . It is easy to prove that (2.30) holds for *random band matrices*, see (1.18) for the definition, with  $\delta_- > 0$ , depending only on  $f$ . We give the proof in Appendix A. For *generalized Wigner matrices*, i.e., Wigner matrices with comparable variances, see (1.62) for the definition, it is easy to check that

$$\delta_{\pm} \geq C_{inf} > 0.$$

The fact that  $\delta_+ > 0$  for generalized Wigner matrices allows better control in terms of the edge behavior of the estimates. This is the main reason why the statement below is different for universal Wigner matrices (see (1.1)) and for generalized Wigner matrices, (1.62).

The precision of the local semicircle law depends on three factors. The first factor is the resolution (scale on which the semicircle holds), this is given by the imaginary part  $\eta = \Im z$  of the spectral parameter  $z$  in the Stieltjes transform. The second factor is the distance to the edge, measured by  $\kappa = ||E| - 2|$ . The last factor is the size of the typical matrix elements, measured by the quantity

$$M := \frac{1}{\max_{ij} \sigma_{ij}^2} \quad (2.31)$$

called the *spread* of the matrix. For example, for Wigner matrices  $M = N$ , for generalized Wigner matrices, (1.1),  $M \sim N$  and for random band matrices (1.18) we have  $M \sim W$ .

**Theorem 2.6 (Strong local semicircle law)** *Let  $H$  be a hermitian  $N \times N$  random matrix with  $\mathbb{E} h_{ij} = 0$ ,  $1 \leq i, j \leq N$ , and assume that the variances  $\sigma_{ij}^2$  satisfy (1.17) and (2.30). Suppose that the distributions of the matrix elements have a uniformly subexponential decay in the sense that there exist constants  $\alpha, \beta > 0$ , independent of  $N$ , such that for any  $x > 0$  and for each  $(i, j)$  we have*

$$\mathbb{P}(|h_{ij}| \geq x^\alpha |\sigma_{ij}|) \leq \beta e^{-x}. \quad (2.32)$$

*We consider universal Wigner matrices and its special class, the generalized Wigner matrices in parallel. The parameter  $A$  will distinguish between the two cases; we set  $A = 2$  for universal Wigner matrices, and  $A = 1$  for generalized Wigner matrices, where the results will be stronger.*

*Define the following domain in  $\mathbb{C}$*

$$D := \left\{ z = E + i\eta \in \mathbb{C} : |z| \leq 10, \eta > 0, \sqrt{M\eta} \geq (\log N)^{C_1} (\kappa + \eta)^{\frac{1}{4}-A} \right\} \quad (2.33)$$

*where  $\kappa := ||E| - 2|$ . Then there exist constants  $C_1, C_2$ , and  $c > 0$ , depending only on  $\alpha, \beta$  and  $\delta_-$  in (2.30), such that for any  $\varepsilon > 0$  and  $K > 0$  the Stieltjes transform of the empirical eigenvalue distribution of  $H$  satisfies*

$$\mathbb{P} \left( \sup_{z \in D} |m(z) - m_{sc}(z)| \geq \frac{N^\varepsilon}{M\eta(\kappa + \eta)^A} \right) \leq \frac{C(\varepsilon, K)}{N^K} \quad (2.34)$$

for sufficiently large  $N$ . Furthermore, the diagonal matrix elements of the Green function  $G_{ii}(z) = (H - z)^{-1}(i, i)$  satisfy that

$$\mathbb{P} \left( \sup_{z \in D} \max_i |G_{ii}(z) - m_{sc}(z)| \geq \frac{(\log N)^{C_2}}{\sqrt{M\eta}} (\kappa + \eta)^{\frac{1}{4} - \frac{A}{2}} \right) \leq CN^{-c(\log \log N)} \quad (2.35)$$

and for the off-diagonal elements we have

$$\mathbb{P} \left( \sup_{z \in D} \max_{i \neq j} |G_{ij}(z)| \geq \frac{(\log N)^{C_2}}{\sqrt{M\eta}} (\kappa + \eta)^{\frac{1}{4}} \right) \leq CN^{-c(\log \log N)} \quad (2.36)$$

for any sufficiently large  $N$ .

*Remark.* The subexponential decay condition (2.32) can be easily weakened if we are not aiming at error estimates faster than any power law of  $N$ . The estimates are not optimal as far as the control on the probability is concerned.

The local semicircle estimates imply that the empirical counting function of the eigenvalues is close to the semicircle counting function (Theorem 2.7) and that the location of the eigenvalues are close to their classical location in mean square deviation sense (Theorem 2.8). This latter result will be used to verify (1.64), or more precisely Assumption III (3.9) later, that will be used as the key input to our hydrodynamical approach for universality.

To formulate these statements precisely, let  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$  be the ordered eigenvalues of a universal Wigner matrix. We define the *normalized empirical counting function* by

$$\mathbf{n}(E) := \frac{1}{N} \#\{\lambda_j \leq E\} \quad (2.37)$$

and the *averaged counting function* by

$$n(E) = \frac{1}{N} \mathbb{E} \#\{\lambda_j \leq E\}. \quad (2.38)$$

Finally, let

$$n_{sc}(E) := \int_{-\infty}^E \varrho_{sc}(x) dx \quad (2.39)$$

be the distribution function of the semicircle law which is very close to the counting function of the  $\gamma$ 's,  $\frac{1}{N} \#\{\gamma_j \leq E\} \approx n_{sc}(E)$ . Recall that  $\gamma_j$ 's are the classical location of the eigenvalues, determined by the semicircle law, see (1.63).

With these notations, we have the following theorems:

**Theorem 2.7** *Let  $A = 2$  for universal Wigner matrices, satisfying (1.17), (2.31) and (2.32) with  $M \geq (\log N)^{24+6\alpha}$ . For generalized Wigner matrices, satisfying (1.17), (2.31), (1.62) and (2.32),*

we set  $A = 2$  and recall  $M = N$  in this case. Then for any  $\varepsilon > 0$  and  $K \geq 1$  there exists a constant  $C(\varepsilon, K)$  such that

$$\mathbb{P}\left\{\sup_{|E| \leq 3} |\mathbf{n}(E) - n_{sc}(E)| [\kappa_E]^A \leq \frac{CN^\varepsilon}{M}\right\} \geq 1 - \frac{C(\varepsilon, K)}{N^K},$$

where the  $\mathbf{n}(E)$  and  $n_{sc}(E)$  were defined in (2.37) and (2.39) and  $\kappa_E = ||E| - 2|$ .

**Theorem 2.8** *Let  $H$  be a generalized Wigner matrix with subexponential decay, i.e., assume that (1.17), (2.31), (1.62) and (2.32) hold. Let  $\lambda_j$  denote the eigenvalues of  $H$  and  $\gamma_j$  be their semiclassical location, defined by (1.63). Then for any  $\varepsilon < 1/7$  and for any  $K > 1$  there exists a constant  $C_K$  such that*

$$\mathbb{P}\left\{\sum_{j=1}^N |\lambda_j - \gamma_j|^2 \leq N^{-\varepsilon}\right\} \geq 1 - \frac{C_K}{N^K}. \quad (2.40)$$

and

$$\sum_{j=1}^N \mathbb{E}|\lambda_j - \gamma_j|^2 \leq CN^{-\varepsilon}. \quad (2.41)$$

These theorems were proven in Theorems 6.3 and 7.1 of [47]. They are consequences of the strong local semicircle law, Theorem 2.6. We will not give the detailed proof, but we mention a very useful formula that allows one to translate Stieltjes transforms to densities. In this context this formula first appeared in [41].

**Lemma 2.9 (Helffer-Sjöstrand, [19])** *Let  $f$  be a real valued  $C^1$  function on  $\mathbb{R}$ . Let  $\chi(y)$  be a smooth cutoff function with support in  $[-1, 1]$ , with  $\chi(y) = 1$  for  $|y| \leq 1/2$  and with bounded derivatives. Let*

$$\tilde{f}(x + iy) = (f(x) + iyf'(x))\chi(y),$$

then

$$f(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\partial_{\bar{z}} \tilde{f}(x + iy)}{\lambda - x - iy} dx dy = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{iyf''(x)\chi(y) + i(f(x) + iyf'(x))\chi'(y)}{\lambda - x - iy} dx dy. \quad (2.42)$$

We will apply this lemma in the following form. Let  $\varrho(d\lambda) \in L^1(\mathbb{R})$  be a real (signed) measure and let  $m(z)$  be its Stieltjes transform

$$m(z) = \int_{\mathbb{R}} \frac{\varrho(d\lambda)}{\lambda - z}.$$

In the application  $\varrho$  will be  $\varrho_N - \varrho_{sc}$ , i.e., the difference of the empirical density and the semicircle density. Since  $f$  is real, we have

$$\begin{aligned}
\left| \int_{\mathbb{R}} f(\lambda) \varrho(d\lambda) \right| &= \left| \operatorname{Re} \int_{\mathbb{R}} f(\lambda) \varrho(d\lambda) \right| \\
&= \left| \frac{1}{2\pi} \operatorname{Re} \int_{\mathbb{R}^2} \partial_{\bar{z}} \tilde{f}(x+iy) m(x+iy) dx dy \right| \\
&\leq \left| \frac{1}{2\pi} \int_{\mathbb{R}^2} y f''(x) \chi(y) \operatorname{Im} m(x+iy) dx dy \right| \\
&\quad + C \int_{\mathbb{R}^2} (|f(x)| + |y| |f'(x)|) |\chi'(y)| |m(x+iy)| dx dy.
\end{aligned} \tag{2.43}$$

In order to get the counting function, we will choose

$$f(\lambda) = f_{E,\eta}(\lambda),$$

where  $f_{E,\eta}$  is the characteristic function of the semi-axis  $(-\infty, E)$ , smoothed out on a scale  $\eta$ , i.e.,  $f_{E,\eta}(\lambda) \equiv 1$  for  $\lambda \leq E - \eta$ ,  $f_{E,\eta}(\lambda) \equiv 0$  for  $\lambda \geq E + \eta$ , and  $|f'| \leq C\eta^{-1}$ ,  $|f''| \leq C\eta^{-2}$  in the interval  $[E - \eta, E + \eta]$ .

The second term in (2.43) is typically harmless, since  $\chi'$  is supported at  $|y| \geq 1/2$ , i.e. this term requires information on the Stieltjes transform far away from the real axis. In the first term, we have only the imaginary part of the Stieltjes transform and it is easy to see that

$$y \Im m(x+iy) \leq \int |\varrho(dx)|.$$

One can perform an integration by parts bringing the second derivative of  $f$  down to the first derivative, which, after integration is under control even if  $\eta$  is very small. For the details, see [41] and [47].

### 2.3.2 Sketch of the proof of the strong semicircle law for matrix elements

For pedagogical reasons we will neglect the edge problem for the presentation, i.e., we will assume that  $E = \Re z$  always satisfies  $\kappa = ||E| - 2| \geq \kappa_0$  for some fixed  $\kappa_0 > 0$  and we do not follow the dependence of the constants on  $\kappa_0$ . We remark that our first two papers [37, 38] made this assumption. The edge behavior was effectively treated first in [39] and substantially improved in Theorem 4.1 of [41], but the bounds were not optimal. The best result (Theorem 2.6) is in [47].

We will thus prove the following partial version of Theorem 2.6:

**Theorem 2.10** *Assume the conditions of Theorem 2.6. For some  $\kappa_0 > 0$ , define the following domain in  $\mathbb{C}$*

$$D := \left\{ z = E + i\eta \in \mathbb{C} : |z| \leq Q, \eta > 0, ||E| - 2| \geq \kappa_0, \sqrt{M\eta} \geq (\log N)^{C_1}, \right\} \tag{2.44}$$

where  $C_1$  and  $Q$  are sufficiently large. Then there exist constants  $C_1, C_2, C$  and  $c > 0$ , depending only on  $\alpha, \beta, \kappa_0$  and  $\delta_-$  in (2.30), such that the diagonal matrix elements of the Green function  $G_{ii}(z) = (H - z)^{-1}(i, i)$  satisfy that

$$\mathbb{P} \left( \sup_{z \in D} \max_i |G_{ii}(z) - m_{sc}(z)| \geq \frac{(\log N)^{C_2}}{\sqrt{M\eta}} \right) \leq CN^{-c(\log \log N)}, \quad (2.45)$$

and for the off-diagonal elements we have

$$\mathbb{P} \left( \sup_{z \in D} \max_{i \neq j} |G_{ij}(z)| \geq \frac{(\log N)^{C_2}}{\sqrt{M\eta}} \right) \leq CN^{-c(\log \log N)} \quad (2.46)$$

for any sufficiently large  $N$ .

We start with determining a system of self-consistent equations for the diagonal matrix elements of the resolvent. The following lemma is a simple combination of the resolvent identities from Section 2.1.

**Lemma 2.11** *The diagonal resolvent matrix elements  $G_{ii} = (H - z)^{-1}(i, i)$  satisfy the following system of self-consistent equations*

$$G_{ii} = \frac{1}{-z - \sum_j \sigma_{ij}^2 G_{jj} + \Upsilon_i}, \quad (2.47)$$

where

$$\Upsilon_j = \Upsilon_j(z) := A_j + h_{jj} - Z_j. \quad (2.48)$$

with

$$A_i := \sigma_{ii}^2 G_{ii} + \sum_{j \neq i} \sigma_{ij}^2 \frac{G_{ij} G_{ji}}{G_{ii}} \quad (2.49)$$

$$Z_i := Z_{ii}^{(i)} - \mathbb{E}_i Z_{ii}^{(i)}, \quad Z_{ii}^{(i)} := \mathbf{a}^i \cdot G^{(i)} \mathbf{a}^i = \sum_{k, l \neq i} \overline{\mathbf{a}}_k^i G_{kl}^{(i)} \mathbf{a}_l^i$$

*Proof.* Introduce

$$Z_{ij}^{(ij)} := \mathbf{a}^i \cdot G^{(ij)} \mathbf{a}^j, \quad Z_{ii}^{(i)} := \mathbf{a}^i \cdot G^{(i)} \mathbf{a}^i,$$

and recall from (2.6)

$$K_{ij}^{(ij)} = h_{ij} - z\delta_{ij} - Z_{ij}^{(ij)}, \quad K_{ii}^{(i)} = h_{ii} - z - Z_{ii}^{(i)}.$$

We can write  $G_{ii}$  as follows (2.9)

$$G_{ii} = (K_{ii}^{(i)})^{-1} = \frac{1}{\mathbb{E}_{\mathbf{a}^i} K_{ii}^{(i)} + K_{ii}^{(i)} - \mathbb{E}_{\mathbf{a}^i} K_{ii}^{(i)}}, \quad (2.50)$$



where  $\mathbb{E}_{\mathbf{a}^i} = \mathbb{E}_i$  denotes the expectation with respect to the elements in the  $i$ -th column of the matrix  $H$ .

Using the fact that  $G^{(i)} = (H^{(i)} - z)^{-1}$  is independent of  $\mathbf{a}^i$  and  $\mathbb{E}_{\mathbf{a}^i} \overline{\mathbf{a}_k^i} \mathbf{a}_l^i = \delta_{kl} \sigma_{ik}^2$ , we obtain

$$\mathbb{E}_{\mathbf{a}^i} K_{ii}^{(i)} = -z - \sum_{j \neq i} \sigma_{ij}^2 G_{jj}^{(i)}$$

and

$$K_{ii}^{(i)} - \mathbb{E}_{\mathbf{a}^i} K_{ii}^{(i)} = h_{ii} - Z_i. \quad (2.51)$$

Use

$$G_{kl} = G_{kl}^{(i)} + \frac{G_{ki} G_{il}}{G_{ii}}$$

from (2.10) to express

$$\mathbb{E}_{\mathbf{a}^i} K_{ii}^{(i)} = -z - \sum_{j \neq i} \sigma_{ij}^2 G_{jj}^{(i)} = -z - \sum_{j \neq i} \sigma_{ij}^2 G_{jj} + \sum_{j \neq i} \sigma_{ij}^2 \frac{G_{ki} G_{il}}{G_{ii}}.$$

With the notation from (2.49), we have

$$\mathbb{E}_{\mathbf{a}^i} K_{ii}^{(i)} = -z - \sum_{j \neq i} \sigma_{ij}^2 G_{jj}^{(i)} = -z - \sum_j \sigma_{ij}^2 G_{jj} + A_i.$$

Combining this with (2.51), from (2.50) we eventually obtain (2.47).  $\square$

Introduce the notations

$$v_i := G_{ii} - m_{sc}, \quad m := \frac{1}{N} \sum_i G_{ii}, \quad \bar{v} := \frac{1}{N} \sum_i v_i = \frac{1}{N} \sum_i (G_{ii} - m_{sc}).$$

We will estimate the following key quantities

$$\Lambda_d := \max_k |v_k| = \max_k |G_{kk} - m_{sc}|, \quad \Lambda_o := \max_{k \neq \ell} |G_{k\ell}|, \quad (2.52)$$

where the subscripts refer to ‘‘diagonal’’ and ‘‘offdiagonal’’ matrix elements. All the quantities defined so far depend on the spectral parameter  $z = E + i\eta$ , but we will mostly omit this fact from the notation. The real part  $E$  will always be kept fixed. For the imaginary part we will use a continuity argument at the end of the proof and then the dependence of  $\Lambda_{d,o}$  on  $\eta$  will be indicated.

Both quantities  $\Lambda_d$  and  $\Lambda_o$  will be typically small for  $z \in D$ ; eventually we will prove that their size is less than  $(M\eta)^{-1/2}$ , modulo logarithmic corrections. We thus define the exceptional event

$$\Omega_\Lambda = \Omega_\Lambda(z) := \left\{ \Lambda_d(z) + \Lambda_o(z) \geq (\log N)^{-C} \right\}$$

with some  $C$  (in this presentation, we will not care about matching all exponents). We will always work in  $\Omega_\Lambda^c$ , and, in particular, we will have

$$\Lambda_d(z) + \Lambda_o(z) \ll 1.$$

It is easy to check from the explicit formula on  $m_{sc}$  that

$$c \leq |m_{sc}(z)| \leq C, \quad z \in D. \quad (2.53)$$

with some positive constants, so from  $G_{ii} = m_{sc}(z) + O(\Lambda_d)$  we have

$$c \leq |G_{ii}(z)| \leq C, \quad z \in D. \quad (2.54)$$

Recalling (2.11)

$$G_{kl}^{(i)} = G_{kl} - \frac{G_{ki}G_{il}}{G_{ii}}, \quad i \neq l, k,$$

together with (2.54) it implies that for any  $i$

$$\max_{k \neq l} |G_{kl}^{(i)}| \leq \Lambda_o + C\Lambda_o^2 \leq C\Lambda_o \quad \text{in } \Omega_\Lambda^c,$$

$$c \leq |G_{kk}^{(i)}| \leq C, \quad \text{for all } k \neq i \text{ and in } \Omega_\Lambda^c \quad (2.55)$$

$$|G_{kk}^{(i)} - m_{sc}| \leq \Lambda_d + C\Lambda_o^2 \quad \text{for all } k \neq i \text{ and in } \Omega_\Lambda^c \quad (2.56)$$

and (see (2.49))

$$|A_i| \leq \frac{C}{M} + C\Lambda_o^2 \quad \text{in } \Omega_\Lambda^c. \quad (2.57)$$

Similarly, with one more expansion step, we get

$$\max_{ij} \max_{k \neq l} |G_{kl}^{(ij)}| \leq C\Lambda_o, \quad \max_{ij} \max_k |G_{kk}^{(ij)}| \leq C \quad \text{in } \Omega_\Lambda^c$$

and

$$|G_{kk}^{(ij)} - m_{sc}| \leq \Lambda_d + C\Lambda_o^2 \quad \text{for all } k \neq i, j \text{ and in } \Omega_\Lambda^c. \quad (2.58)$$

Using these estimates, the following lemma shows that  $Z_i$  and  $Z_{ij}^{(ij)}$  are small assuming  $\Lambda_d + \Lambda_o$  is small and the  $h_{ij}$ 's are not too large. These bounds hold uniformly in  $D$ .

**Lemma 2.12** *Define the exceptional events*

$$\begin{aligned}
\Omega_1 &:= \left\{ \max_{1 \leq i, j \leq N} |h_{ij}| \geq (\log N)^C |\sigma_{ij}| \right\} \\
\Omega_d(z) &:= \left\{ \max_i |Z_i(z)| \geq \frac{(\log N)^C}{\sqrt{M\eta}} \right\} \\
\Omega_o(z) &:= \left\{ \max_{i \neq j} |Z_{ij}^{(ij)}(z)| \geq \frac{(\log N)^C}{\sqrt{M\eta}} \right\}
\end{aligned} \tag{2.59}$$

and we let

$$\Omega := \Omega_1 \cup \bigcup_{z \in D} \left[ (\Omega_d(z) \cup \Omega_o(z)) \cap \Omega_\lambda^c(z) \right] \tag{2.60}$$

to be the set of all exceptional events. Then we have

$$\mathbb{P}(\Omega) \leq CN^{-c(\log \log N)}. \tag{2.61}$$

*Proof:* Under the assumption of (2.32), analogously to (2.22), we have

$$\mathbb{P}(\Omega_1) \leq CN^{-c \log \log N}, \tag{2.62}$$

so we can work on the complement set  $\Omega_1^c$  and assume that

$$\max_{ij} |h_{ij}| \leq \frac{(\log N)^C}{\sqrt{M}}. \tag{2.63}$$

We now prove that for any fixed  $z \in D$ , we have

$$\mathbb{P}\left(\Omega_\lambda^c(z) \cap \left\{ \max_i |Z_i(z)| \geq \frac{(\log N)^C}{\sqrt{M\eta}} \right\}\right) \leq CN^{-c \log \log N} \tag{2.64}$$

and

$$\mathbb{P}\left(\Omega_\lambda^c(z) \cap \left\{ \max_{i \neq j} |Z_{ij}^{(ij)}(z)| \geq \frac{(\log N)^C}{\sqrt{M\eta}} \right\}\right) \leq CN^{-c \log \log N}. \tag{2.65}$$

Recall that  $Z^{(ij)}$  is a quadratic form in the components of the random vectors  $\mathbf{a}^i$  and  $\mathbf{a}^j$ . For such functions, we have the following general large deviation result. The proof relies on the Burkholder martingale inequality and it will be given in Appendix B.

**Lemma 2.13** *Let  $a_i$  ( $1 \leq i \leq N$ ) be  $N$  independent random complex variables with mean zero, variance  $\sigma^2$  and having the uniform subexponential decay*

$$\mathbb{P}(|a_i| \geq x^\alpha) \leq \beta e^{-x}. \quad (2.66)$$

for some  $\alpha$  and  $\beta$  and for all  $x$ . Let  $A_i, B_{ij} \in \mathbb{C}$  ( $1 \leq i, j \leq N$ ). Then we have that

$$\mathbb{P} \left\{ \left| \sum_{i=1}^N a_i A_i \right| \geq (\log N)^{\frac{3}{2} + \alpha} \sigma \left( \sum_{i=1}^N |A_i|^2 \right)^{1/2} \right\} \leq CN^{-\log \log N}, \quad (2.67)$$

$$\mathbb{P} \left\{ \left| \sum_{i=1}^N \bar{a}_i B_{ii} a_i - \sum_{i=1}^N \sigma^2 B_{ii} \right| \geq (\log N)^{\frac{3}{2} + 2\alpha} \sigma^2 \left( \sum_{i=1}^N |B_{ii}|^2 \right)^{1/2} \right\} \leq CN^{-\log \log N}, \quad (2.68)$$

$$\mathbb{P} \left\{ \left| \sum_{i \neq j} \bar{a}_i B_{ij} a_j \right| \geq (\log N)^{3 + 2\alpha} \sigma^2 \left( \sum_{i \neq j} |B_{ij}|^2 \right)^{1/2} \right\} \leq CN^{-\log \log N}, \quad (2.69)$$

for some constants  $C$  depending on  $\alpha$  and  $\beta$  in (2.32).

To see (2.64), we apply the estimate (2.68) and we obtain that

$$|Z_i| \leq (\log N)^C \left( \sum_{k, l \neq i} \left| \sigma_{ik} G_{kl}^{(i)} \sigma_{li} \right|^2 \right)^{1/2} \quad (2.70)$$

holds with a probability larger than  $1 - CN^{-c(\log \log N)}$  for sufficiently large  $N$ .

Denote by  $u_\alpha^{(i)}$  and  $\lambda_\alpha^{(i)}$  ( $\alpha = 1, 2, \dots, N-1$ ) the eigenvectors and eigenvalues of  $H^{(i)}$ . Let  $u_\alpha^{(i)}(l)$  denote the  $l$ -th coordinate of  $u_\alpha^{(i)}$ . Then, using  $\sigma_{il}^2 \leq 1/M$  and (2.56), we have

$$\begin{aligned} \sum_{k, l \neq i} \left| \sigma_{ik} G_{kl}^{(i)} \sigma_{li} \right|^2 &\leq \frac{1}{M} \sum_{k \neq i} \sigma_{ik}^2 \sum_{l \neq i} |G_{kl}^{(i)}|^2 \\ &\leq \frac{1}{M} \sum_{k \neq i} \sigma_{ik}^2 \left( |G^{(i)}|^2 \right)_{kk} \\ &= \frac{1}{M} \sum_{k \neq i} \sigma_{ik}^2 \sum_{\alpha} \frac{|u_\alpha^{(i)}(k)|^2}{|\lambda_\alpha^{(i)} - z|^2} \leq \frac{1}{M} \sum_{k \neq i} \sigma_{ik}^2 \frac{\text{Im } G_{kk}^{(i)}(z)}{\eta} \\ &\leq \frac{C}{M\eta} \quad \text{in } \Omega_\Lambda^c. \end{aligned} \quad (2.71)$$

Here we defined  $|A|^2 := A^*A$  for any matrix  $A$  and we used (2.55). Together with (2.70) we have proved (2.64) for a fixed  $z$ . The offdiagonal estimate (2.65), for  $i \neq j$ , is proven similarly, using (2.69) instead of (2.68).

We thus proved that for each fixed  $z \in D$ , the sets  $\Omega_d(z)$ ,  $\Omega_o(z)$  have small probability. In order to prove (2.61), in principle, we have to take the union of uncountable many events. But it is easy to see that the quantities  $Z_i(z)$  and  $Z_{ij}^{(ij)}(z)$  defining these events are Lipschitz continuous functions with a Lipschitz constant  $\eta^{-1} \leq N$ . Thus controlling them on a sufficiently dense but finite net of points will control them for every  $z \in D$ . This proves Lemma 2.12.  $\square$

The key step in the proof is the following lemma, which says that if  $\Lambda$  is somewhat small, like  $(\log N)^{-C}$ , then the estimate can be boosted to show that it is much smaller, like  $(M\eta)^{-1/2}$ .

**Lemma 2.14** *Recall  $\Lambda_d$ ,  $\Lambda_o$  and  $\Omega$  defined in (2.52) and (2.60) and recall the set  $D$  from (2.44). Then we for any  $z \in D$ , in the event  $\Omega^c$ , if*

$$\Lambda_o(z) + \Lambda_d(z) \leq (\log N)^{-C} \quad (2.72)$$

then

$$\Lambda_o(z) + \Lambda_d(z) \leq \frac{(\log N)^C}{\sqrt{M\eta}}. \quad (2.73)$$

*Proof of Lemma 2.14.* Choosing  $C_1$  in (2.44) sufficiently large, we can ensure from Lemma 2.12 that

$$Z_{ij}^{(ij)} \ll 1, \quad Z_i \ll 1.$$

We first estimate the offdiagonal term  $G_{ij}$ . Under the condition (2.72), from (2.8), (2.54) and (2.55) we have

$$|G_{ij}| = |G_{ii}| |G_{jj}^{(i)}| |K_{ij}^{(ij)}| \leq C \left( |h_{ij}| + |Z_{ij}^{(ij)}| \right), \quad i \neq j.$$

By Lemma 2.12, we have, in the event  $\Omega^c$ ,

$$|G_{ij}| \leq \frac{(\log N)^C}{\sqrt{M}} + \frac{(\log N)^C}{\sqrt{M\eta}} \leq \frac{(\log N)^C}{\sqrt{M\eta}}. \quad (2.74)$$

This proves the estimate (2.73) for the summand  $\Lambda_o$ .

Now we estimate the diagonal terms. Recalling  $\Upsilon_i = A_i + h_{ii} - Z_i$  from (2.48), with (2.57), (2.74) and Lemma 2.12 we have,

$$\Upsilon := \max_i |\Upsilon_i| \leq C \frac{(\log N)^C}{\sqrt{M}} + \frac{(\log N)^C}{\sqrt{M\eta}} \leq \frac{(\log N)^C}{\sqrt{M\eta}} \ll 1 \quad \text{in } \Omega^c \quad (2.75)$$

(in the last step we used  $z \in D$  and that  $C_1$  is large). From (2.47) we have the identity

$$v_i = G_{ii} - m_{sc} = \frac{1}{-z - m_{sc} - \left( \sum_j \sigma_{ij}^2 v_j - \Upsilon_i \right)} - m_{sc}. \quad (2.76)$$

Using that  $(m_{sc} + z) = -m_{sc}^{-1}$ , the fact that  $|m_{sc} + z| \geq 1$  and that  $\Lambda_d + \Upsilon \ll 1$  we can expand (2.76) as

$$v_i = m_{sc}^2 \left( \sum_j \sigma_{ij}^2 v_j - \Upsilon_i \right) + O \left( \sum_j \sigma_{ij}^2 v_j - \Upsilon_i \right)^2 = m_{sc}^2 \left( \sum_j \sigma_{ij}^2 v_j - \Upsilon_i \right) + O \left( (\Lambda_d + \Upsilon)^2 \right). \quad (2.77)$$

Summing up this formula for all  $i$  and recalling the definition  $\bar{v} := \frac{1}{N} \sum_i v_i = m - m_{sc}$  yield

$$\bar{v} = m_{sc}^2(z) \bar{v} - \frac{m_{sc}^2(z)}{N} \sum_i \Upsilon_i + O \left( (\Lambda_d + \Upsilon)^2 \right).$$

Introducing the notations  $\zeta := m_{sc}^2(z)$ ,  $\bar{\Upsilon} := \frac{1}{N} \sum_i \Upsilon_i$  for simplicity, we have (using  $\Lambda_d \leq 1$ )

$$\bar{v} = -\frac{\zeta}{1-\zeta} \bar{\Upsilon} + O \left( \frac{\zeta}{1-\zeta} (\Lambda_d + \Upsilon)^2 \right) = O \left( \frac{\zeta}{1-\zeta} (\Lambda_d^2 + \Upsilon) \right). \quad (2.78)$$

Recall that  $\Sigma$  denotes the matrix of covariances,  $\Sigma_{ij} = \sigma_{ij}^2$ , and we know that 1 is a simple eigenvalue with the constant vector  $\mathbf{e} = N^{-1/2}(1, 1, \dots, 1)$  as the eigenvector. Let  $Q := I - |\mathbf{e}\rangle\langle\mathbf{e}|$  be the projection onto the orthogonal complement of  $\mathbf{e}$ , note that  $\Sigma$  and  $Q$  commute. Let  $\|\cdot\|_{\infty \rightarrow \infty}$  denote the  $\ell^\infty \rightarrow \ell^\infty$  matrix norm. With these notations, (2.77) can be written as

$$v_i - \bar{v} = \zeta \sum_j \Sigma_{ij} (v_j - \bar{v}) - \zeta (\Upsilon_i - \bar{\Upsilon}) + O \left( |\zeta| (\Lambda_d^2 + \Upsilon) \right) + O \left( (\Lambda_d + \Upsilon)^2 \right),$$

and the error terms sum up to zero. Therefore, with  $\Upsilon \leq 1$ , we have

$$\begin{aligned} v_i - \bar{v} &= - \sum_j \left( \frac{\zeta}{1-\zeta\Sigma} \right)_{ij} (\Upsilon_j - \bar{\Upsilon}) + O \left( \left\| \frac{\zeta Q}{1-\zeta\Sigma} \right\|_{\infty \rightarrow \infty} (\Lambda_d^2 + \Upsilon) \right) \\ &= \left\| \frac{\zeta Q}{1-\zeta\Sigma} \right\|_{\infty \rightarrow \infty} O(\Lambda_d^2 + \Upsilon). \end{aligned} \quad (2.79)$$

Combining (2.78) with (2.79), we have

$$\max_i |v_i| \leq C \left( \left\| \frac{\zeta Q}{1-\zeta\Sigma} \right\|_{\infty \rightarrow \infty} + \left| \frac{\zeta}{1-\zeta} \right| \right) (\Lambda_d^2 + \Upsilon). \quad (2.80)$$

From this calculation it is clear how the estimates deteriorate at the edge. It is easy to check that

$$1 - \zeta = 1 - m_{sc}^2(z) \sim \sqrt{\kappa + \eta}, \quad z = E + i\eta, \quad \kappa = ||E| - 2|.$$

However, as we remarked at the beginning of the proof, we assume  $\kappa \geq \kappa_0 > 0$  for simplicity, so we will not follow this deterioration.

To estimate the norm of the resolvent, we recall the following elementary lemma (Lemma 5.3 in [46]).

**Lemma 2.15** Let  $\delta_- > 0$  be a given constant. Then there exist small real numbers  $\tau \geq 0$  and  $c_1 > 0$ , depending only on  $\delta_-$ , such that for any positive number  $\delta_+$ , we have

$$\max_{x \in [-1+\delta_-, 1-\delta_+]} \left\{ \left| \tau + x m_{sc}^2(z) \right|^2 \right\} \leq (1 - c_1 q(z)) (1 + \tau)^2 \quad (2.81)$$

with

$$q(z) := \max\{\delta_+, |1 - \operatorname{Re} m_{sc}^2(z)|\}. \quad (2.82)$$

□

**Lemma 2.16** Suppose that  $\Sigma$  satisfies (2.29), i.e.,  $\operatorname{Spec}(Q\Sigma) \subset [-1 + \delta_-, 1 - \delta_+]$ . Then we have

$$\left\| \frac{Q}{1 - m_{sc}^2(z)\Sigma} \right\|_{\infty \rightarrow \infty} \leq \frac{C(\delta_-) \log N}{q(z)} \quad (2.83)$$

with some constant  $C(\delta_-)$  depending on  $\delta_-$  and with  $q$  defined in (2.82).

From these lemmas one can see that the deterioration of the estimates at the edge can be offset by assuming  $\delta_+ > 0$ , in which case  $q(z)$  remains  $O(1)$  uniformly in  $z$ .

*Proof:* Let  $\|\cdot\|$  denote the usual  $\ell^2 \rightarrow \ell^2$  matrix norm and introduce  $\zeta = m_{sc}^2(z)$ . Rewrite

$$\left\| \frac{Q}{1 - \zeta\Sigma} \right\| = \frac{1}{1 + \tau} \left\| \frac{Q}{1 - \frac{\zeta\Sigma + \tau}{1 + \tau}} \right\|$$

with  $\tau$  given in (2.81). By (2.81), we have

$$\left\| \frac{\zeta\Sigma + \tau}{1 + \tau} Q \right\| \leq \sup_{x \in [-1+\delta_-, 1-\delta_+]} \left| \frac{\zeta x + \tau}{1 + \tau} \right| \leq (1 - c_1 q(z))^{1/2}.$$

To estimate the  $\ell^\infty \rightarrow \ell^\infty$  norm of this matrix, recall that  $|\zeta| = |m_{sc}|^2 \leq 1$  and  $\sum_j |\Sigma_{ij}| = \sum_j \Sigma_{ij} = \sum_j \sigma_{ij}^2 = 1$ . Thus we have

$$\left\| \frac{\zeta\Sigma + \tau}{1 + \tau} Q \right\|_{\infty \rightarrow \infty} = \max_i \sum_j \left| \left( \frac{\zeta\Sigma + \tau}{1 + \tau} \right)_{ij} \right| \leq \frac{1}{1 + \tau} \max_i \sum_j |\zeta\Sigma_{ij} + \tau\delta_{ij}| \leq \frac{|\zeta| + \tau}{1 + \tau} \leq 1.$$

To see (2.83), we can expand

$$\begin{aligned} \left\| \frac{1}{1 - \frac{\zeta\Sigma + \tau}{1 + \tau}} Q \right\|_{\infty \rightarrow \infty} &\leq \sum_{n < n_0} \left\| \frac{\zeta\Sigma + \tau}{1 + \tau} Q \right\|_{\infty \rightarrow \infty}^n + \sum_{n \geq n_0} \left\| \left( \frac{\zeta\Sigma + \tau}{1 + \tau} \right)^n Q \right\|_{\infty \rightarrow \infty} \\ &\leq n_0 + \sqrt{N} \sum_{n \geq n_0} \left\| \left( \frac{\zeta\Sigma + \tau}{1 + \tau} \right)^n Q \right\| = n_0 + \sqrt{N} \sum_{n \geq n_0} (1 - c_1 q(z))^{n/2} \\ &= n_0 + C\sqrt{N} \frac{(1 - c_1 q(z))^{n_0/2}}{q(z)} \leq \frac{C \log N}{q(z)}. \end{aligned}$$

Choosing  $n_0 = C \log N/q(z)$ , we have proved the Lemma.  $\square$

We now return to the proof of Lemma 2.14, recall that we are in the set  $\Omega^c$ . First, inserting (2.83) into (2.80), we obtain

$$\Lambda_d = \max_i |v_i| \leq C(\Lambda_d^2 + \Upsilon) \log N.$$

By the assumption (2.72), we have  $C\Lambda_d \log N \leq 1/2$ , so the quadratic term can be absorbed in the linear term on the right hand side, and we get

$$\Lambda_d \leq C\Upsilon \log N.$$

Using the bound (2.75) on  $\Upsilon$ , we obtain

$$\Lambda_d \leq \frac{(\log N)^C}{\sqrt{M\eta}}, \quad (2.84)$$

which, together with (2.74), completes the proof of (2.73).  $\square$

*Proof of Theorem 2.10.* Lemma 2.14 states that, in the event  $\Omega^c$ , if  $\Lambda_d(z) + \Lambda_o(z) \leq R(z)$  holds for some  $z \in D$ , then  $\Lambda_d(z) + \Lambda_o(z) \leq S(z)$ , where we introduced the notation

$$R(z) := (\log N)^{-C}, \quad S(z) := \frac{(\log N)^C}{\sqrt{M\eta}}.$$

By assumption (2.44) of Theorem 2.10, we have  $S(z) < R(z)$  for any  $z \in D$  and these functions are continuous. Clearly,  $\Lambda_d(z) + \Lambda_o(z) \leq 3/\eta \leq R(z)$  for  $\eta = (\log N)^C$ . Thus by a continuity argument,  $\Lambda_d(z) + \Lambda_o(z) \leq S(z)$  in the set  $\Omega^c$  as long as the condition (2.44) is satisfied. This proves Theorem 2.10.  $\square$

Note that in the last step we were not fully rigorous. Using  $\eta = (\log N)^C$  requires choosing  $Q$  in the definition of the set  $D$  (2.44) to grow logarithmically, on the other hand that would imply that the constants in the estimate (2.53) deteriorate logarithmically. In the actual proof [47] we followed these additional logarithmic factors.

### 2.3.3 Sketch of the proof of the strong semicircle law for Stieltjes transform

In this section we strengthen the estimate of Theorem 2.10 for the Stieltjes transform  $m(z) = \frac{1}{N} \sum_i G_{ii}$ . The key improvement is that  $|m - m_{sc}|$  will be estimated with a precision  $(M\eta)^{-1}$  while the  $|G_{ii} - m_{sc}|$  was controlled by a precision  $(M\eta)^{-1/2}$  only (modulo logarithmic terms and terms expressing the deterioration of the estimate near the edge). This will prove a partial version of (2.34) of Theorem 2.6.



**Theorem 2.17** *Assume the conditions of Theorem 2.6 and recall the notations  $\kappa = \kappa_E := ||E| - 2|$  and recall the definition of the domain  $D$  from (2.44). Then for any  $\varepsilon > 0$  and  $K > 0$  there exists a constant  $C = C(\varepsilon, K)$  such that*

$$\mathbb{P}\left(\sup_{z \in D} |m(z) - m_{sc}(z)| \geq \frac{N^\varepsilon}{M\eta}\right) \leq \frac{C(\varepsilon, K)}{N^K}. \quad (2.85)$$

*Proof of Theorem 2.17.* Fix  $z \in D$  and we can assume that we work in the complement of the small probability events estimated in (2.45) and (2.46). In particular, the estimate (2.73) is available. As in (2.78), where  $\bar{v} = m(z) - m_{sc}(z)$ , we have that

$$m - m_{sc} = -\frac{\zeta}{1-\zeta} \frac{1}{N} \sum_i \Upsilon_i + O\left(\frac{\zeta}{1-\zeta} (\Lambda_d + \Upsilon)^2\right)$$

holds with a very high probability, but now we keep the first term and estimate it better than the most trivial bound used in (2.78) to extract some cancellation from the fluctuating sum. Recall that  $\zeta = m_{sc}^2(z)$  and we mostly omit the argument  $z$  from the notations. The quantities  $\Lambda_d$ ,  $\Upsilon_i$  and  $\Upsilon$  were defined in (2.52), (2.48) and (2.75). Then with (2.75) and (2.84) we have

$$m(z) - m_{sc}(z) = O\left(\frac{1}{1-\zeta} \frac{1}{N} \sum_j \Upsilon_j\right) + O\left(\frac{N^\varepsilon}{M\eta}\right)$$

holds with a very high probability for any small  $\varepsilon > 0$ . Recall that  $\Upsilon_i = A_i + h_{ii} - Z_i$ . We have, from (2.49), (2.54) and  $\sigma_{ij}^2 \leq M^{-1}$ ,

$$|A_j| \leq \frac{C}{M} + C\Lambda_o^2 \leq \frac{CN^\varepsilon}{M\eta},$$

where we used (2.73) to bound  $\Lambda_o$ .

We thus obtain that

$$m - m_{sc} = O\left(\frac{1}{1-\zeta} \left(\frac{1}{N} \sum_i Z_i - \frac{1}{N} \sum_i h_{ii}\right)\right) + O\left(\frac{N^\varepsilon}{M\eta}\right) \quad (2.86)$$

holds with a very high probability. Since  $h_{ii}$ 's are independent, applying the first estimate in the large deviation Lemma 2.13, we have

$$\mathbb{P}\left(\left|\frac{1}{N} \sum_i h_{ii}\right| \geq (\log N)^{C_\alpha} \frac{1}{\sqrt{MN}}\right) \leq CN^{-c \log \log N}. \quad (2.87)$$

On the complement event, the estimate  $(\log N)^{C_\alpha} (MN)^{-1/2}$  can be included in the last error term in (2.86). It only remains to bound

$$\frac{1}{N} \sum_i Z_i,$$

whose moment is bounded in the next lemma, and we will comment on its proof below.

**Lemma 2.18** For fixed  $z$  in domain  $D$  and any natural number  $s$ , we have

$$\mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N Z_i \right|^{2s} \leq C_s \left( \frac{1}{M\eta} \right)^{2s}$$

for sufficiently large  $N$ .

Using this lemma, we have that for any  $\varepsilon > 0$  and  $K > 0$ ,

$$\mathbb{P} \left( \frac{1}{N} \left| \sum_{i=1}^N Z_i \right| \geq \frac{N^\varepsilon}{M\eta} \right) \leq N^{-K}$$

for sufficiently large  $N$ . Combining this with (2.87) and (2.86), we obtain (2.85) and complete the proof of Theorem 2.17.  $\square$

*Sketch of the proof of Lemma 2.18.* The proof is a long and carefully organized high moment calculation, similar to the second moment bound in (2.23), but now we extract an additional factor from the sum. Note that boosting the second moment calculation (2.23) to higher moments (and recalling that for universal Wigner matrices  $M$  replaces  $N$ ) we can prove

$$|Z_i| \lesssim \frac{1}{\sqrt{M\eta}}. \quad (2.88)$$

(we will indicate the proof in Lemma 2.19 below). If  $Z_i$ 's were independent, then the central limit theorem would imply that

$$\left| \frac{1}{N} \sum_i Z_i \right| \lesssim \frac{1}{\sqrt{M}} \frac{1}{\sqrt{M\eta}},$$

which would be more than enough. They are not quite independent, but almost. The dependence among different  $Z_i$ 's can be extracted from using the resolvent formulas in Section 2.1.

We will sketch the second moment calculation, i.e.,  $s = 1$ , more details and higher moments are given in Section 8 and 9 of [47].

**Definition 2.1** Define the operator  $\mathbb{I}\mathbb{E}_i$  as

$$\mathbb{I}\mathbb{E}_i := \mathbb{I} - \mathbb{E}_{\mathbf{a}_i} \quad (2.89)$$

where  $\mathbb{I}$  is the identity operator.

The variance of  $\frac{1}{N} \sum_{i=1}^N Z_i$  is given by

$$\frac{1}{N^2} \mathbb{E} \left| \sum_{i=1}^N Z_i \right|^2 = \frac{1}{N^2} \mathbb{E} \sum_{\alpha \neq \beta} \overline{Z_\alpha} Z_\beta + \frac{1}{N^2} \mathbb{E} \sum_{\alpha} |Z_\alpha|^2 \quad (2.90)$$

We start by estimating the first term of (2.90) for  $\alpha = 1$  and  $\beta = 2$ . The basic idea is to rewrite  $G_{kl}^{(1)}$  as

$$G_{kl}^{(1)} = P_{kl}^{(12)} + P_{kl}^{(1)} \quad (2.91)$$

with  $P_{kl}^{(12)}$  independent of  $\mathbf{a}^1, \mathbf{a}^2$  and  $P_{kl}^{(1)}$  independent of  $\mathbf{a}^1$ . To construct this decomposition for  $k, l \notin \{1, 2\}$ , by (2.10) or (2.11) we can rewrite  $G_{kl}^{(1)}$  as

$$G_{kl}^{(1)} = G_{kl}^{(12)} + \frac{G_{k2}^{(1)} G_{2l}^{(1)}}{G_{22}^{(1)}}, \quad k, l \notin \{1, 2\}. \quad (2.92)$$

The first term on r.h.s is independent of  $\mathbf{a}^2$ . Applying Theorem 2.10 for the minors, we get that  $G_{kl}^{(1)} \lesssim (M\eta)^{-1/2}$  for  $k \neq l \neq 1$  and  $G_{kk}^{(2)} \geq c > 0$ , thus

$$\left| \frac{G_{k2}^{(1)} G_{2l}^{(1)}}{G_{22}^{(1)}} \right| \lesssim \frac{1}{M\eta} \quad (2.93)$$

holds with very high probability. We note that this bound is the order of the square of the bound  $G_{kl}^{(1)} \lesssim (M\eta)^{-1/2}$  from Theorem 2.10.

Now we define  $P^{(1)}$  and  $P^{(12)}$  (for  $k, l \neq 1$ ) as

1. If  $k, l \neq 2$ ,

$$P_{kl}^{(12)} = G_{kl}^{(12)}, \quad P_{kl}^{(1)} = \frac{G_{k2}^{(1)} G_{2l}^{(1)}}{G_{22}^{(1)}} = G_{kl}^{(1)} - G_{kl}^{(12)} \quad (2.94)$$

2. if  $k = 2$  or  $l = 2$ ,

$$P_{kl}^{(12)} = 0, \quad P_{kl}^{(1)} = G_{kl}^{(1)}. \quad (2.95)$$

Hence (2.91) holds and  $P_{kl}^{(12)}$  is independent of  $\mathbf{a}^2$ .

With this convention, we have the following expansion of  $Z_1$

$$Z_1 = \mathbb{E}_1 \overline{\mathbf{a}^1} \cdot P^{(12)} \mathbf{a}^1 + \mathbb{E}_1 \overline{\mathbf{a}^1} \cdot P^{(1)} \mathbf{a}^1. \quad (2.96)$$

**Lemma 2.19** For  $N^{-1} \leq \eta \leq 1$  and fixed  $p \in \mathbb{N}$ , we have the following estimates

$$\mathbb{E} \left| \overline{\mathbf{a}^1} \cdot P^{(1)} \mathbf{a}^1 \right|^p \lesssim \frac{C_p}{(M\eta)^p} \quad (2.97)$$

$$\mathbb{E} \left| \overline{\mathbf{a}^1} \cdot P^{(12)} \mathbf{a}^1 \right|^p \lesssim \frac{C_p}{(M\eta)^{p/2}} \quad (2.98)$$

Note that this lemma also implies that

$$\mathbb{E}|Z_i|^p \lesssim \frac{C_p}{(M\eta)^{p/2}}, \quad 1 \leq i \leq N \quad (2.99)$$

i.e. it also proves (2.88) and it estimates the second term in (2.90) by  $N^{-1}(M\eta)^{-1} \leq (M\eta)^{-2}$ .

*Proof of Lemma 2.19.* First we rewrite  $\overline{\mathbf{a}^1} \cdot P^{(1)}\mathbf{a}^1$  as follows

$$\overline{\mathbf{a}^1} \cdot P^{(1)}\mathbf{a}^1 = \sum_{k,l \neq 2} \overline{\mathbf{a}_k^1} \cdot \left( \frac{G_{k2}^{(1)} G_{2l}^{(1)}}{G_{2,2}^{(1)}} \right) \mathbf{a}_l^1 + \sum_{k \neq 2} \overline{\mathbf{a}_k^1} \cdot G_{k2}^{(1)} \mathbf{a}_2^1 + \sum_{l \neq 2} \overline{\mathbf{a}_2^1} \cdot G_{2l}^{(1)} \mathbf{a}_l^1 + \overline{\mathbf{a}_2^1} G_{22}^{(1)} \mathbf{a}_2^1. \quad (2.100)$$

By the large deviation estimate (2.69), we have

$$\mathbb{P} \left( \left| \sum_{k,l \neq 2} \overline{\mathbf{a}_k^1} \cdot \left( \frac{G_{k2}^{(1)} G_{2l}^{(1)}}{G_{2,2}^{(1)}} \right) \mathbf{a}_l^1 \right| \geq \frac{(\log N)^C}{M\eta} \right) \leq N^{-c \log \log N} \quad (2.101)$$

Similarly, from (2.67) and the fact that  $\|\mathbf{a}^1\|_\infty \lesssim M^{-1/2}$  we get that the second and third terms in (2.100) are bounded by  $\frac{1}{\sqrt{M}} \frac{1}{\sqrt{M\eta}} \leq (M\eta)^{-1}$  (modulo logarithmic factors). The last term is even smaller, this is of order  $1/M$ , with a very high probability.

We have thus proved that

$$\mathbb{P} \left( \left| \overline{\mathbf{a}^1} \cdot P^{(1)}\mathbf{a}^1 \right| \leq \frac{(\log N)^C}{M\eta} \right) \geq 1 - N^{-c \log \log N}. \quad (2.102)$$

This inequality implies the desired inequality (2.97) except on the exceptional set with probability less than any power of  $1/N$ . Since all Green functions are bounded by  $\eta^{-1} \leq N$ , the contribution from the exceptional set is negligible and this proves (2.97). Finally, similar proof yields (2.98).  $\square$

Exchange the index 1 and 2, we can define  $P^{(21)}$  and  $P^{(2)}$  and expand  $Z_2$  as

$$Z_2 = \mathbb{I}\mathbb{E}_2 \overline{\mathbf{a}^2} \cdot P^{(21)}\mathbf{a}^2 + \mathbb{I}\mathbb{E}_2 \overline{\mathbf{a}^2} \cdot P^{(2)}\mathbf{a}^2 \quad (2.103)$$

Here  $P_{kl}^{(21)}$  is independent of  $\mathbf{a}^2$  and  $\mathbf{a}^1$ ;  $P_{kl}^{(2)}$  is independent of  $\mathbf{a}^2$ . Combining (2.103) with (2.96), we have for the  $\alpha = 1, \beta = 2$  cross term in (2.90) that

$$\mathbb{E} \overline{Z}_1 Z_2 = \mathbb{E} \left[ \left( \mathbb{I}\mathbb{E}_1 \left\{ \mathbf{a}^1 \cdot \overline{P^{(12)}\mathbf{a}^1} + \mathbf{a}^1 \cdot \overline{P^{(1)}\mathbf{a}^1} \right\} \right) \left( \mathbb{I}\mathbb{E}_2 \left\{ \overline{\mathbf{a}^2} \cdot P^{(21)}\mathbf{a}^2 + \overline{\mathbf{a}^2} \cdot P^{(2)}\mathbf{a}^2 \right\} \right) \right]. \quad (2.104)$$

Note that if  $X^{(i)}$  is a random variable, independent of  $\mathbf{a}^i$ , then for any random variable  $Y$ , we have

$$\mathbb{I}\mathbb{E}_i [Y X^{(i)}] = X^{(i)} \mathbb{I}\mathbb{E}_i Y$$

in particular  $\mathbb{E}\mathbb{E}_i X^{(i)} = 0$  (with  $Y = 1$ ). Thus

$$\mathbb{E}\left[\left(\mathbb{E}\mathbb{E}_1 X^{(2)}\right)\left(\mathbb{E}\mathbb{E}_2 X^{(1)}\right)\right] = \mathbb{E}\left[\mathbb{E}\mathbb{E}_1\left[\left(\mathbb{E}\mathbb{E}_2 X^{(1)}\right)X^{(2)}\right]\right] = 0$$

since  $\mathbb{E}\mathbb{E}_i = 0$ . Using this idea, one can easily see that the only non-vanishing term on the right hand side of (2.104) is

$$\mathbb{E}\left(\mathbb{E}\mathbb{E}_1 \overline{\mathbf{a}^1 P^{(1)} \mathbf{a}^1}\right)\left(\mathbb{E}\mathbb{E}_2 \overline{\mathbf{a}^2 P^{(2)} \mathbf{a}^2}\right). \quad (2.105)$$

By the Cauchy-Schwarz inequality and Lemma 2.19, we obtain

$$|\mathbb{E}\overline{Z}_1 Z_2| \lesssim \frac{1}{(M\eta)^2}. \quad (2.106)$$

Since the indices 1 and 2 can be replaced by  $\alpha \neq \beta$ , together with (2.90) we have thus proved Lemma 2.18 for  $p = 2$ . □

## 2.4 Delocalization of eigenvectors

Let  $H$  be a universal Wigner matrix with subexponential decay (2.32). Let  $\mathbf{v}$  be an  $\ell^2$ -normalized eigenvector of  $H$ , then the size of the  $\ell^p$ -norm of  $\mathbf{v}$ , for  $p > 2$ , gives information about delocalization. Complete delocalization occurs when  $\|\mathbf{v}\|_p \lesssim N^{-1/2+1/p}$  (note that  $\|\mathbf{v}\|_p \geq CN^{-1/2+1/p}\|\mathbf{v}\|_2$ ). The following result shows that for generalized Wigner matrices (1.62), the eigenvectors are fully delocalized with a very high probability. For universal Wigner matrices with spread  $M$  (2.31), the eigenvectors are delocalized on scale at least  $M$ .

**Theorem 2.20** *Under the conditions of Theorem 2.6, for any  $E$  with  $\kappa = |E - 2| \geq \kappa_0$ , we have*

$$\mathbb{P}\left\{\exists \mathbf{v} : H\mathbf{v} = \lambda\mathbf{v}, |\lambda - E| \leq \frac{1}{M}, \|\mathbf{v}\|_2 = 1, \|\mathbf{v}\|_\infty \geq \frac{(\log N)^C}{\sqrt{M}}\right\} \leq CN^{-c \log \log N}. \quad (2.107)$$

We remark that  $\|\mathbf{v}\|_\infty \lesssim M^{-1/2}$  indicates that the eigenvector has to be supported in a region of size at least  $M$ , i.e. the localization length of  $\mathbf{v}$  is at least  $M$ . We note that the delocalization conjecture predicts, that the localization length is in fact  $M^2$ , i.e. the optimal bound should be

$$\|\mathbf{v}\|_\infty \lesssim \frac{1}{M}$$

with a high probability. As it was explained in Section 1.3, this is an open question. Only some partial results are available, i.e. we proved in [34, 35] that for random band matrices (1.18) with band width  $W \sim M$ , the localization length is at least  $M^{1+\frac{1}{6}}$ .

*Proof.* We will only prove (2.107). We again neglect the dependence of the estimate on  $\kappa_0$  (this can be traced back from the proof). The estimate (2.45) guarantees that

$$|G_{ii}(z)| \leq C$$

for any  $z = E + i\eta$  with  $M\eta \geq (\log N)^C$  with a very high probability. Choose  $\eta = (\log N)^C/M$ . Let  $\mathbf{v}_\alpha$  be the eigenvectors of  $H$  and let  $\mathbf{v}$  be an eigenvector with eigenvalue  $|\lambda - E| \leq 1/M$ . Thus

$$C \geq |\Im G_{ii}| = \sum_{\alpha} \frac{\eta |\mathbf{v}_\alpha(i)|^2}{(\lambda_\alpha - E)^2 + \eta^2} \geq \frac{\eta |\mathbf{v}(i)|^2}{M^{-2} + \eta^2} \geq \eta^{-1} |\mathbf{v}(i)|^2$$

i.e.

$$\|\mathbf{v}\|_\infty \leq \frac{(\log N)^C}{\sqrt{M}}.$$

□

Note that the proof was very easy since pointwise bounds on the diagonal elements of the resolvent were available. We remark that it is possible to prove this theorem relying only on the semicircle law, in fact this was our traditional path in [37, 38, 39]. For example, in [39] we proved:

**Theorem 2.21** [39, Corollary 3.2] *Let  $H$  be a Wigner matrix with single entry distribution with a Gaussian decay. Then for any  $|E| < 2$ , fixed  $K$  and  $2 < p < \infty$  we have*

$$\mathbb{P} \left\{ \exists \mathbf{v} : H\mathbf{v} = \lambda\mathbf{v}, |\lambda - E| \leq \frac{K}{N}, \|\mathbf{v}\|_2 = 1, \|\mathbf{v}\|_p \geq QN^{-\frac{1}{2} + \frac{1}{p}} \right\} \leq Ce^{-c\sqrt{Q}}$$

for  $Q$  and  $N$  large enough.

*Sketch of the proof.* We will give the proof of a weaker result, where logarithmic factors are allowed. Suppose that  $\lambda \in [-2 + \kappa, 2 - \kappa]$  and that  $H\mathbf{v} = \lambda\mathbf{v}$ . Consider the decomposition

$$H = \begin{pmatrix} h & \mathbf{a}^* \\ \mathbf{a} & H^{(1)} \end{pmatrix} \quad (2.108)$$

introduced in Section 2.1, i.e. here  $\mathbf{a} = (h_{12}, \dots, h_{1N})^*$  and  $H^{(1)}$  is the  $(N-1) \times (N-1)$  matrix obtained by removing the first row and first column from  $H$ . Let  $\mu_\alpha$  and  $\mathbf{u}_\alpha$  (for  $\alpha = 1, 2, \dots, N-1$ ) denote the eigenvalues and the normalized eigenvectors of  $H^{(1)}$ . From the eigenvalue equation  $H\mathbf{v} = \lambda\mathbf{v}$  we find

$$\begin{aligned} (h - \lambda)v_1 + \mathbf{a} \cdot \mathbf{v}' &= 0 \\ v_1\mathbf{a} + (H^{(1)} - \lambda)\mathbf{v}' &= 0 \end{aligned}$$

where we decomposed the eigenvector  $\mathbf{v} = (v_1, v')$ ,  $v_1 \in \mathbb{R}$ ,  $v' \in \mathbb{R}^{N-1}$ . Solving the second equation for  $v'$  we get  $v' = v_1(\lambda - H^{(1)})^{-1}\mathbf{a}$ . From the normalization condition,  $\|\mathbf{v}\|^2 = v_1^2 + \|v'\|^2 = 1$  we thus obtain for the first component of  $\mathbf{v}$  that

$$|v_1|^2 = \frac{1}{1 + \mathbf{a} \cdot (\lambda - H^{(1)})^{-2}\mathbf{a}} = \frac{1}{1 + \frac{1}{N} \sum_{\alpha} \frac{\xi_{\alpha}}{(\lambda - \mu_{\alpha})^2}} \leq \frac{4N\eta^2}{\sum_{\mu_{\alpha} \in I} \xi_{\alpha}}, \quad (2.109)$$

where in the second equality we set  $\xi_{\alpha} = |\sqrt{N}\mathbf{a} \cdot \mathbf{u}_{\alpha}|^2$  and used the spectral representation of  $H^{(1)}$ . We also chose an interval  $I$  of length  $\eta = |I| = Q/N$ . Is it easy to check that  $\mathbb{E}\xi_{\alpha} = 1$  and that different  $\xi_{\alpha}$ 's are essentially independent and they satisfy the following large deviation estimate:

$$\mathbb{P}\left(\sum_{\alpha \in \mathcal{I}} \xi_{\alpha} \leq \frac{m}{2}\right) \leq e^{-c\sqrt{m}}$$

where  $m = |\mathcal{I}|$  is the cardinality of the index set. There are several proofs, depending on the condition on the single site distribution, e.g., under Gaussian decay condition, see Lemma 4.7 of [39] that relies on the Hanson-Wright theorem [59].

Let now  $\mathcal{N}_I$  denote the number of eigenvalues of  $H$  in  $I$ . From the local semicircle (e.g. Theorem 1.8) we know that  $\mathcal{N}_I$  is of order  $N|I|$  for any interval  $I$  away from the edge and  $|I| \gg 1/N$ . We recall that the eigenvalues of  $H$ ,  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$ , and the eigenvalues of  $H^{(1)}$  are interlaced. This means that there exist at least  $\mathcal{N}_I - 1$  eigenvalues of  $H^{(1)}$  in  $I$ . Therefore, using that the components of any eigenvector are identically distributed, we have

$$\begin{aligned} & \mathbb{P}\left(\exists \mathbf{v} \text{ with } H\mathbf{v} = \lambda\mathbf{v}, \|\mathbf{v}\| = 1, \lambda \in I \text{ and } \|\mathbf{v}\|_{\infty} \geq \frac{Q}{N^{1/2}}\right) \\ & \leq N^2 \mathbb{P}\left(\exists \mathbf{v} \text{ with } H\mathbf{v} = \lambda\mathbf{v}, \|\mathbf{v}\| = 1, \lambda \in I \text{ and } |v_1|^2 \geq \frac{Q^2}{N}\right) \\ & \leq CN^2 \mathbb{P}\left(\sum_{\mu_{\alpha} \in I} \xi_{\alpha} \leq \frac{4N^2\eta^2}{Q^2}\right) \\ & \leq CN^2 \mathbb{P}\left(\sum_{\mu_{\alpha} \in I} \xi_{\alpha} \leq \frac{4N^2\eta^2}{Q^2} \text{ and } \mathcal{N}_I \geq cN|I|\right) + CN^2 \mathbb{P}(\mathcal{N}_I \leq cN|I|) \\ & \leq CN^2 e^{-c\sqrt{N|I|}} + CN^2 e^{-c\sqrt{N|I|}} \\ & \leq Ce^{-c\sqrt{Q}}, \end{aligned} \quad (2.110)$$

assuming that  $4N^2\eta^2/Q^2 \leq cN|I| = cN|I|$ , i.e. that  $Q \geq \sqrt{N\eta}$ .

Here we used that the deviation from the semicircle law is subexponentially penalized,

$$\mathbb{P}(\mathcal{N}_I \leq cN|I|) \leq e^{-c'\sqrt{N|I|}}$$

for sufficiently small  $c$  and  $c'$  if  $I$  is away from the edge. Such a strong subexponential bound does not directly follow from our strong semicircle law Theorem 2.6, but it can be proved for Wigner matrices with Gaussian decay, Theorem 1.8.  $\square$

### 3 Universality for Gaussian divisible matrix ensembles

#### 3.1 Strong local ergodicity of the Dyson Brownian Motion

In this section, we consider the following general question. Suppose  $\mu = e^{-N\mathcal{H}}/Z$  is a probability measure on the configuration space  $\mathbb{R}^N$  characterized by some Hamiltonian  $\mathcal{H} : \mathbb{R}^N \rightarrow \mathbb{R}$ , where  $Z = \int e^{-N\mathcal{H}(\mathbf{x})} d\mathbf{x} < \infty$  is the normalization. We will always assume that  $\mathcal{H}$  is symmetric under permutation of the variables  $\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$ . The typical example to keep in mind is the Hamiltonian of the general  $\beta$ -ensembles (1.44), or the specific GOE ( $\beta = 1$ ) or GUE ( $\beta = 2$ ) cases.

We consider time dependent permutational symmetric probability measures with density  $f_t(\mathbf{x})$ ,  $t \geq 0$ , with respect to the measure  $\mu(d\mathbf{x}) = \mu(\mathbf{x})d\mathbf{x}$ . The dynamics is characterized by the forward equation

$$\partial_t f_t = Lf_t, \quad t \geq 0, \quad (3.1)$$

with a given permutation symmetric initial data  $f_0$ . Here the generator  $L$  is defined via the Dirichlet form as

$$D(f) = D_\mu(f) = - \int f Lf d\mu = \sum_{j=1}^N \frac{1}{2N} \int (\partial_j f)^2 d\mu, \quad \partial_j = \partial_{x_j}. \quad (3.2)$$

Formally, we have  $L = \frac{1}{2N}\Delta - \frac{1}{2}(\nabla\mathcal{H})\nabla$ . We will ignore the question of the precise form and operator domain of this operator, we just mention that  $D(f)$  is a semibounded quadratic form, so  $L$  can be defined via the Friedrichs extension on  $L^2(d\mu)$  and it can be extended to  $L^1$  as well. In any case, the dynamics is well defined for any  $f_0 \in L^1(d\mu)$  initial data. For more details, see Appendix A of [44].

Strictly speaking, we will consider a sequence of Hamiltonians  $\mathcal{H}_N$  and corresponding dynamics  $L_N$  and  $f_{t,N}$  parametrized by  $N$ , but the  $N$ -dependence will be omitted. All results will concern the  $N \rightarrow \infty$  limit. Alternatively, one could describe the dynamics by a coupled system of stochastic differential equations (1.61) as mentioned in Section 1.6.2, but we will not use this formalism here.

For any  $k \geq 1$  we define the  $k$ -point correlation functions (marginals) of the probability measure  $f_t d\mu$  by

$$p_{t,N}^{(k)}(x_1, x_2, \dots, x_k) = \int_{\mathbb{R}^{N-k}} f_t(\mathbf{x}) \mu(\mathbf{x}) dx_{k+1} \dots dx_N. \quad (3.3)$$

The correlation functions of the equilibrium measure are denoted by

$$p_{\mu,N}^{(k)}(x_1, x_2, \dots, x_k) = \int_{\mathbb{R}^{N-k}} \mu(\mathbf{x}) dx_{k+1} \dots dx_N.$$



We now list our main assumptions on the initial distribution  $f_0$  and on its evolution  $f_t$ . This formalism is adjusted to generalized Wigner matrices; random covariance matrices require some minor modifications that we do not pursue here (see [44] for details). We first define the subdomain

$$\Sigma_N := \{\mathbf{x} \in \mathbb{R}^N, x_1 < x_2 < \dots < x_N\} \quad (3.4)$$

of ordered sets of points  $\mathbf{x}$ .

**Assumption I.** The Hamiltonian  $\mathcal{H}$  of the equilibrium measure has the form

$$\mathcal{H} = \mathcal{H}_N(\mathbf{x}) = \beta \left[ \sum_{j=1}^N U(x_j) - \frac{1}{N} \sum_{i < j} \log |x_i - x_j| \right], \quad (3.5)$$

where  $\beta \geq 1$ . The function  $U : \mathbb{R} \rightarrow \mathbb{R}$  is smooth with  $\inf_{\mathbb{R}} U'' > -\infty$ , and

$$U(x) \geq C|x|^\delta \quad \text{for some } \delta > 0 \text{ and } |x| \text{ large.}$$

Note that this assumption is automatic for symmetric and hermitian Wigner matrices with the GOE or GUE being the invariant measure.

Near the  $x_{i+1} = x_i$  boundary component of  $\Sigma_N$ , the generator has the form

$$L = \frac{1}{4N} \left( \partial_u^2 + \frac{\beta}{u} \partial_u \right) + \text{regular operator}$$

in the relative coordinates  $u = \frac{1}{2}(x_{i+1} - x_i)$  when  $u \ll 1$ . It is known (Appendix A of [44]) that for  $\beta \geq 1$  the repelling force of the singular diffusion is sufficiently strong to prevent the particle from falling into the origin, i.e., in the original coordinates the trajectories of the points do not cross. In particular, the ordering will be preserved under the dynamics (for a stochastic proof, see Lemma 4.3.3 of [4]). In the sequel we will thus assume that  $f_t$  is a probability measure on  $\Sigma_N$ . We continue to use the notation  $f$  and  $\mu$  for the restricted measure. Note that the correlation functions  $p^{(k)}$  from (3.3) are still defined on  $\mathbb{R}^k$ , i.e., their arguments remain unordered.

It follows from Assumption I that the Hessian matrix of  $\mathcal{H}$  satisfies the following bound:

$$\langle \mathbf{v}, \nabla^2 \mathcal{H}(\mathbf{x}) \mathbf{v} \rangle \geq \frac{\beta}{N} \sum_{i < j} \frac{(v_i - v_j)^2}{(x_i - x_j)^2}, \quad \mathbf{v} = (v_1, \dots, v_N) \in \mathbb{R}^N, \quad \mathbf{x} \in \Sigma_N. \quad (3.6)$$

This convexity bound is the key assumption; our method works for a broad class of general Hamiltonians as long as (3.6) holds. In particular, an arbitrary many-body potential function  $V(\mathbf{x})$  can be added to the Hamiltonians (3.5) as long as  $V$  is convex on the open sets  $\Sigma_N$ .

**Assumption II.** There exists a continuous, compactly supported density function  $\varrho(x) \geq 0$ ,  $\int_{\mathbb{R}} \varrho = 1$ , on the real line, independent of  $N$ , such that for any fixed  $a, b \in \mathbb{R}$

$$\lim_{N \rightarrow \infty} \sup_{t \geq 0} \left| \int \frac{1}{N} \sum_{j=1}^N \mathbf{1}(x_j \in [a, b]) f_t(\mathbf{x}) d\mu(\mathbf{x}) - \int_a^b \varrho(x) dx \right| = 0. \quad (3.7)$$

Let  $\gamma_j = \gamma_{j,N}$  denote the location of the  $j$ -th point under the limiting density, i.e.,  $\gamma_j$  is defined by

$$N \int_{-\infty}^{\gamma_j} \varrho(x) dx = j, \quad 1 \leq j \leq N, \quad \gamma_j \in \text{supp} \varrho. \quad (3.8)$$

We will call  $\gamma_j$  the *classical location* of the  $j$ -th point. Note that  $\gamma_j$  may not be uniquely defined if the support of  $\varrho$  is not connected but in this case the next Assumption III will not be satisfied anyway.

**Assumption III.** There exists an  $\varepsilon > 0$  such that

$$\sup_{t \geq 0} \int \frac{1}{N} \sum_{j=1}^N (x_j - \gamma_j)^2 f_t(d\mathbf{x}) \mu(d\mathbf{x}) \leq CN^{-1-2\varepsilon} \quad (3.9)$$

with a constant  $C$  uniformly in  $N$ .

Under Assumption II, the typical spacing between neighboring points is of order  $1/N$  away from the spectral edges, i.e., in the vicinity of any energy  $E$  with  $\varrho(E) > 0$ . Assumption III guarantees that typically the random points  $x_j$  remain in the  $N^{-1/2-\varepsilon}$  vicinity of their classical location.

The final assumption is an upper bound on the local density. For any  $I \in \mathbb{R}$ , let

$$\mathcal{N}_I := \sum_{i=1}^N \mathbf{1}(x_i \in I)$$

denote the number of points in  $I$ .

**Assumption IV.** For any compact subinterval  $I_0 \subset \{E : \varrho(E) > 0\}$ , and for any  $\delta > 0$ ,  $\sigma > 0$  there are constants  $C_n$ ,  $n \in \mathbb{N}$ , depending on  $I_0$ ,  $\delta$  and  $\sigma$  such that for any interval  $I \subset I_0$  with  $|I| \geq N^{-1+\sigma}$  and for any  $K \geq 1$ , we have

$$\sup_{\tau \geq N^{-2\varepsilon+\delta}} \int \mathbf{1}\{\mathcal{N}_I \geq KN|I|\} f_\tau d\mu \leq C_n K^{-n}, \quad n = 1, 2, \dots, \quad (3.10)$$

where  $\varepsilon$  is the exponent from Assumption III.

Note that for the symmetric or hermitian Wigner matrices, Assumption I is automatic, Assumption II is the (global) semicircle law (1.27) and Assumption IV is the upper bound on the density (Lemma 1.7). The really serious condition to check is (3.9).

The main general theorem is the following:

**Theorem 3.1** *Suppose that the Hamiltonian given in (3.5) satisfies Assumption I. Let the initial density  $f_0$  satisfy a bounded entropy condition, i.e.,*

$$S_\mu(f_0) := \int f_0 \log f_0 d\mu \leq CN^m \quad (3.11)$$

*with some fixed exponent  $m$  and suppose that Assumptions II, III and IV hold for the solution  $f_t$  of the forward equation (3.1). Let  $E \in \mathbb{R}$  and  $b > 0$  such that  $\min\{\varrho(x) : x \in [E - b, E + b]\} > 0$ . Then for any  $\delta > 0$ , for any integer  $k \geq 1$  and for any compactly supported continuous test function  $O : \mathbb{R}^k \rightarrow \mathbb{R}$  we have*

$$\begin{aligned} \lim_{N \rightarrow \infty} \sup_{t \geq N^{-2\varepsilon + \delta}} \int_{E-b}^{E+b} dE' \int_{\mathbb{R}^k} d\alpha_1 \dots d\alpha_k O(\alpha_1, \dots, \alpha_k) \\ \times \frac{1}{\varrho(E)^k} \left( p_{t,N}^{(k)} - p_{\mu,N}^{(k)} \right) \left( E' + \frac{\alpha_1}{N\varrho(E)}, \dots, E' + \frac{\alpha_k}{N\varrho(E)} \right) = 0 \end{aligned} \quad (3.12)$$

where  $\varepsilon > 0$  is the exponent from Assumption III.

This theorem shows that the local statistics of the points  $x_j$  in the bulk with respect to the time evolved distribution  $f_t$  coincides with the local statistics with respect to the equilibrium distribution  $\mu$  as long as  $t$  is much larger than  $N^{-2\varepsilon}$ .

In many applications, the local equilibrium statistics can be explicitly computed and in the  $b \rightarrow 0$  limit it becomes independent of  $E$ , in particular this is the case for the classical matrix ensembles. The simplest explicit formula is for the GUE case, when the correlation functions are given by the sine kernel (1.34).

### 3.2 The local relaxation flow

The main idea behind the proof of Theorem 3.1 is to analyze the relaxation to equilibrium of the dynamics (3.1). The equilibrium is given by an essentially convex Hamiltonian  $\mathcal{H}$  so the Bakry-Emery method applies. This method was first used in the context of the Dyson Brownian motion in Section 5.1 of [41].

To explain the idea, assume, temporarily, that the potential  $U$  in (3.5) is uniformly convex, i.e.

$$U''(x) \geq U_0 > 0.$$

This is certainly the case for the Gaussian ensembles when  $U(x) = \frac{1}{4}x^2$ . Then we have the following lower bound on the Hessian of  $\mathcal{H}$

$$\text{Hess } \mathcal{H} \geq \beta U_0 \quad (3.13)$$

on the set  $\Sigma_N$  (3.4) since the logarithmic potential is convex. It is essential at this point to stress that (3.13) holds only in the *open* set  $\Sigma_N$ , since the second derivatives of the logarithmic interactions have a delta function singularity (with the “wrong” sign) on the boundary. It requires a separate technical argument to show that for  $\beta \geq 1$  the points sufficiently repel each other so that the Dyson Brownian motion never leaves the open set  $\Sigma_N$  and thus the Bakry-Emery method [8] applies. See a remark after Theorem 3.2.

We devote the next pedagogical section to recall the Bakry-Emery criterion in a general setup on  $\mathbb{R}^N$ .

### 3.2.1 Bakry-Emery method

Let the probability measure  $\mu$  on  $\mathbb{R}^N$  be given by a strictly convex Hamiltonian  $\mathcal{H}$ :

$$d\mu(\mathbf{x}) = \frac{e^{-\mathcal{H}(\mathbf{x})}}{Z} d\mathbf{x}, \quad \nabla^2 \mathcal{H}(\mathbf{x}) = \text{Hess } \mathcal{H}(\mathbf{x}) \geq K > 0 \quad (3.14)$$

with some constant  $K$ , and let  $L$  be the generator of the dynamics associated with the Dirichlet form

$$D(f) = D_\mu(f) = - \int f L f d\mu := \frac{1}{2} \sum_j \int (\partial_j f)^2 d\mu, \quad \partial_j = \partial_{x_j}$$

(note that in this section we neglect the prefactor  $N$  in front of  $\mathcal{H}$  in the definition of  $\mu$  and the prefactor  $1/N$  originally present in (3.2)). Formally we have  $L = \frac{1}{2}\Delta - \frac{1}{2}(\nabla \mathcal{H})\nabla$ . The operator  $L$  is symmetric with respect to the measure  $d\mu$ , i.e.

$$\int f L g d\mu = \int (L f) g d\mu = -\frac{1}{2} \int \nabla f \cdot \nabla g d\mu. \quad (3.15)$$

We define the relative entropy of any probability measure  $f$  with  $\int f d\mu = 1$  by

$$S_\mu(f) = S(f) = \int f(\log f) d\mu.$$

Both the Dirichlet form and the entropy are non-negative, they are zero only for  $f \equiv 1$  and they measure the distance of  $f$  from equilibrium  $f = 1$ . The entropy can be used to control the total variation norm directly via the *entropy inequality*

$$\int |f - 1| d\mu \leq \sqrt{2S_\mu(f)}. \quad (3.16)$$

Let  $f_t$  be the solution to the evolution equation

$$\partial_t f_t = Lf_t, \quad t > 0 \quad (3.17)$$

with a given initial condition  $f_0$  and consider the evolution of the entropy  $S(f_t)$  and the Dirichlet form  $D(\sqrt{f_t})$ . Simple calculation shows

$$\partial_t S(f_t) = \int (Lf_t) \log f_t d\mu + \int f_t \frac{Lf_t}{f_t} d\mu = -\frac{1}{2} \int \frac{(\nabla f_t)^2}{f_t} d\mu = -4D(\sqrt{f_t}), \quad (3.18)$$

where we used that  $\int Lf_t d\mu = 0$  by (3.15). Similarly, we can compute the evolution of the Dirichlet form. Let  $h := \sqrt{f}$  for simplicity, then

$$\partial_t h_t = \frac{1}{2h_t} \partial h_t^2 = \frac{1}{2h_t} Lh_t^2 = Lh_t + \frac{1}{2h_t} (\nabla h_t)^2.$$

In the last step we used that  $Lh^2 = (\nabla h)^2 + 2hLh$  that can be seen either directly from  $L = \frac{1}{2}\Delta - \frac{1}{2}(\nabla \mathcal{H})\nabla$  or from the following identity for any test function  $g$ :

$$\int gLh^2 d\mu = -\frac{1}{2} \int \nabla g \cdot \nabla (h^2) d\mu = \int [-\nabla(hg) + g\nabla h] \nabla h d\mu = \int g [(\nabla h)^2 + 2hLh] d\mu.$$

We compute (dropping the  $t$  subscript for brevity)

$$\begin{aligned} \partial_t D(\sqrt{f_t}) &= \frac{1}{2} \partial_t \int (\nabla h)^2 d\mu \\ &= \int (\nabla h)(\nabla Lh) d\mu + \frac{1}{2} \int (\nabla h) \cdot \nabla \frac{(\nabla h)^2}{h} d\mu \\ &= \int (\nabla h)[\nabla, L]h d\mu + \int (\nabla h)L(\nabla h) d\mu + \frac{1}{2} \int \sum_{ij} \partial_i h \left[ \frac{2(\partial_j h)\partial_i \partial_j h}{h} - \frac{(\partial_j h)^2 \partial_i h}{h^2} \right] d\mu \\ &= -\frac{1}{2} \int (\nabla h)(\nabla^2 \mathcal{H})\nabla h d\mu - \frac{1}{2} \int \sum_{ij} (\partial_i \partial_j h)^2 d\mu + \frac{1}{2} \int \sum_{ij} \left[ \frac{2(\partial_j h)(\partial_i h)\partial_{ij} h}{h} - \frac{(\partial_j h)^2 (\partial_i h)^2}{h^2} \right] d\mu \\ &= -\frac{1}{2} \int (\nabla h)(\nabla^2 \mathcal{H})\nabla h d\mu - \frac{1}{2} \int \sum_{ij} \left( \partial_{ij} h - \frac{(\partial_i h)(\partial_j h)}{h} \right)^2 d\mu, \end{aligned} \quad (3.19)$$

where we used the commutator

$$[\nabla, L] = -\frac{1}{2}(\nabla^2 \mathcal{H})\nabla.$$

Therefore, under the convexity condition (3.14), we have

$$\partial_t D(\sqrt{f_t}) \leq -KD(\sqrt{f_t}). \quad (3.20)$$

Combining (3.18) and (3.20),

$$\partial_t D(\sqrt{f_t}) \leq \frac{K}{4} \partial_t S(f_t). \quad (3.21)$$

At  $t = \infty$  the equilibrium is achieved,  $f_\infty = 1$ , and both the entropy and the Dirichlet form are zero. After integrating (3.21) back from  $t = \infty$ , we get the *logarithmic Sobolev inequality*

$$S(f_t) \leq \frac{4}{K} D(\sqrt{f_t}) \quad (3.22)$$

for any  $t \geq 0$ , in particular for any initial distribution  $f = f_0$ . Inserting this back to (3.18), we have

$$\partial_t S(f_t) \leq -KS(f_t).$$

Integrating from time 0, we obtain the *exponential relaxation of the entropy on time scale  $t \sim 1/K$*

$$S(f_t) \leq e^{-tK} S(f_0). \quad (3.23)$$

Finally, we can integrate (3.18) from time  $t/2$  to  $t$  to get

$$S(f_t) - S(f_{t/2}) = -4 \int_{t/2}^t D(\sqrt{f_\tau}) d\tau.$$

Using the positivity of the entropy  $S(f_t) \geq 0$  on the left side and the monotonicity of the Dirichlet form (from (3.20)) on the right side, we get

$$D(\sqrt{f_t}) \leq \frac{2}{t} S(f_{t/2}), \quad (3.24)$$

thus, using (3.23), we obtain *exponential relaxation of the Dirichlet form on time scale  $t \sim 1/K$*

$$D(\sqrt{f_t}) \leq \frac{2}{t} e^{-tK/2} S(f_0).$$

We summarize the result of this calculation:

**Theorem 3.2** [8] *Assuming the convexity bound on the Hamiltonian,  $\nabla^2 \mathcal{H} \geq K$  with some positive constant  $K$ , the measure  $\mu = e^{-\mathcal{H}}/Z$  satisfies the logarithmic Sobolev inequality*

$$S(f) \leq \frac{4}{K} D(\sqrt{f}), \quad \text{for any density } \int f d\mu = 1, \quad (3.25)$$

*and the dynamics (3.17) relaxes to equilibrium on the time scale  $t \sim 1/K$  both in the sense of entropy and Dirichlet form:*

$$S(f_t) \leq e^{-tK} S(f_0), \quad D(\sqrt{f_t}) \leq \frac{2}{t} e^{-tK/2} S(f_0). \quad (3.26)$$

□

**Technical remark.** In our application, the dynamics will be restricted to the subset  $\Sigma_N = \{\mathbf{x} : x_1 < x_2 < \dots < x_N\}$ , and thus we need to check that the boundary term in the integration by parts

$$\int_{\partial\Sigma} \partial_i h \partial_{i_j}^2 h e^{-\mathcal{H}} = 0 \quad (3.27)$$

(from the third line to the fourth line in (3.19)) vanishes. The role of  $\mathcal{H}$  will be played by  $N\mathcal{H}_N$  where  $\mathcal{H}_N$  is defined in (3.5). Although the density function of the measure  $e^{-N\mathcal{H}_N}$  behaves as  $(x_{i+1} - x_i)^\beta$  near the  $x_{i+1} = x_i$  component of the boundary, hence it vanishes at the boundary, the function  $f_t$  is the solution to a parabolic equation with a singular drift, so in principle it may blow up at the boundary. Further complication is that  $h = \sqrt{f}$  and the derivative of the square root is singular. Nevertheless, by using parabolic regularity theory and cutoff functions properly subordinated to the geometry of the set  $\Sigma_N$ , we can prove that (3.27) vanishes. This is one reason why the restriction  $\beta \geq 1$  is necessary. For the details, see Appendix B of [44].

### 3.2.2 Universality of gap distribution of the Dyson Brownian Motion for Short Time

Using the convexity bound (3.13) for the Dyson Brownian motion, the Bakry-Emery method guarantees that  $\mu$  satisfies the logarithmic Sobolev inequality and the relaxation time to equilibrium is of order one. The following result is the main theorem of Section 3.2. It shows that the relaxation time is in fact much shorter than order one at least locally and for observables that depend only on the eigenvalue *differences*.

**Theorem 3.3 (Universality of Dyson Brownian Motion for Short Time)** *Suppose that the Hamiltonian  $\mathcal{H}$  given in (3.5) satisfies the convexity bound (3.6) with  $\beta \geq 1$ . Let  $f_t$  be the solution of the forward equation (3.1) with initial density  $f_0$  that satisfies  $S_\mu(f_0) := \int f_0(\log f_0)d\mu \leq CN^m$  for some  $m$  fixed. Set*

$$Q := \sup_{t \geq 0} \sum_j \int (x_j - \gamma_j)^2 f_t d\mu, \quad (3.28)$$

and assume that  $Q \leq CN^m$  with some exponent  $m$ . Fix  $n \geq 1$  and an array of positive integers,  $\mathbf{m} = (m_1, m_2, \dots, m_n) \in \mathbb{N}_+^n$ . Let  $G : \mathbb{R}^n \rightarrow \mathbb{R}$  be a bounded smooth function with compact support and define

$$\mathcal{G}_{i,\mathbf{m}}(\mathbf{x}) := G\left(N(x_i - x_{i+m_1}), N(x_{i+m_1} - x_{i+m_2}), \dots, N(x_{i+m_{n-1}} - x_{i+m_n})\right). \quad (3.29)$$

Then for any sufficiently small  $\varepsilon' > 0$ , there exist constants  $C, c$ , depending only on  $\varepsilon'$  and  $G$  such that for any  $J \subset \{1, 2, \dots, N - m_n\}$  and any  $\tau > 0$ , we have

$$\left| \int \frac{1}{N} \sum_{i \in J} \mathcal{G}_{i,\mathbf{m}}(\mathbf{x}) f_\tau d\mu - \int \frac{1}{N} \sum_{i \in J} \mathcal{G}_{i,\mathbf{m}}(\mathbf{x}) d\mu \right| \leq CN^{\varepsilon'} \sqrt{Q\tau^{-1}} + Ce^{-cN^{\varepsilon'}}. \quad (3.30)$$

In Section 3.2.3 we explain the intuition behind the proof. The precise proof will be given in Section 3.2.4.

### 3.2.3 Intuitive proof of Theorem 3.3

The key idea is that we can “speed up” the convergence to local equilibrium by modifying the dynamics by adding an auxiliary potential  $W(\mathbf{x})$  to the Hamiltonian. It will have the form

$$W(\mathbf{x}) = \sum_{j=1}^N W_j(x_j), \quad W_j(x) = \frac{1}{2R^2}(x_j - \gamma_j)^2, \quad (3.31)$$

i.e. it is a quadratic confinement on scale  $R$  for each eigenvalue near its classical location, and we define

$$\tilde{\mathcal{H}} = \mathcal{H} + W. \quad (3.32)$$

The new measure is denoted by

$$\omega(d\mathbf{x}) = e^{-N\tilde{\mathcal{H}}(\mathbf{x})} / \tilde{Z} \, d\mathbf{x}$$

and it will be called *the local relaxation measure*. We will typically choose  $R \ll 1$ , so that the additional term  $W$  substantially increases the lower bound (3.13) on the Hessian, hence speeding up the dynamics from relaxation time on scale  $O(1)$  to  $O(R^2)$ . This is the **first step** of the proof and it will be formulated in Theorem 3.4 whose proof basically follows the Bakry-Emery argument from Section 3.2.1.

In the **second step**, we consider an arbitrary measure of the form  $q\omega$  and we control the difference of expectation values

$$\int \mathbf{G}q d\omega - \int \mathbf{G} d\omega \quad (3.33)$$

of observables

$$\mathbf{G} = \frac{1}{N} \sum_{i \in J} \mathcal{G}_{i,\mathbf{m}}$$

in terms of the entropy and the Dirichlet form of  $q$  with respect to  $\omega$ . Eventually, this will enable us to compare the expectations

$$\int \mathbf{G}q d\omega - \int \mathbf{G}q' d\omega$$

for any two measures  $q\omega$  and  $q'\omega$ , in particular for the measures  $f_\tau\mu$  and  $\mu$  that will be written in this form by defining  $q = f_\tau\mu/\omega$  and  $q' = \mu/\omega$ .

Here we face with an  $N$ -problem: both the entropy and the Dirichlet form are extensive quantities. A direct application of the entropy inequality (3.16) to (3.33) would estimate the observable  $\mathbf{G}$ , an order one quantity, by a quantity of order  $O(\sqrt{N})$ . We can run the new dynamics up to some time  $\tau$  and write

$$\int \mathbf{G}q d\omega - \int \mathbf{G} d\omega = \int \mathbf{G}(q - q_\tau) d\omega + \int \mathbf{G}(q_\tau - 1) d\omega.$$



If  $\tau$  is larger than the relaxation time of the new dynamics, then the second term is exponentially small by the entropy inequality. In the first term we can integrate back the time derivative to extract a factor proportional with  $\tau$ , but it will require to differentiate the observable that brings in an additional  $N$  factor due to its scaling, which looks bad. However, due to the lower bound (3.6) on the Hessian, the dynamics relaxes even much faster in certain directions, namely for observables *depending only on differences of  $x_i$ 's*. This mechanism can be extracted from the Bakry-Emery argument (see (3.42)) and it can be used to gain an additional  $N$ -factor; this is the content of Theorem 3.5.

Finally, in the **third step**, we have to compare the original dynamics with the new one in the sense of entropy and Dirichlet form since  $D_\omega(\sqrt{f_\tau\mu/\omega})$  and  $S_\omega(f_\tau\mu/\omega)$  need to be computed for the estimates in the second step. These would be given by the standard decay to equilibrium estimates (3.26) if  $f_\tau\mu/\omega$  were evolving with respect to the modified dynamics, but  $f_\tau$  evolves by the original dynamics. We thus need to show that the error due to the modification of the dynamics by adding  $W$  is negligible.

It turns out that the key quantity that determines how much error was made by the additional potential is the  $H^1$  norm of  $W$ , i.e.

$$\Lambda_t = \int (\nabla W)^2 f_t d\mu.$$

Due to the explicit form of  $W$ , we have

$$\Lambda_t = R^{-4} \sum_i \int (x_i - \gamma_i)^2 f_t d\mu \leq CN^{-2\varepsilon} R^{-4} \quad (3.34)$$

using Assumption III (3.9). Given  $\varepsilon > 0$ , we can therefore choose an  $R \ll 1$  so that we still have  $\Lambda \ll 1$ . This will complete the proof.

At first it is surprising that such an idea has any chance to work; eventually one either modifies the dynamics substantially or does not. But the point is that the speed of convergence is determined by the *second* derivative of the auxiliary potential, while the modification in the Dirichlet form and the entropy is determined by the  $(\nabla W)^2$ , i.e., there is some room to play. So one can speed up the dynamics and still compare Dirichlet forms and entropies of the two equilibrium measures if a strong apriori bound (3.34) on  $\Lambda$  is given. This is one of the reasons why the method works.

The other key observation is the effective use of the convexity bound (3.6) which exploits a crucial property of the dynamics of the Dyson Brownian motion (1.61). The logarithmic interaction potential gives rise to a singular force

$$F(x_i) = -\frac{1}{4}x_i - \frac{1}{2N} \sum_{j \neq i} \frac{1}{x_i - x_j} \quad (3.35)$$

acting on the  $i$ -th particle. Formally  $F(x_i)$  is a mean field force, and if  $x_j$  were distributed according to the semicircle law, then the bulk of the sum would cancel the  $-\frac{1}{4}x_i$  term. However, the effect of

the neighboring particles,  $j = i \pm 1$ , is huge: they exert a force of order one on  $x_i$ . Such a force may move the particle  $x_i$  by a distance of order  $1/N$  within a very short time of order  $1/N$ . Note that by the order preserving property of the dynamics, the  $O(1/N)$  interval is roughly the whole space available for  $x_i$ , at least in the bulk. Thus  $x_i$  is likely to relax to its equilibrium within a time scale of order  $1/N$  due to the strong repulsive force from its neighbors. Of course this argument is not a proof since the other particles move as well. However, our observables involve only eigenvalue differences and in the difference coordinates the strong drift is persistently present. This indicates that the eigenvalue differences may relax to their local equilibrium on a time scale almost of order  $1/N$ .

For other observables, the relaxation time is necessary longer. In particular, it can happen that there is no precise cancellation from the bulk in (3.35), in which case the neighboring particles all feel the same mean field drift and will move collectively. In fact, if the initial density profile substantially differs from the semicircle, the relaxation to the semicircle may even take order one time (although such scenario is excluded in our case by (3.9)).

### 3.2.4 Detailed proof of Theorem 3.3

Every constant in this proof depends on  $\varepsilon'$  and  $G$ , and we will not follow the precise dependence. Given  $\tau > 0$ , we define  $R := \tau^{1/2} N^{-\varepsilon'/2}$ .

We now introduce the local relaxation measure,  $\omega_N = \omega = \psi\mu$ , defined by

$$\psi = \frac{Z}{\tilde{Z}} \exp(-NW),$$

where  $\tilde{Z}$  is chosen such that  $\omega$  is a probability measure, in particular  $\omega = e^{-N\tilde{\mathcal{H}}}/\tilde{Z}$  with

$$\tilde{\mathcal{H}} = \mathcal{H} + W.$$

The potential  $W$  was defined in (3.31) and it confines the  $j$ -th point  $x_j$  near its classical location.

The local relaxation flow is defined to be the reversible dynamics w.r.t.  $\omega$ . The dynamics is described by the generator  $\tilde{L}$  defined by

$$\int f \tilde{L}g d\omega = -\frac{1}{2N} \sum_j \int (\partial_j f)(\partial_j g) d\omega. \quad (3.36)$$

Explicitly,  $\tilde{L}$  is given by

$$L = \tilde{L} + \sum_j b_j \partial_j, \quad b_j = W_j'(x_j) = \frac{x_j - \gamma_j}{R^2}. \quad (3.37)$$

Since the additional potential  $W_j$  is uniformly convex with

$$\inf_j \inf_{x \in \mathbb{R}} W_j''(x) \geq R^{-2}, \quad (3.38)$$

by (3.6) and  $\beta \geq 1$  we have

$$\left\langle \mathbf{v}, \nabla^2 \tilde{\mathcal{H}}(\mathbf{x}) \mathbf{v} \right\rangle \geq \frac{1}{R^2} \|\mathbf{v}\|^2 + \frac{1}{N} \sum_{i < j} \frac{(v_i - v_j)^2}{(x_i - x_j)^2}, \quad \mathbf{v} \in \mathbb{R}^N. \quad (3.39)$$

The  $R^{-2}$  in the first term comes from the additional convexity of the local interaction and it enhances the “local Dirichlet form dissipation”. In particular we have the uniform lower bound

$$\nabla^2 \tilde{\mathcal{H}} = \text{Hess}(-\log \omega) \geq R^{-2}.$$

This guarantees that the relaxation time to  $\omega$  for the  $\tilde{L}$  dynamics is bounded above by  $CR^2$ .

The first ingredient to prove Theorem 3.3 is the analysis of the local relaxation flow which satisfies the logarithmic Sobolev inequality and the following dissipation estimate. Its proof follows the standard argument presented in Section 3.2.1. The key additional input is the convexity bound (3.6) which gives rise to the second term on the r.h.s of (3.41) from the last line of (3.19).

**Theorem 3.4** *Suppose (3.39) holds. Consider the forward equation*

$$\partial_t q_t = \tilde{L} q_t, \quad t \geq 0, \quad (3.40)$$

*with initial condition  $q_0 = q$  and with reversible measure  $\omega$ . Assume that  $q \in L^\infty(d\omega)$ . Then we have the following estimates*

$$\partial_t D_\omega(\sqrt{q_t}) \leq -\frac{1}{R^2} D_\omega(\sqrt{q_t}) - \frac{1}{2N^2} \int \sum_{i,j=1}^N \frac{(\partial_i \sqrt{q_t} - \partial_j \sqrt{q_t})^2}{(x_i - x_j)^2} d\omega, \quad (3.41)$$

$$\frac{1}{2N^2} \int_0^\infty ds \int \sum_{i,j=1}^N \frac{(\partial_i \sqrt{q_s} - \partial_j \sqrt{q_s})^2}{(x_i - x_j)^2} d\omega \leq D_\omega(\sqrt{q}) \quad (3.42)$$

*and the logarithmic Sobolev inequality*

$$S_\omega(q) \leq CR^2 D_\omega(\sqrt{q}) \quad (3.43)$$

*with a universal constant  $C$ . Thus the time to equilibrium is of order  $R^2$ :*

$$S_\omega(q_t) \leq e^{-Ct/R^2} S_\omega(q_0). \quad (3.44)$$

*Proof.* This theorem can be proved following the standard argument presented in Section 3.2.1. The key additional input is the convexity bound (3.6) which gives rise to the second term on the r.h.s of (3.41) from the last line of (3.19). In particular, this serves as an upper bound on  $\partial_t D_\omega(\sqrt{q_t})$ , thus integrating (3.41) one obtains (3.42)  $\square$

The estimate (3.42) on the second term in (3.39) plays a key role in the next theorem.

**Theorem 3.5** Suppose that Assumption I holds and let  $q \in L^\infty$  be a density,  $\int q d\omega = 1$ . Fix  $n \geq 1$ ,  $\mathbf{m} \in \mathcal{N}_+^n$ , let  $G : \mathbb{R}^n \rightarrow \mathbb{R}$  be a bounded smooth function with compact support and recall the definition of  $\mathcal{G}_{i,\mathbf{m}}$  from (3.29). Then for any  $J \subset \{1, 2, \dots, N - n\}$  and any  $\tau > 0$  we have

$$\left| \int \frac{1}{N} \sum_{i \in J} \mathcal{G}_{i,\mathbf{m}}(\mathbf{x}) q d\omega - \int \frac{1}{N} \sum_{i \in J} \mathcal{G}_{i,\mathbf{m}}(\mathbf{x}) d\omega \right| \leq C \left( \frac{D_\omega(\sqrt{q})\tau}{N} \right)^{1/2} + C \sqrt{S_\omega(q)} e^{-c\tau/R^2}. \quad (3.45)$$

*Proof.* For simplicity, we will consider the case when  $\mathbf{m} = (1, 2, \dots, n)$ , the general case easily follows by appropriately redefining the function  $G$ . Let  $q_t$  satisfy

$$\partial_t q_t = \tilde{L} q_t, \quad t \geq 0,$$

with an initial condition  $q_0 = q$ .

We write

$$\int \left[ \frac{1}{N} \sum_{i \in J} \mathcal{G}_{i,\mathbf{m}} \right] (q - 1) d\omega = \int \left[ \frac{1}{N} \sum_{i \in J} \mathcal{G}_{i,\mathbf{m}} \right] (q - q_\tau) d\omega + \int \left[ \frac{1}{N} \sum_{i \in J} \mathcal{G}_{i,\mathbf{m}} \right] (q_\tau - 1) d\omega.$$

The second term can be estimated by the entropy inequality (3.16), by the decay of the entropy (3.44) and the boundedness of  $G$ , this gives the second term in (3.45).

To compare  $q_0 = q$  with  $q_\tau$ , by differentiation, we have

$$\begin{aligned} & \int \frac{1}{N} \sum_i \mathcal{G}_{i,\mathbf{m}}(\mathbf{x}) q_\tau d\omega - \int \frac{1}{N} \sum_i \mathcal{G}_{i,\mathbf{m}}(\mathbf{x}) q_0 d\omega \\ &= \int_0^\tau ds \int \frac{1}{N} \sum_i \sum_{k=1}^n \partial_k G \left( N(x_i - x_{i+1}), \dots, N(x_{i+n-1} - x_{i+n}) \right) [\partial_{i+k-1} q_s - \partial_{i+k} q_s] d\omega. \end{aligned}$$

From the Schwarz inequality and  $\partial q = 2\sqrt{q}\partial\sqrt{q}$ , the last term is bounded by

$$\begin{aligned} & 2 \sum_{k=1}^n \left[ \int_0^\tau ds \int \sum_i \left[ \partial_k G \left( N(x_i - x_{i+1}), \dots, N(x_{i+n-1} - x_{i+n}) \right) \right]^2 (x_{i+k-1} - x_{i+k})^2 q_s d\omega \right]^{1/2} \\ & \times \left[ \int_0^\tau ds \int \frac{1}{N^2} \sum_i \frac{1}{(x_{i+k-1} - x_{i+k})^2} [\partial_{i+k-1} \sqrt{q_s} - \partial_{i+k} \sqrt{q_s}]^2 d\omega \right]^{1/2} \\ & \leq C \sqrt{\frac{D_\omega(\sqrt{q_0})\tau}{N}}, \end{aligned} \quad (3.46)$$

where we have used (3.42) and that

$$\left[ \partial_k G \left( N(x_i - x_{i+1}), \dots, N(x_{i+k-1} - x_{i+k}), \dots, N(x_{i+n-1} - x_{i+n}) \right) \right]^2 (x_{i+k-1} - x_{i+k})^2 \leq CN^{-2},$$

since  $G$  is smooth and compactly supported. □

As a comparison to Theorem 3.5, we state the following result which can be proved in a similar way.

**Lemma 3.6** *Let  $G : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded smooth function with compact support and let  $E$  be fixed. Then we have, for any  $\tau > 0$ ,*

$$\left| \frac{1}{N} \sum_i \int G(N(x_i - E)) q d\omega - \int G(N(x_i - E)) d\omega \right| \leq C \left( S_\omega(q) [\tau + e^{-c\tau/R^2}] \right)^{1/2} = CR \sqrt{S_\omega(q)} \quad (3.47)$$

after optimizing for  $\tau$  in the last step.

Notice that by exploiting the local Dirichlet form dissipation coming from the second term on the r.h.s. of (3.41), we have gained the crucial factor  $N^{-1/2}$  in (3.45) compared with (3.47).

The final ingredient to prove Theorem 3.3 is the following entropy and Dirichlet form estimates.

**Theorem 3.7** *Suppose that (3.6) holds and recall  $\tau = R^2 N^{\varepsilon'}$ . Let  $g_t = f_t/\psi$ . Assume that  $S_\omega(g_0) \leq CN^m$  with some fixed  $m$ . Then the entropy and the Dirichlet form satisfy the estimates:*

$$S_\omega(g_{\tau/2}) \leq CNR^{-2}Q, \quad D_\omega(\sqrt{g_\tau}) \leq CNR^{-4}Q. \quad (3.48)$$

*Proof.* Recall that  $\partial_t f_t = L f_t$ . The standard estimate on the entropy of  $f_t$  with respect to the invariant measure is obtained by differentiating the entropy twice and using the logarithmic Sobolev inequality (see Section 3.2.1). The entropy and the Dirichlet form in (3.48) are, however, computed with respect to the measure  $\omega$ . This yields the additional second term in the identity (3.18) and we use the following identity [102] that holds for any probability density  $\psi_t$ :

$$\partial_t S_\mu(f_t|\psi_t) = -\frac{2}{N} \sum_j \int (\partial_j \sqrt{g_t})^2 \psi_t d\mu + \int g_t (L - \partial_t) \psi_t d\mu,$$

where  $g_t = f_t/\psi_t$  and

$$S_\mu(f_t|\psi_t) := \int f_t \log \frac{f_t}{\psi_t} d\mu$$

is the relative entropy.

In our application we set  $\psi_t$  to be time independent,  $\psi_t = \psi = \omega/\mu$ , hence  $S_\mu(f_t|\psi) = S_\omega(g_t)$  and we have, by using (3.37),

$$\partial_t S_\omega(g_t) = -\frac{2}{N} \sum_j \int (\partial_j \sqrt{g_t})^2 d\omega + \int \tilde{L} g_t d\omega + \sum_j \int b_j \partial_j g_t d\omega.$$

Since  $\omega$  is invariant, the middle term on the right hand side vanishes, and from the Schwarz inequality

$$\partial_t S_\omega(g_t) \leq -D_\omega(\sqrt{g_t}) + CN \sum_j \int b_j^2 g_t d\omega \leq -D_\omega(\sqrt{g_t}) + CNA, \quad (3.49)$$

where we defined

$$\Lambda := QR^{-4} = \sup_{t \geq 0} R^{-4} \sum_j \int (x_j - \gamma_j)^2 f_t d\mu. \quad (3.50)$$

Together with (3.43), we have

$$\partial_t S_\omega(g_t) \leq -CR^{-2} S_\omega(g_t) + CNA, \quad (3.51)$$

which, after integrating it from  $t = 0$  to  $\tau/2$ , proves the first inequality in (3.48) using that  $\tau = R^2 N^{\varepsilon'}$  and  $S_\omega(g_0) \leq CN^m$ . In fact, we have

$$S_\omega(g_s) \leq CR^2 N \Lambda \quad (3.52)$$

for any  $s \geq \tau/2$ . The second inequality in (3.48) can be obtained from the first one by integrating (3.49) from  $t = \tau/2$  to  $t = \tau$  and using the monotonicity of the Dirichlet form in time, similarly to (3.24).  $\square$

Finally, we complete the proof Theorem 3.3. Recall that  $\tau = R^2 N^{\varepsilon'}$ , choose  $q := g_\tau = f_\tau / \psi$  in Theorem 3.5. The entropy can be estimated as  $S_\omega(q) = S_\omega(g_\tau) \leq CN^m$ , that follows from (3.52). The condition  $q \in L^\infty$  can be guaranteed by the approximation argument. Then Theorem 3.7, Theorem 3.5 and the fact that  $\Lambda\tau = Q\tau^{-1}N^{2\varepsilon'}$  directly imply

$$\left| \int \frac{1}{N} \sum_{i \in J} \mathcal{G}_{i,\mathbf{m}} f_\tau d\mu - \int \frac{1}{N} \sum_{i \in J} \mathcal{G}_{i,\mathbf{m}} d\omega \right| \leq CN^{\varepsilon'} \sqrt{Q\tau^{-1}} + Ce^{-cN^{\varepsilon'}}, \quad (3.53)$$

i.e., the local statistics of  $q\omega = f_\tau\mu$  and  $\omega$  can be compared. The condition  $S_\omega(g_0) \leq CN^m$  needed to apply Theorem 3.7 is guaranteed by  $S_\mu(f_0) \leq CN^m$  and by  $|\int f_0(\log \psi) d\mu| \leq CN^m$  that follows from the robust bound on  $\Lambda \leq CN^m$  among the assumptions in Theorem 3.3. Clearly, equation (3.53) also holds for the special choice  $f_0 = 1$  (for which  $f_\tau = 1$ ), i.e., local statistics of  $\mu$  and  $\omega$  can also be compared. This completes the proof of Theorem 3.3.  $\square$

### 3.3 From gap distribution to correlation functions: Sketch of the proof of Theorem 3.1.

Our main result will follow from Theorem 3.3 and from the fact that in case  $\tau \geq N^{-2\varepsilon+\delta}$ , the assumption (3.9) guarantees that

$$N^{\varepsilon'} \sqrt{Q\tau^{-1}} = N^{\varepsilon'-\delta/2} = N^{-\delta/6} \rightarrow 0$$

with the choice  $\varepsilon' = \delta/3$ . Therefore the local statistics of observables involving eigenvalue differences coincide in the  $N \rightarrow \infty$  limit.

To complete the proof of Theorem 3.1, we will have to show that the convergence of the observables  $\mathcal{G}_{i,\mathbf{m}}$  is sufficient to identify the correlation functions of the  $x_i$ 's in the sense prescribed in Theorem 3.1. This is a fairly standard technical argument, here we just summarize the main points.

Theorem 3.3 detects the joint behavior of eigenvalue differences on the correct scale  $1/N$ , due to the factor  $N$  in the argument of  $\mathcal{G}$  in (3.29). The slight technical problem is that the observable (3.29), and its averaged version (3.30), involve *fixed indices* of eigenvalues, while correlation functions involve cumulative statistics.

To understand this subtlety, consider  $n = 1$  for simplicity and let  $m_1 = 1$ , say. The observable (3.30) answers to the question: “What is the empirical distribution of differences of consecutive eigenvalues?”, in other words (3.30) directly identifies the *gap distribution* defined in Section 1.5.1. Correlation functions answer to the question: “What is the probability that there are two eigenvalues at a fixed distance away from each other?”, in other words they are not directly sensitive to the possible other eigenvalues in between. Of course these two questions are closely related and it is easy to deduce the answer from each other. This is exactly what the calculation (1.37) has achieved in one direction, now we need to go in the other direction: identify correlation functions from (generalized) gap distributions.

In fact this direction is easier and the essence is given by the following formula:

$$\begin{aligned} & \int_{E-b}^{E+b} dE' \int_{\mathbb{R}^n} d\alpha_1 \dots d\alpha_n O(\alpha_1, \dots, \alpha_n) p_{\tau, N}^{(n)}\left(E' + \frac{\alpha_1}{N\varrho(E)}, \dots, E' + \frac{\alpha_n}{N\varrho(E)}\right) \\ &= \int_{E-b}^{E+b} dE' \int \sum_{i_1 \neq i_2 \neq \dots \neq i_n} \tilde{O}(N(x_{i_1} - E'), N(x_{i_1} - x_{i_2}), \dots, N(x_{i_{n-1}} - x_{i_n})) f_{\tau} d\mu, \\ &= \int_{E-b}^{E+b} dE' \int \sum_{\mathbf{m} \in S_n} \sum_{i=1}^N Y_{i,\mathbf{m}}(E', \mathbf{x}), \end{aligned} \quad (3.54)$$

where we let  $S_n$  denote the set of increasing positive integers,  $\mathbf{m} = (m_2, m_3, \dots, m_n) \in \mathbb{N}_+^{n-1}$ ,  $m_2 < m_3 < \dots < m_n$  and we introduced

$$\begin{aligned} Y_{i,\mathbf{m}}(E', \mathbf{x}) &:= \tilde{O}(N(x_i - E'), N(x_i - x_{i+m_2}), \dots, N(x_i - x_{i+m_n})) \\ \tilde{O}(u_1, u_2, \dots, u_n) &:= O(\varrho(E)u_1, \varrho(E)(u_1 - u_2), \dots). \end{aligned} \quad (3.55)$$

The first equality of (3.54) is just the definition of the correlation function after a trivial rescaling. From the second to the third line we changed indices  $i = i_1$ ,  $i_2 = i + m_2$ ,  $i_3 = i + m_3, \dots$ , and performed a resummation over all index differences encoded in  $\mathbf{m}$ . Apart from the first variable  $N(x_{i_1} - E')$ , the function  $Y_{i,\mathbf{m}}$  is of the form (3.29), so Theorem 3.3 will apply. The dependence on the first variable will be negligible after the  $dE'$  integration on a macroscopic interval.

To control the error terms in this argument, especially to show that even the error terms in the potentially infinite sum over  $\mathbf{m} \in S_n$  converge, one needs an apriori bound on the local density. This is where Assumption IV (3.10) is used. For the details, see Section 7 of [44].  $\square$

## 4 The Green function comparison theorems

We now state our main comparison theorem for the matrix elements of the Green functions of two Wigner ensembles. As in the paper of Tao and Vu [91], we assume conditions on four moments. It will lead quickly to Theorem 4.2 stating that the correlation functions of eigenvalues of two matrix ensembles are identical up to scale  $1/N$  provided that the first four moments of all matrix elements of these two ensembles are almost identical. Here we do not assume that the real and imaginary parts are i.i.d., hence the  $k$ -th moment of  $h_{ij}$  is understood as the collection of numbers  $\int \bar{h}^s h^{k-s} \nu_{ij}(dh)$ ,  $s = 0, 1, 2, \dots, k$ . The main result in [91] compares the joint distribution of individual eigenvalues — which is not covered by our Theorem 4.1 — but it does not address directly the matrix elements of Green functions. The key input for both theorems is the local semicircle law on the almost optimal scale  $N^{-1+\varepsilon}$ . The eigenvalue perturbation used in [91] requires certain estimates on the eigenvalue level repulsion; the proof of Theorem 4.1 is a straightforward resolvent perturbation theory.

**Theorem 4.1 (Green function comparison)** *Consider two generalized  $N \times N$  Wigner matrices,  $H^{(v)}$  and  $H^{(w)}$ , with matrix elements  $h_{ij}$  given by the random variables  $N^{-1/2}v_{ij}$  and  $N^{-1/2}w_{ij}$ , respectively, with  $v_{ij}$  and  $w_{ij}$  satisfying the uniform subexponential decay condition*

$$\mathbb{P}(|v_{ij}| \geq x^\alpha) \leq \beta e^{-x}, \quad \mathbb{P}(|w_{ij}| \geq x^\alpha) \leq \beta e^{-x},$$

with some  $\alpha, \beta > 0$ . Fix a bijective ordering map on the index set of the independent matrix elements,

$$\phi : \{(i, j) : 1 \leq i \leq j \leq N\} \rightarrow \{1, \dots, \gamma(N)\}, \quad \gamma(N) := \frac{N(N+1)}{2},$$

and denote by  $H_\gamma$  the generalized Wigner matrix whose matrix elements  $h_{ij}$  follow the  $v$ -distribution if  $\phi(i, j) \leq \gamma$  and they follow the  $w$ -distribution otherwise; in particular  $H^{(v)} = H_0$  and  $H^{(w)} = H_{\gamma(N)}$ . Let  $\kappa > 0$  be arbitrary and suppose that, for any small parameter  $\tau > 0$  and for any  $y \geq N^{-1+\tau}$ , we have the following estimate on the diagonal elements of the resolvent

$$\mathbb{P} \left( \max_{0 \leq \gamma \leq \gamma(N)} \max_{1 \leq k \leq N} \max_{|E| \leq 2^{-\kappa}} \left| \left( \frac{1}{H_\gamma - E - iy} \right)_{kk} \right| \leq N^{2\tau} \right) \geq 1 - CN^{-c \log \log N} \quad (4.1)$$

with some constants  $C, c$  depending only on  $\tau, \kappa$ . Moreover, we assume that the first three moments of  $v_{ij}$  and  $w_{ij}$  are the same, i.e.

$$\mathbb{E} \bar{v}_{ij}^s v_{ij}^u = \mathbb{E} \bar{w}_{ij}^s w_{ij}^u, \quad 0 \leq s + u \leq 3,$$



and the difference between the fourth moments of  $v_{ij}$  and  $w_{ij}$  satisfies

$$\left| \mathbb{E} \bar{v}_{ij}^s v_{ij}^{4-s} - \mathbb{E} \bar{w}_{ij}^s w_{ij}^{4-s} \right| \leq N^{-\delta}, \quad s = 0, 1, 2, 3, 4, \quad (4.2)$$

for some given  $\delta > 0$ . Let  $\varepsilon > 0$  be arbitrary and choose an  $\eta$  with  $N^{-1-\varepsilon} \leq \eta \leq N^{-1}$ . For any sequence of positive integers  $k_1, \dots, k_n$ , set complex parameters  $z_j^m = E_j^m \pm i\eta$ ,  $j = 1, \dots, k_m$ ,  $m = 1, \dots, n$ , with  $|E_j^m| \leq 2 - 2\kappa$  and with an arbitrary choice of the  $\pm$  signs. Let  $G^{(v)}(z) = (H^{(v)} - z)^{-1}$  denote the resolvent and let  $F(x_1, \dots, x_n)$  be a function such that for any multi-index  $\alpha = (\alpha_1, \dots, \alpha_n)$  with  $1 \leq |\alpha| \leq 5$  and for any  $\varepsilon' > 0$  sufficiently small, we have

$$\max \left\{ |\partial^\alpha F(x_1, \dots, x_n)| : \max_j |x_j| \leq N^{\varepsilon'} \right\} \leq N^{C_0 \varepsilon'} \quad (4.3)$$

and

$$\max \left\{ |\partial^\alpha F(x_1, \dots, x_n)| : \max_j |x_j| \leq N^2 \right\} \leq N^{C_0} \quad (4.4)$$

for some constant  $C_0$ .

Then, there is a constant  $C_1$ , depending on  $\alpha, \beta, \sum_m k_m$  and  $C_0$  such that for any  $\eta$  with  $N^{-1-\varepsilon} \leq \eta \leq N^{-1}$  and for any choices of the signs in the imaginary part of  $z_j^m$ , we have

$$\left| \mathbb{E} F \left( \frac{1}{N} \text{Tr} \left[ \prod_{j=1}^{k_1} G^{(v)}(z_j^1) \right], \dots, \frac{1}{N} \text{Tr} \left[ \prod_{j=1}^{k_n} G^{(v)}(z_j^n) \right] \right) - \mathbb{E} F \left( G^{(v)} \rightarrow G^{(w)} \right) \right| \leq C_1 N^{-1/2+C_1\varepsilon} + C_1 N^{-\delta+C_1\varepsilon}, \quad (4.5)$$

where the arguments of  $F$  in the second term are changed from the Green functions of  $H^{(v)}$  to  $H^{(w)}$  and all other parameters remain unchanged.

*Remark 1:* We formulated Theorem 4.1 for functions of traces of monomials of the Green function because this is the form we need in the application. However, the result (and the proof we are going to present) holds directly for matrix elements of monomials of Green functions as well, namely, for any choice of  $\ell_1, \dots, \ell_{2n}$ , we have

$$\left| \mathbb{E} F \left( \frac{1}{N^{k_1-1}} \left[ \prod_{j=1}^{k_1} G^{(v)}(z_j^1) \right]_{\ell_1, \ell_2}, \dots, \frac{1}{N^{k_n-1}} \left[ \prod_{j=1}^{k_n} G^{(v)}(z_j^n) \right]_{\ell_{2n-1}, \ell_{2n}} \right) - \mathbb{E} F \left( G^{(v)} \rightarrow G^{(w)} \right) \right| \leq C_1 N^{-1/2+C_1\varepsilon} + C_1 N^{-\delta+C_1\varepsilon}. \quad (4.6)$$

We also remark that Theorem 4.1 holds for generalized Wigner matrices if  $C_{sup} = \sup_{ij} N \sigma_{ij}^2 < \infty$ . The positive lower bound on the variances,  $C_{inf} > 0$  in (1.62) is not necessary for this theorem.

*Remark 2:* Although we state Theorem 4.1 for hermitian and symmetric ensembles, similar results hold for real and complex sample covariance ensembles; the modification of the proof is obvious and we omit the details.

The following is the main corollary of Theorem 4.1. It will be proven later in the section, but we mention, for the orientation of the reader, that the important statement is Theorem 4.1, the proof of its corollary is a fairly straightforward technicality. The basic ideas are explained at the beginning of each proofs.

**Theorem 4.2 (Correlation function comparison theorem)** *Suppose the assumptions of Theorem 4.1 hold. Let  $p_{v,N}^{(k)}$  and  $p_{w,N}^{(k)}$  be the  $k$ -point functions of the eigenvalues w.r.t. the probability law of the matrix  $H^{(v)}$  and  $H^{(w)}$ , respectively. Then for any  $|E| < 2$ , any  $k \geq 1$  and any compactly supported continuous test function  $O : \mathbb{R}^k \rightarrow \mathbb{R}$  we have*

$$\int_{\mathbb{R}^k} d\alpha_1 \dots d\alpha_k O(\alpha_1, \dots, \alpha_k) \left( p_{v,N}^{(k)} - p_{w,N}^{(k)} \right) \left( E + \frac{\alpha_1}{N}, \dots, E + \frac{\alpha_k}{N} \right) = 0. \quad (4.7)$$

#### 4.1 Proof of the Green function comparison Theorem 4.1

The basic idea is that we estimate the effect of changing one matrix elements on the resolvent by a resolvent expansion. Since each matrix element has a typical size of  $N^{-1/2}$  and the resolvents are essentially bounded thanks to (4.1), a resolvent expansion up to the fourth order will identify the change with a precision  $O(N^{-5/2})$  (modulo some tiny corrections of order  $N^{O(\tau)}$ ). The expectation values of the terms up to fourth order involve only the first four moments of the single entry distribution, which can be directly compared. The error terms are negligible even when we sum them up  $N^2$  times, the number of comparison steps needed to replace all matrix elements.

To start the detailed proof, we first need an estimate of the type (4.1) for the resolvents of all intermediate matrices. From the trivial bound

$$\Im \left( \frac{1}{H - E - i\eta} \right)_{jj} \leq \left( \frac{y}{\eta} \right) \Im \left( \frac{1}{H - E - iy} \right)_{jj}, \quad \eta \leq y,$$

and from (4.1) we have the following apriori bound

$$\mathbb{P} \left( \max_{0 \leq \gamma \leq \gamma(N)} \max_{1 \leq k \leq N} \max_{|E| \leq 2 - \kappa} \sup_{\eta \geq N^{-1-\varepsilon}} \left| \Im \left( \frac{1}{H_\gamma - E \pm i\eta} \right)_{kk} \right| \leq N^{3\tau+\varepsilon} \right) \geq 1 - CN^{-c \log \log N}. \quad (4.8)$$

Note that the supremum over  $\eta$  can be included by establishing the estimate first for a fine grid of  $\eta$ 's with spacing  $N^{-10}$  and then extend the bound for all  $\eta$  by using that the Green functions are Lipschitz continuous in  $\eta$  with a Lipschitz constant  $\eta^{-2}$ .

Let  $\lambda_m$  and  $u_m$  denote the eigenvalues and eigenvectors of  $H_\gamma$ , then by the definition of the Green function, we have

$$\left| \left( \frac{1}{H_\gamma - z} \right)_{jk} \right| \leq \sum_{m=1}^N \frac{|u_m(j)||u_m(k)|}{|\lambda_m - z|} \leq \left[ \sum_{m=1}^N \frac{|u_m(j)|^2}{|\lambda_m - z|} \right]^{1/2} \left[ \sum_{m=1}^N \frac{|u_m(k)|^2}{|\lambda_m - z|} \right]^{1/2}.$$

Define a dyadic decomposition

$$U_n = \{m : 2^{n-1}\eta \leq |\lambda_m - E| < 2^n\eta\}, \quad n = 1, 2, \dots, n_0 := C \log N, \quad (4.9)$$

$$U_0 = \{m : |\lambda_m - E| < \eta\}, \quad U_\infty := \{m : 2^{n_0}\eta \leq |\lambda_m - E|\},$$

and divide the summation over  $m$  into  $\cup_n U_n$

$$\sum_{m=1}^N \frac{|u_m(j)|^2}{|\lambda_m - z|} \leq C \sum_n \sum_{m \in U_n} \Im \frac{|u_m(j)|^2}{\lambda_m - E - i2^n\eta} \leq C \sum_n \Im \left( \frac{1}{H_\gamma - E - i2^n\eta} \right)_{jj}.$$

Using (4.1) for  $n = 0, 1, \dots, n_0$  and a trivial bound of  $O(1)$  for  $n = \infty$ , we have proved that

$$\mathbb{P} \left( \sup_{0 \leq \gamma \leq \gamma(N)} \sup_{1 \leq k, \ell \leq N} \max_{|E| \leq 2^{-\kappa}} \sup_{\eta \geq N^{-1-\varepsilon}} \left| \left( \frac{1}{H_\gamma - E \pm i\eta} \right)_{k\ell} \right| \leq N^{4\tau+\varepsilon} \right) \geq 1 - CN^{-c \log \log N}. \quad (4.10)$$

Now we turn to the one by one replacement. For notational simplicity, we will consider the case when the test function  $F$  has only  $n = 1$  variable and  $k_1 = 1$ , i.e., we consider the trace of a first order monomial; the general case follows analogously. Consider the telescopic sum of differences of expectations

$$\begin{aligned} \mathbb{E} F \left( \frac{1}{N} \text{Tr} \frac{1}{H^{(v)} - z} \right) - \mathbb{E} F \left( \frac{1}{N} \text{Tr} \frac{1}{H^{(w)} - z} \right) \\ = \sum_{\gamma=1}^{\gamma(N)} \left[ \mathbb{E} F \left( \frac{1}{N} \text{Tr} \frac{1}{H_\gamma - z} \right) - \mathbb{E} F \left( \frac{1}{N} \text{Tr} \frac{1}{H_{\gamma-1} - z} \right) \right]. \end{aligned} \quad (4.11)$$

Let  $E^{(ij)}$  denote the matrix whose matrix elements are zero everywhere except at the  $(i, j)$  position, where it is 1, i.e.,  $E_{k\ell}^{(ij)} = \delta_{ik}\delta_{j\ell}$ . Fix an  $\gamma \geq 1$  and let  $(i, j)$  be determined by  $\phi(i, j) = \gamma$ . We will compare  $H_{\gamma-1}$  with  $H_\gamma$ . Note that these two matrices differ only in the  $(i, j)$  and  $(j, i)$  matrix elements and they can be written as

$$\begin{aligned} H_{\gamma-1} &= Q + \frac{1}{\sqrt{N}}V, & V &:= v_{ij}E^{(ij)} + v_{ji}E^{(ji)} \\ H_\gamma &= Q + \frac{1}{\sqrt{N}}W, & W &:= w_{ij}E^{(ij)} + w_{ji}E^{(ji)}, \end{aligned}$$

with a matrix  $Q$  that has zero matrix element at the  $(i, j)$  and  $(j, i)$  positions and where we set  $v_{ji} := \bar{v}_{ij}$  for  $i < j$  and similarly for  $w$ . Define the Green functions

$$R := \frac{1}{Q - z}, \quad S := \frac{1}{H_\gamma - z}.$$

We first claim that the estimate (4.10) holds for the Green function  $R$  as well. To see this, we have, from the resolvent expansion,

$$R = S + N^{-1/2}SVS + \dots + N^{-9/5}(SV)^9S + N^{-5}(SV)^{10}R.$$

Since  $V$  has only at most two nonzero element, when computing the  $(k, \ell)$  matrix element of this matrix identity, each term is a finite sum involving matrix elements of  $S$  or  $R$  and  $v_{ij}$ , e.g.  $(SVS)_{k\ell} = S_{ki}v_{ij}S_{j\ell} + S_{kj}v_{ji}S_{i\ell}$ . Using the bound (4.10) for the  $S$  matrix elements, the subexponential decay for  $v_{ij}$  and the trivial bound  $|R_{ij}| \leq \eta^{-1}$ , we obtain that the estimate (4.10) holds for  $R$ .

We can now start proving the main result by comparing the resolvents of  $H^{(\gamma-1)}$  and  $H^{(\gamma)}$  with the resolvent  $R$  of the reference matrix  $Q$ . By the resolvent expansion,

$$S = R - N^{-1/2}RVR + N^{-1}(RV)^2R - N^{-3/2}(RV)^3R + N^{-2}(RV)^4R - N^{-5/2}(RV)^5S,$$

so we can write

$$\frac{1}{N}\text{Tr} S = \widehat{R} + \xi, \quad \xi = \sum_{m=1}^4 N^{-m/2}\widehat{R}^{(m)} + N^{-5/2}\Omega$$

with

$$\widehat{R} = \frac{1}{N}\text{Tr} R, \quad \widehat{R}^{(m)} = (-1)^m \frac{1}{N}\text{Tr} (RV)^m R, \quad \Omega = -\frac{1}{N}\text{Tr} (RV)^5 S.$$

For each diagonal element in the computation of these traces, the contribution to  $\widehat{R}$ ,  $\widehat{R}^{(m)}$  and  $\Omega$  is a sum of a few terms. E.g.

$$\widehat{R}^{(2)} = \frac{1}{N} \sum_k \left[ R_{ki}v_{ij}R_{jj}v_{ji}R_{ik} + R_{ki}v_{ij}R_{ji}v_{ij}R_{jk} + R_{kj}v_{ji}R_{ii}v_{ij}R_{jk} + R_{kj}v_{ji}R_{ij}v_{ji}R_{ik} \right]$$

and similar formulas hold for the other terms. Then we have

$$\begin{aligned} \mathbb{E}F \left( \frac{1}{N}\text{Tr} \frac{1}{H_\gamma - z} \right) &= \mathbb{E}F \left( \widehat{R} + \xi \right) \\ &= \mathbb{E} \left[ F(\widehat{R}) + F'(\widehat{R})\xi + F''(\widehat{R})\xi^2 + \dots + F^{(5)}(\widehat{R} + \xi')\xi^5 \right] \\ &= \sum_{m=0}^5 N^{-m/2}\mathbb{E}A^{(m)}, \end{aligned} \tag{4.12}$$

where  $\xi'$  is a number between 0 and  $\xi$  and it depends on  $\widehat{R}$  and  $\xi$ ; the  $A^{(m)}$ 's are defined as

$$A^{(0)} = F(\widehat{R}), \quad A^{(1)} = F'(\widehat{R})\widehat{R}^{(1)}, \quad A^{(2)} = F''(\widehat{R})(\widehat{R}^{(1)})^2 + F'(\widehat{R})\widehat{R}^{(2)},$$

and similarly for  $A^{(3)}$  and  $A^{(4)}$ . Finally,

$$A^{(5)} = F'(\widehat{R})\Omega + F^{(5)}(\widehat{R} + \xi')(\widehat{R}^{(1)})^5 + \dots$$

The expectation values of the terms  $A^{(m)}$ ,  $m \leq 4$ , with respect to  $v_{ij}$  are determined by the first four moments of  $v_{ij}$ , for example

$$\begin{aligned} \mathbb{E} A^{(2)} &= F'(\widehat{R}) \left[ \frac{1}{N} \sum_k R_{ki} R_{jj} R_{ik} + \dots \right] \mathbb{E} |v_{ij}|^2 + F''(\widehat{R}) \left[ \frac{1}{N^2} \sum_{k,\ell} R_{ki} R_{j\ell} R_{\ell j} R_{ik} + \dots \right] \mathbb{E} |v_{ij}|^2 \\ &\quad + F'(\widehat{R}) \left[ \frac{1}{N} \sum_k R_{ki} R_{ji} R_{jk} + \dots \right] \mathbb{E} v_{ij}^2 + F''(\widehat{R}) \left[ \frac{1}{N^2} \sum_{k,\ell} R_{ki} R_{j\ell} R_{\ell i} R_{jk} + \dots \right] \mathbb{E} v_{ij}^2. \end{aligned}$$

Note that the coefficients involve up to four derivatives of  $F$  and normalized sums of matrix elements of  $R$ . Using the estimate (4.10) for  $R$  and the derivative bounds (4.3) for the typical values of  $\widehat{R}$ , we see that all these coefficients are bounded by  $N^{C(\tau+\varepsilon)}$  with a very large probability, where  $C$  is an explicit constant. We use the bound (4.4) for the extreme values of  $\widehat{R}$  but this event has a very small probability by (4.10). Therefore, the coefficients of the moments  $\mathbb{E} \bar{v}_{ij}^s v_{ij}^u$ ,  $u + s \leq 4$ , in the quantities  $A^{(0)}, \dots, A^{(4)}$  are essentially bounded, modulo a factor  $N^{C(\tau+\varepsilon)}$ . Notice that the fourth moment of  $v_{ij}$  appears only in the  $m = 4$  term that already has a prefactor  $N^{-2}$  in (4.12). Therefore, to compute the  $m \leq 4$  terms in (4.12) up to a precision  $o(N^{-2})$ , it is sufficient to know the first three moments of  $v_{ij}$  exactly and the fourth moment only with a precision  $N^{-\delta}$ ; if  $\tau$  and  $\varepsilon$  are chosen such that  $C(\tau + \varepsilon) < \delta$ , then the discrepancy in the fourth moment is irrelevant.

Finally, we have to estimate the error term  $A^{(5)}$ . All terms without  $\Omega$  can be dealt with as before; after estimating the derivatives of  $F$  by  $N^{C(\tau+\varepsilon)}$ , one can perform the expectation with respect to  $v_{ij}$  that is independent of  $\widehat{R}^{(m)}$ . For the terms involving  $\Omega$  one can argue similarly, by appealing to the fact that the matrix elements of  $S$  are also essentially bounded by  $N^{C(\tau+\varepsilon)}$ , see (4.10), and that  $v_{ij}$  has subexponential decay. Alternatively, one can use Hölder inequality to decouple  $S$  from the rest and use (4.10) directly, for example:

$$\mathbb{E} |F'(\widehat{R})\Omega| = \frac{1}{N} \mathbb{E} |F'(\widehat{R}) \text{Tr} (RV)^5 S| \leq \frac{1}{N} \left[ \mathbb{E} (F'(\widehat{R}))^2 \text{Tr} S^2 \right]^{1/2} \left[ \mathbb{E} \text{Tr} (RV)^5 (VR^*)^5 \right]^{1/2} \leq CN^{C(\tau+\varepsilon)}.$$

Note that exactly the same perturbation expansion holds for the resolvent of  $H_{\gamma-1}$ , just  $v_{ij}$  is replaced with  $w_{ij}$  everywhere. By the moment matching condition, the expectation values  $\mathbb{E} A^{(m)}$  of terms for  $m \leq 3$  in (4.12) are identical and the  $m = 4$  term differs by  $N^{-\delta+C(\tau+\varepsilon)}$ . Choosing  $\tau = \varepsilon$ , we have

$$\left| \mathbb{E} F \left( \frac{1}{N} \text{Tr} \frac{1}{H_\gamma - z} \right) - \mathbb{E} F \left( \frac{1}{N} \text{Tr} \frac{1}{H_{\gamma-1} - z} \right) \right| \leq CN^{-5/2+C\varepsilon} + CN^{-2-\delta+C\varepsilon}.$$

After summing up in (4.11) we have thus proved that

$$\left| \mathbb{E} F \left( \frac{1}{N} \text{Tr} \frac{1}{H^{(v)} - z} \right) - \mathbb{E} F \left( \frac{1}{N} \text{Tr} \frac{1}{H^{(w)} - z} \right) \right| \leq CN^{-1/2+C\varepsilon} + CN^{-\delta+C\varepsilon}.$$

The proof can be easily generalized to functions of several variables. This concludes the proof of Theorem 4.1.  $\square$

## 4.2 Proof of the correlation function comparison Theorem 4.2

Define an approximate delta function (times  $\pi$ ) at the scale  $\eta$  by

$$\theta_\eta(x) = \Im \frac{1}{x - i\eta}.$$

We will choose  $\eta \sim N^{-1-\varepsilon}$ , i.e. slightly smaller than the typical eigenvalue spacing. This means that an observable of the form  $\theta_\eta$  have sufficient resolution to detect individual eigenvalues. Moreover, polynomials of such observables detect correlation functions. On the other hand,

$$\frac{1}{N} \Im \text{Tr} G(E + i\eta) = \frac{1}{N} \sum_i \theta_\eta(\lambda_i - E),$$

therefore expectation values of such observables are covered by (4.5) of Theorem 4.1. The rest of the proof consists of making this idea precise. There are two technicalities to resolve. First, correlation functions involve *distinct* eigenvalues (see (4.13) below), while polynomials of resolvent include an overcounting of coinciding eigenvalues. Thus an exclusion-inclusion formula will be needed. Second, although  $\eta$  is much smaller than the relevant scale  $1/N$ , it still does not give pointwise information on the correlation functions. However, the correlation functions in (1.34) are identified only as a weak limit, i.e., tested against a continuous function  $O$ . The continuity of  $O$  can be used to show that the difference between the exact correlation functions and the smeared out ones on the scale  $\eta \sim N^{-1-\varepsilon}$  is negligible. This last step requires an a-priori upper bound on the density to ensure that not too many eigenvalues fall into an irrelevantly small interval; this bound is provided by the local semicircle law.

For notational simplicity, the detailed proof will be given only for the case of three point correlation functions; the proof is analogous for the general case. By definition of the correlation function, for any fixed  $E, \alpha_1, \alpha_2, \alpha_3$ ,

$$\begin{aligned} & \mathbb{E}_w \frac{1}{N(N-1)(N-2)} \sum_{i \neq j \neq k} \theta_\eta \left( \lambda_i - E - \frac{\alpha_1}{N} \right) \theta_\eta \left( \lambda_j - E - \frac{\alpha_2}{N} \right) \theta_\eta \left( \lambda_k - E - \frac{\alpha_3}{N} \right) \\ &= \int dx_1 dx_2 dx_3 p_{w,N}^{(3)}(x_1, x_2, x_3) \theta_\eta(x_1 - E_1) \theta_\eta(x_2 - E_2) \theta_\eta(x_3 - E_3), \quad E_j := E + \frac{\alpha_j}{N}. \end{aligned} \tag{4.13}$$

By the exclusion-inclusion principle,

$$\mathbb{E}_{\mathbf{w}} \frac{1}{N(N-1)(N-2)} \sum_{i \neq j \neq k} \theta_{\eta}(x_1 - E_1) \theta_{\eta}(x_2 - E_2) \theta_{\eta}(x_3 - E_3) = \mathbb{E}_{\mathbf{w}} A_1 + \mathbb{E}_{\mathbf{w}} A_2 + \mathbb{E}_{\mathbf{w}} A_3, \quad (4.14)$$

where

$$A_1 := \frac{1}{N(N-1)(N-2)} \prod_{j=1}^3 \left[ \frac{1}{N} \sum_i \theta_{\eta}(\lambda_i - E_j) \right],$$

$$A_3 := \frac{2}{N(N-1)(N-2)} \sum_i \theta_{\eta}(\lambda_i - E_1) \theta_{\eta}(\lambda_i - E_2) \theta_{\eta}(\lambda_i - E_3) + \dots$$

and

$$A_2 := B_1 + B_2 + B_3, \quad \text{with} \quad B_3 = -\frac{1}{N(N-1)(N-2)} \sum_i \theta_{\eta}(\lambda_i - E_1) \theta_{\eta}(\lambda_i - E_2) \sum_k \theta_{\eta}(\lambda_k - E_3),$$

and similarly,  $B_1$  consists of terms with  $j = k$ , while  $B_2$  consists of terms with  $i = k$ .

Notice that, modulo a trivial change in the prefactor,  $\mathbb{E}_{\mathbf{w}} A_1$  can be approximated by

$$\mathbb{E}_{\mathbf{w}} F \left( \frac{1}{N} \Im \text{Tr} \frac{1}{H^{(v)} - z_1}, \dots, \frac{1}{N} \Im \text{Tr} \frac{1}{H^{(v)} - z_3} \right),$$

where the function  $F$  is chosen to be  $F(x_1, x_2, x_3) := x_1 x_2 x_3$  if  $\max_j |x_j| \leq N^{\varepsilon}$  and it is smoothly cutoff to go to zero in the regime  $\max_j |x_j| \geq N^{2\varepsilon}$ . The difference between the expectation of  $F$  and  $A_1$  is negligible, since it comes from the regime where  $N^{\varepsilon} \leq \max_j \frac{1}{N} |\Im \text{Tr} (H^{(v)} - z_j)^{-1}| \leq N^2$ , which has an exponentially small probability by (4.10) (the upper bound on the Green function always holds since  $\eta \geq N^{-2}$ ). Here the arguments of  $F$  are imaginary parts of the trace of the Green function, but this type of function is allowed when applying Theorem 4.1, since

$$\Im \text{Tr} G(z) = \frac{1}{2} [\text{Tr} G(z) - \text{Tr} G(\bar{z})].$$

We remark that the main assumption (4.1) for Theorem 4.1 is satisfied by using Theorem 2.6 with the choice of  $M \sim N$ .

Similarly, we can approximate  $\mathbb{E}_{\mathbf{w}} B_3$  by

$$\mathbb{E}_{\mathbf{w}} F^* \left( \frac{1}{N^2} \text{Tr} \left\{ \Im \frac{1}{H^{(v)} - z_1} \Im \frac{1}{H^{(v)} - z_2} \right\}, \frac{1}{N} \Im \text{Tr} \frac{1}{H^{(v)} - z_3} \right),$$

where  $F^*(x_1, x_2) = x_1 x_2$  with an appropriate cutoff for large arguments, and there are similar expressions for  $B_1, B_2$  and also for  $A_3$ , the latter involving the trace of the product of three resolvents. By Theorem 4.1, these expectations w.r.t.  $\mathbf{w}$  in the approximations of  $\mathbb{E}_{\mathbf{w}} A_i$  can be replaced by

expectations w.r.t.  $\mathbf{v}$  with only negligible errors provided that  $\eta \geq N^{-1-\varepsilon}$ . We have thus proved that

$$\lim_{N \rightarrow \infty} \int dx_1 dx_2 dx_3 [p_{w,N}^{(3)}(x_1, x_2, x_3) - p_{v,N}^{(3)}(x_1, x_2, x_3)] \theta_\eta(x_1 - E_1) \theta_\eta(x_2 - E_2) \theta_\eta(x_3 - E_3) = 0. \quad (4.15)$$

Set  $\eta = N^{-1-\varepsilon}$  for the rest of the proof. We now show that the validity of (4.15) for any choice of  $E, \alpha_1, \alpha_2, \alpha_3$  (recall  $E_j = E + \alpha_j/N$ ) implies that the rescaled correlation functions,  $p_{w,N}^{(3)}(E + \beta_1/N, \dots, E + \beta_3/N)$  and  $p_{v,N}^{(3)}(E + \beta_1/N, \dots, E + \beta_3/N)$ , as functions of the variables  $\beta_1, \beta_2, \beta_3$ , have the same weak limit.

Let  $O$  be a smooth, compactly supported test function and let

$$O_\eta(\beta_1, \beta_2, \beta_3) := \frac{1}{(\pi N)^3} \int_{\mathbb{R}^3} d\alpha_1 d\alpha_2 d\alpha_3 O(\alpha_1, \alpha_2, \alpha_3) \theta_\eta\left(\frac{\beta_1 - \alpha_1}{N}\right) \dots \theta_\eta\left(\frac{\beta_3 - \alpha_3}{N}\right)$$

be its smoothing on scale  $N\eta$ . Then we can write

$$\begin{aligned} & \int_{\mathbb{R}^3} d\beta_1 d\beta_2 d\beta_3 O(\beta_1, \beta_2, \beta_3) p_{w,N}^{(3)}\left(E + \frac{\beta_1}{N}, \dots, E + \frac{\beta_3}{N}\right) \\ &= \int_{\mathbb{R}^3} d\beta_1 d\beta_2 d\beta_3 O_\eta(\beta_1, \beta_2, \beta_3) p_{w,N}^{(3)}\left(E + \frac{\beta_1}{N}, \dots, E + \frac{\beta_3}{N}\right) \\ &+ \int_{\mathbb{R}^3} d\beta_1 d\beta_2 d\beta_3 (O - O_\eta)(\beta_1, \beta_2, \beta_3) p_{w,N}^{(3)}\left(E + \frac{\beta_1}{N}, \dots, E + \frac{\beta_3}{N}\right). \end{aligned} \quad (4.16)$$

The first term on the right side, after the change of variables  $x_j = E + \beta_j/N$ , is equal to

$$\int_{\mathbb{R}^3} d\alpha_1 d\alpha_2 d\alpha_3 O(\alpha_1, \alpha_2, \alpha_3) \int_{\mathbb{R}^3} dx_1 dx_2 dx_3 p_{w,N}^{(3)}(x_1, x_2, x_3) \theta_\eta(x_1 - E_1) \theta_\eta(x_2 - E_2) \theta_\eta(x_3 - E_3), \quad (4.17)$$

i.e., it can be written as an integral of expressions of the form (4.15) for which limits with  $p_{w,N}$  and  $p_{v,N}$  coincide.

Finally, the second term on the right hand side of (4.16) is negligible. To see this, notice that for any test function  $Q$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^3} d\beta_1 d\beta_2 d\beta_3 Q(\beta_1, \beta_2, \beta_3) p_{w,N}^{(3)}\left(E + \frac{\beta_1}{N}, \dots, E + \frac{\beta_3}{N}\right) \\ &= N^3 \int_{\mathbb{R}^3} dx_1 dx_2 dx_3 Q(N(x_1 - E), N(x_2 - E), N(x_3 - E)) p_{w,N}^{(3)}(x_1, x_2, x_3) \\ &= \left(1 - \frac{1}{N}\right) \left(1 - \frac{2}{N}\right) \mathbb{E}_{\mathbf{w}} \sum_{i \neq j \neq k} Q(N(\lambda_i - E), N(\lambda_j - E), N(\lambda_k - E)). \end{aligned} \quad (4.18)$$



If the test function  $Q$  were supported on a ball of size  $N^{\varepsilon'}$ ,  $\varepsilon' > 0$ , then this last term were bounded by

$$\|Q\|_{\infty} \mathbb{E}_{\mathbf{w}} \mathcal{N}_{CN^{-1+\varepsilon'}}^3(E) \leq C \|Q\|_{\infty} N^{4\varepsilon'}.$$

Here  $\mathcal{N}_{\tau}(E)$  denotes the number of eigenvalues in the interval  $[E - \tau, E + \tau]$  and in the estimate we used the local semicircle law on intervals of size  $\tau \geq N^{-1+\varepsilon'}$ .

Set now  $Q := O - O_{\eta}$ . From the definition of  $O_{\eta}$ , it is easy to see that the function

$$Q_1(\beta_1, \beta_2, \beta_3) = O(\beta_1, \beta_2, \beta_3) - O_{\eta}(\beta_1, \beta_2, \beta_3) \prod_{j=1}^3 1(|\beta_j| \leq N^{\varepsilon'})$$

satisfies the bound  $\|Q_1\|_{\infty} \leq \|Q\|_{\infty} = \|O - O_{\eta}\|_{\infty} \leq CN\eta = CN^{-\varepsilon}$ . So choosing  $\varepsilon' < \varepsilon/4$ , the contribution of  $Q_1$  is negligible. Finally,  $Q_2 = Q - Q_1$  is given by

$$Q_2(\beta_1, \beta_2, \beta_3) = -O_{\eta}(\beta_1, \beta_2, \beta_3) \left[ 1 - \prod_{j=1}^3 1(|\beta_j| \leq N^{\varepsilon'}) \right]$$

and

$$\begin{aligned} |Q_2| &\leq C \left[ \frac{1}{1 + \beta_1^2} \right] \left[ \frac{1}{1 + \beta_2^2} \right] \left[ \frac{1}{1 + \beta_3^2} \right] \left\{ 1(|\beta_1| \geq N^{\varepsilon'}) + \dots \right\} \\ &\leq C \left\{ N^{-\varepsilon'} \left[ \frac{N^{\varepsilon'}}{N^{2\varepsilon'} + \beta_1^2} \right] \left[ \frac{1}{1 + \beta_2^2} \right] \left[ \frac{1}{1 + \beta_3^2} \right] + \dots \right\}. \end{aligned} \quad (4.19)$$

Hence the contribution of  $Q_2$  in the last term of (4.18) is bounded by

$$CN^{-3-\varepsilon'} \mathbb{E}_{\mathbf{w}} \sum_{i,j,k} \left\{ \left[ \frac{N^{-1+\varepsilon'}}{N^{-2+2\varepsilon'} + (\lambda_i - E)^2} \right] \left[ \frac{N^{-1}}{N^{-2} + (\lambda_j - E)^2} \right] \left[ \frac{N^{-1}}{N^{-2} + (\lambda_k - E)^2} \right] + \dots \right\}. \quad (4.20)$$

From the local semicircle law, Theorem 2.6, the last term is bounded by  $N^{-\varepsilon'}$  up to some logarithmic factor. To see this, note that the Riemann sums for eigenvalues in (4.20) can be replaced with an integral because the resolution scale of the functions involved is at least  $N^{-1}$ . This completes the proof of Theorem 4.2.  $\square$

Finally, we sketch the proof of Theorem 1.5 to point out the differences between the two four moment theorems.

*Sketch of the proof of Theorem 1.5.* We again interpolate between  $H$  and  $H'$  step by step, by replacing the distribution of the matrix elements of  $H$  from  $\nu$  to  $\nu'$  one by one according to a fixed ordering. Let  $H^{(\tau)}(h)$  be the matrix where the first  $\tau - 1$  elements follow  $\nu'$  distribution, the  $\tau$ -th

entry is  $h$  and the remaining ones follow  $\nu$ . Denote by  $\lambda_i(H^{(\tau)}(h))$  the  $i$ -th eigenvalue of  $H^{(\tau)}(h)$ . Let

$$\mathcal{F}_\tau(h) := F\left(N\lambda_{i_1}(H^{(\tau)}(h)), N\lambda_{i_2}(H^{(\tau)}(h)), \dots\right)$$

and we prove that

$$|\mathbb{E}\mathcal{F}_\tau(h) - \mathbb{E}'\mathcal{F}_\tau(h')| \leq CN^{-2-c_0}. \quad (4.21)$$

Since the number of replacement steps is of order  $N^2$ , this will show (1.76). Let  $\tau$  represent the  $(pq)$  matrix element and we can drop the index  $\tau$ .

We will prove that

$$\left|\frac{\partial^n \mathcal{F}}{\partial h^n}\right| \leq CN^{O(c_0)+o(1)} \quad (4.22)$$

for any  $n \leq 5$ . Then, by Taylor expansion,

$$\mathcal{F}(h) = \sum_{n=0}^4 \frac{1}{n!} \frac{\partial^n \mathcal{F}}{\partial h^n}(0) h^n + N^{-5/2+O(c_0)+o(1)}, \quad (4.23)$$

since  $|h| \leq N^{-5/2+o(1)}$  with very high probability. After taking expectations for  $h$  with respect to  $\nu$  and  $\nu'$ , since the first four moments match, the contributions of the fully expanded terms in (4.23) coincide and this proves (4.21).

To see (4.22), we assume for simplicity that  $F$  has only one variable and  $i_1 = i$ . Then

$$\frac{\partial \mathcal{F}}{\partial h}(h) = NF'(\lambda) \frac{\partial \lambda_i}{\partial h}(h).$$

By standard first order perturbation theory, with  $h = h_{pq}$ ,

$$\frac{\partial \lambda_i}{\partial h} = 2\Re \mathbf{e} u_i(p) \bar{u}_i(q), \quad (4.24)$$

where  $\mathbf{u}_i = (u_i(1), u_i(2), \dots, u_i(N))$  is the eigenfunction belonging to  $\lambda_i$ . Since the eigenvectors are delocalized,

$$\|\mathbf{u}\|_\infty \approx N^{-1/2} \quad (4.25)$$

(modulo logarithmic corrections, see (1.31)), so we obtain

$$\left|\frac{\partial \lambda_i}{\partial h}\right| \lesssim O(N^{-1}) \quad (4.26)$$

and thus

$$\left|\frac{\partial \mathcal{F}}{\partial h}\right| \lesssim N \cdot N^{c_0} \cdot N^{-1} = N^{c_0}.$$

For higher order derivatives, we have to differentiate the eigenfunctions as well. This gives rise to the resonances, for example with  $h = h_{pq}$

$$\frac{\partial u_i(p)}{\partial h_{pq}} = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} u_j(p) [\bar{u}_j(p) u_i(q) + \bar{u}_j(q) u_i(p)]$$

thus

$$\left| \frac{\partial u_i(p)}{\partial h} \right| \leq N^{-1/2} \frac{1}{N} \sum_{j \neq i} \frac{1}{|\lambda_i - \lambda_j|} \lesssim N^{-1/2+c_0},$$

assuming that the eigenvalues regularly follow the semicircle law and no two neighboring eigenvalues get closer than  $N^{-1-c_0}$ , see (1.77). Substituting this bound into the derivative of (4.24), we have

$$\left| \frac{\partial^2 \lambda_i}{\partial h^2} \right| \lesssim CN^{-1+c_0}.$$

Combining this bound with (4.26), and reinstating the general case when  $F$  has more than one variable, we have

$$\left| \frac{\partial^2 \mathcal{F}}{\partial h^2} \right| \lesssim CN \cdot |\partial_{ii} F| |\partial^2 \lambda_i| + CN^2 |\partial_{ij} F| |\partial \lambda_j| |\partial \lambda_i| \leq CN^{2c_0}.$$

The argument for the higher derivatives is similar. The key technical inputs are the delocalization bound on the eigenvectors (4.25) that can be obtained from local semicircle law and the lower tail estimate (1.77).

## 5 Universality for Wigner matrices: putting it together

In this section we put the previous information together to prove our main result, which is Theorem 5.1 below. Recall that  $p_N(\lambda_1, \lambda_2, \dots, \lambda_N)$  denotes the symmetric joint density of the eigenvalues of the  $N \times N$  Wigner matrix  $H$ . For simplicity we will use the formalism as if the joint distribution of the eigenvalues were absolutely continuous with respect to the Lebesgue measure, but it is not necessary for the proof. Recall that for any  $k \geq 1$  we defined the  $k$ -point correlation functions (marginals) by

$$p_N^{(k)}(\lambda_1, \dots, \lambda_k) = \int_{\mathbb{R}^{N-k}} p_N(\lambda_1, \lambda_2, \dots, \lambda_N) d\lambda_{k+1} \dots d\lambda_N. \quad (5.1)$$

We will use the notation  $p_{N,GUE}^{(k)}$  and  $p_{N,GOE}^{(k)}$  for the correlation functions of the GUE and GOE ensembles.

We consider the rescaled correlation functions about a fixed energy  $E$  under a scaling that guarantees that the local density is one. The sine-kernel universality for the GUE ensemble

states that the rescaled correlation functions converge weakly to the determinant of the sine-kernel,  $K(x) = \frac{\sin \pi x}{\pi x}$ , i.e.,

$$\frac{1}{[\varrho_{sc}(E)]^k} p_{N,GUE}^{(k)} \left( E + \frac{\alpha_1}{N \varrho_{sc}(E)}, \dots, E + \frac{\alpha_k}{N \varrho_{sc}(E)} \right) \rightarrow \det (K(\alpha_\ell - \alpha_j))_{\ell,j=1}^k \quad (5.2)$$

as  $N \rightarrow \infty$  for any fixed energy  $|E| < 2$  in the bulk of the spectrum [70, 20]. Similar result holds for the GOE case; the sine kernel being replaced with a similar but somewhat more complicated universal function, see [68]. Our main result is that universality (5.2) holds for generalized hermitian or symmetric Wigner matrices after averaging a bit in the energy  $E$ :

**Theorem 5.1** [40, 46, 47] *Let  $H$  be an  $N \times N$  symmetric or hermitian generalized Wigner matrix (1.62). In the hermitian case we assume that the real and imaginary parts are i.i.d. Suppose that the distribution  $\nu$  of the rescaled matrix elements  $\sqrt{N}h_{ij}$  have subexponential decay (2.32). Let  $k \geq 1$  and  $O : \mathbb{R}^k \rightarrow \mathbb{R}$  be a continuous, compactly supported function. Then for any  $|E| < 2$ , we have*

$$\begin{aligned} & \lim_{b \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{2b} \int_{E-b}^{E+b} dv \int_{\mathbb{R}^k} d\alpha_1 \dots d\alpha_k O(\alpha_1, \dots, \alpha_k) \\ & \times \frac{1}{[\varrho_{sc}(v)]^k} \left( p_N^{(k)} - p_{N,\#}^{(k)} \right) \left( v + \frac{\alpha_1}{N \varrho_{sc}(v)}, \dots, v + \frac{\alpha_k}{N \varrho_{sc}(v)} \right) = 0, \end{aligned} \quad (5.3)$$

where  $\#$  stands for GOE or GUE for the symmetric or hermitian cases, respectively.

*Proof.* For definiteness, we consider the symmetric Wigner case, i.e.,  $\mathbb{E}h_{ij}^2 = 1/N$  and the limit will be the Gaussian Orthogonal Ensemble (GOE), corresponding to the parameter  $\beta = 1$  in the general formalism. The joint distribution of the eigenvalues  $\mathbf{x} = (x_1, x_2, \dots, x_N)$  is given by the following measure

$$\mu = \mu_N(d\mathbf{x}) = \frac{e^{-\mathcal{H}(\mathbf{x})}}{Z} d\mathbf{x}, \quad \mathcal{H}(\mathbf{x}) = N \left[ \sum_{i=1}^N \frac{x_i^2}{4} - \frac{1}{N} \sum_{i < j} \log |x_j - x_i| \right], \quad (5.4)$$

and we assume that the eigenvalues are ordered, i.e.,  $\mu$  is restricted to  $\Sigma_N$  (3.4).

Let  $\widehat{H}$  be a symmetric Wigner matrix with a single entry distribution satisfying the subexponential decay (2.32). We let the matrix evolve according to the matrix valued Ornstein-Uhlenbeck process:

$$H_t = e^{-t/2} \widehat{H} + (1 - e^{-t})^{1/2} V, \quad (5.5)$$

where  $V$  is an independent GOE matrix. The distribution  $\nu_t(dx) = u_t(x)dx$  of the matrix elements evolves according to the Ornstein-Uhlenbeck process on  $\mathbb{R}$ , i.e.,

$$\partial_t u_t = \mathcal{L} u_t, \quad \mathcal{L} = \frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{x}{2} \frac{\partial}{\partial x}. \quad (5.6)$$

Note that the initial distribution  $\nu = \nu_0$  may be singular, but for any  $t > 0$  the distribution  $\nu_t$  is absolutely continuous.

The Ornstein-Uhlenbeck process (5.6) induces [29] the Dyson Brownian motion on the eigenvalues with a generator given by

$$L = \sum_{i=1}^N \frac{1}{2N} \partial_i^2 + \sum_{i=1}^N \left( -\frac{1}{4} x_i + \frac{1}{2N} \sum_{j \neq i} \frac{1}{x_i - x_j} \right) \partial_i \quad (5.7)$$

acting on  $L^2(\mu)$ . The measure  $\mu$  is invariant and reversible with respect to the dynamics generated by  $L$ .

Denote the distribution of the eigenvalues at time  $t$  by  $f_t(\mathbf{x})\mu(d\mathbf{x})$ . Then  $f_t$  satisfies

$$\partial_t f_t = L f_t \quad (5.8)$$

with initial condition  $f_0$  given by the eigenvalue density of the Wigner matrix  $\widehat{H}$ . With the previous notations,  $p_N = f_0 \mu_N$ , where the  $p_N$  and hence  $f_0$  may be singular with respect to the Lebesgue measure. Due to  $\beta \geq 1$ , the eigenvalues do not cross, i.e., the dynamics (5.8) is well defined on  $\Sigma_N$ . By using a straightforward symmetrization, one can extend the equilibrium measure, the density functions and the dynamics to the whole  $\mathbb{R}^N$ . We will use the formalism of ordered eigenvalues everywhere, except in the definition of the correlation functions (5.1), where the symmetrized version is easier. With a small abuse of notations, we will disregard this difference.

Theorem 5.1 for Wigner matrices was originally proved in [40]. Here we present a more streamlined proof, following [46, 47] that also extends to generalized Wigner matrices. The main part of the proof of Theorem 5.1 consists of three steps:

*Step 1.* First we show that there exists an  $\varepsilon_0 > 0$  such that the correlation functions of *any* Wigner ensemble with a Gaussian convolution of variance  $t \sim N^{-\varepsilon_0}$  coincide with the GOE. In other words, *any* ensemble of the form (5.5) with  $t \geq N^{-\varepsilon_0}$  (and with subexponential decay on the matrix elements of  $\widehat{H}$ ) has a universal local statistics. The  $\varepsilon_0$  depends on the parameters  $\alpha, \beta$  in the subexponential decay condition.

*Step 2.* We then show that for any given Wigner matrix  $H$  we can find another Wigner matrix  $\widehat{H}$  and a time  $t = N^{-\varepsilon_0}$  such that the first three moments of  $H$  and  $H_t = e^{-t/2} \widehat{H} + (1 - e^{-t})^{1/2} V$  coincide and the fourth moments are close by  $O(N^{-\varepsilon_0})$ .

*Step 3.* Theorem 4.2, which was the corollary of the Green function comparison theorem, shows that the local correlation functions of  $H$  and  $H_t$  from Step 2 coincide. Together with Step 1, this will complete the proof of Theorem 5.1.  $\square$

Now we formulate the statements in Step 1 and Step 2 more precisely, Step 3 is already completed.

## 5.1 Step 1: Universality for Gaussian convolutions

This step is just an application of Theorem 3.1, we formulate it for our special case:

**Theorem 5.2** *Suppose that the probability distribution of the initial symmetric Wigner matrix  $\widehat{H}$  has subexponential decay (2.32) with some parameters  $\alpha, \beta$  and let  $H_t$  be given by the Gaussian convolution (5.5). Let  $p_{t,N}^{(k)}$  denote the  $k$ -point correlation function of the eigenvalues of  $H_t$ . Then there exists an  $\varepsilon_0 > 0$ , depending on the parameters in (2.32) such that for any  $t \geq N^{-\varepsilon_0}$  we have*

$$\begin{aligned} \lim_{b \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{2b} \int_{E-b}^{E+b} dv \int_{\mathbb{R}^k} d\alpha_1 \dots d\alpha_k O(\alpha_1, \dots, \alpha_k) \\ \times \frac{1}{[\varrho_{sc}(v)]^k} \left( p_{t,N}^{(k)} - p_{N,GOE}^{(k)} \right) \left( v + \frac{\alpha_1}{N\varrho_{sc}(v)}, \dots, v + \frac{\alpha_k}{N\varrho_{sc}(v)} \right) = 0 \end{aligned} \quad (5.9)$$

for any continuous, compactly supported test function  $O$ .

*Proof.* We just have to check that the assumptions of Theorem 3.1 are satisfied. First, the Hamiltonian of the equilibrium measure is (5.4), and it is clearly of the form (3.5), so Assumption I is automatic. The entropy assumption (3.11) on the initial state  $f_0$  may not be satisfied since  $f_0$  can even be singular. However, by the semigroup property of the OU flow, one can consider the initial condition  $f_{t_0}$  for the flow  $f_t$ ,  $t \geq t_0$ , for some  $t_0 \leq N^{-\varepsilon_0}$  since the statement of Theorem 3.1 concerns only the time  $t \geq N^{-\varepsilon_0}$ . Thus it is sufficient to show that the entropy condition is satisfied for some very small  $t_0 \ll N^{-\varepsilon_0}$ .

To see this, let  $\nu_t$  denote the single entry distribution of  $H$  and  $\bar{\nu}_t$  the probability measure of the matrix  $H_t$ . Let  $\bar{\mu}$  denote the probability measure of the GOE and  $\mu$  the probability measure of its  $ij$ -th element which is a Gaussian measure with mean zero and variance  $1/N$ . Since the dynamics of matrix elements are independent (subject to the symmetry condition), and the entropy is additive, we have the identity

$$\int \log \left( \frac{d\bar{\nu}_t}{d\bar{\mu}} \right) d\bar{\nu}_t = \sum_{i \leq j} \int \log \left( \frac{d\nu_t}{d\mu} \right) d\nu_t \leq CN^2 \int \log \left( \frac{d\nu_t}{d\mu} \right) d\nu_t \quad (5.10)$$

since the summation runs over the indices of all independent elements  $1 \leq i \leq j \leq N$ . Clearly, the process  $t \rightarrow \nu_t$  is an Ornstein-Uhlenbeck process and each entropy term on the right hand side of (5.10) is bounded by  $CN$  provided that  $t \geq t_0 := 1/N$  and  $\nu_0$  has a subexponential decay. Since the entropy of the marginal distribution on the eigenvalues is bounded by the entropy of the total measure on the matrix, we have proved that

$$\int f_{1/N} \log f_{1/N} d\mu \leq CN^3, \quad (5.11)$$

and this verifies (3.11). Therefore, in order to apply Theorem 3.1, we only have to verify the Assumptions II, III and IV. Clearly, Assumptions II and IV follow from the local semicircle law,

Theorem 2.6 with  $\varrho(E) = \varrho_{sc}(E)$  (note that in the bounded variance case  $M \sim N$ ), and Assumption III was proven in Theorem 2.8. Now we can apply Theorem 3.1 and we get (5.9) with any  $\varepsilon_0 > 2\varepsilon$ , where  $\varepsilon$  is obtained from Theorem 2.8.  $\square$

## 5.2 Step 2: Matching Lemma

For any real random variable  $\xi$ , denote by  $m_k(\xi) = \mathbb{E}\xi^k$  its  $k$ -th moment. By Schwarz inequality, the sequence of moments,  $m_1, m_2, \dots$  are not arbitrary numbers, for example  $|m_1|^k \leq m_k$  and  $m_2^2 \leq m_4$ , etc, but there are more subtle relations. For example, if  $m_1 = 0$ , then

$$m_4 m_2 - m_3^2 \geq m_2^3 \quad (5.12)$$

which can be obtained by

$$m_3^2 = [\mathbb{E}\xi^3]^2 = [\mathbb{E}\xi(\xi^2 - 1)]^2 \leq [\mathbb{E}\xi^2][\mathbb{E}(\xi^2 - 1)^2] = m_2(m_4 - 2m_2^2 + 1)$$

and noticing that (5.12) is scale invariant, so it is sufficient to prove it for  $m_2 = 1$ . In fact, it is easy to see that (5.12) saturates if and only if the support of  $\xi$  consists of exactly two points (apart from the trivial case when  $\xi \equiv 0$ ).

This restriction shows that given a sequence of four admissible moments,  $m_1 = 0$ ,  $m_2 = 1$ ,  $m_3, m_4$ , there may not exist a Gaussian divisible random variable  $\xi$  with these moments; e.g. the moment sequence  $(m_1, m_2, m_3, m_4) = (0, 1, 0, 1)$  uniquely characterizes the standard Bernoulli variable ( $\xi = \pm 1$  with  $1/2$ - $1/2$  probability). However, if we allow a bit room in the fourth moment, then one can match any four admissible moments with a small Gaussian convolution. This is the content of the next lemma (from Lemma 3.4 of [47]) and this completes Step 2.

**Lemma 5.3** *Let  $m_3$  and  $m_4$  be two real numbers such that*

$$m_4 - m_3^2 - 1 \geq 0, \quad m_4 \leq C_2 \quad (5.13)$$

*for some positive constant  $C_2$ . Let  $\xi^G$  be a Gaussian random variable with mean 0 and variance 1. Then for any sufficient small  $\gamma > 0$  (depending on  $C_2$ ), there exists a real random variable  $\xi_\gamma$  with subexponential decay and independent of  $\xi^G$ , such that the first four moments of*

$$\xi' = (1 - \gamma)^{1/2} \xi_\gamma + \gamma^{1/2} \xi^G \quad (5.14)$$

*are  $m_1(\xi') = 0$ ,  $m_2(\xi') = 1$ ,  $m_3(\xi') = m_3$  and  $m_4(\xi')$ , and*

$$|m_4(\xi') - m_4| \leq C\gamma \quad (5.15)$$

*for some  $C$  depending on  $C_2$ .*

*Proof.* It is easy to see by an explicit construction that:

**Claim:** For any given numbers  $m_3, m_4$ , with  $m_4 - m_3^2 - 1 \geq 0$  there is a random variable  $X$  with first four moments  $0, 1, m_3, m_4$  and with subexponential decay.

For any real random variable  $\zeta$ , independent of  $\xi^G$ , and with the first 4 moments being  $0, 1, m_3(\zeta)$  and  $m_4(\zeta) < \infty$ , the first 4 moments of

$$\zeta' = (1 - \gamma)^{1/2}\zeta + \gamma^{1/2}\xi^G \quad (5.16)$$

are  $0, 1,$

$$m_3(\zeta') = (1 - \gamma)^{3/2}m_3(\zeta) \quad (5.17)$$

and

$$m_4(\zeta') = (1 - \gamma)^2m_4(\zeta) + 6\gamma - 3\gamma^2. \quad (5.18)$$

Using the Claim, we obtain that for any  $\gamma > 0$  there exists a real random variable  $\xi_\gamma$  such that the first four moments are  $0, 1,$

$$m_3(\xi_\gamma) = (1 - \gamma)^{-3/2}m_3 \quad (5.19)$$

and

$$m_4(\xi_\gamma) = m_3(\xi_\gamma)^2 + (m_4 - m_3^2).$$

With  $m_4 \leq C_2$ , we have  $m_3^2 \leq C_2^{3/2}$ , thus

$$|m_4(\xi_\gamma) - m_4| \leq C\gamma \quad (5.20)$$

for some  $C$  depending on  $C_2$ . Hence with (5.17) and (5.18), we obtain that  $\xi' = (1 - \gamma)^{1/2}\xi_\gamma + \gamma^{1/2}\xi^G$  satisfies  $m_3(\xi') = m_3$  and (5.15). This completes the proof of Lemma 5.3.  $\square$

## A Spectral condition for band matrices

**Lemma A.1** *Let  $B = (\sigma_{ij})$  satisfying (1.17) and (1.18) with  $W \geq 1$  and with  $f$  being a nonnegative symmetric function with  $\int f = 1$  and  $f \in L^\infty(\mathbb{R})$ . Then we have*

$$-1 + \delta \leq B \leq 1 \quad (A.1)$$

for some  $\delta > 0$  and  $W$  large enough, depending on  $f$ .

*Proof.* The upper bound in (A.1) is a trivial Schwarz inequality:

$$\|B\mathbf{u}\|^2 = \sum_i \left| \sum_j \sigma_{ij}^2 u_j \right|^2 \leq \sum_{ijj'} \sigma_{ij}^2 \sigma_{ij'}^2 |u_j| |u_{j'}| \leq \frac{1}{2} \sum_{ijj'} \sigma_{ij}^2 \sigma_{ij'}^2 [|u_j|^2 + |u_{j'}|^2] \leq \|\mathbf{u}\|_2^2,$$

by using (1.17) twice and the symmetry  $\sigma_{ij}^2 = \sigma_{ji}^2$ .



For the lower bound, recall that the discrete Fourier transform in  $d = 1$  dimensions is defined as follows. Let  $\varepsilon := 1/N$  and

$$\Lambda_\varepsilon := \Lambda = \varepsilon\mathbf{Z}/\mathbf{Z}$$

be the periodic one dimensional lattice (torus) of size 1 and spacing  $\varepsilon$  with its dual lattice being

$$\Lambda_\varepsilon^* := \Lambda^* := \left(2\pi\mathbf{Z}\right) / \left(\frac{2\pi}{\varepsilon}\mathbf{Z}\right).$$

Let  $\psi$  be a function on  $\Lambda$ . Then its Fourier transform  $\mathcal{F}_N\psi$  is a function on  $\Lambda^*$  defined as

$$\mathcal{F}_N\psi(p) = \varepsilon \sum_{x \in \Lambda} \psi(x) e^{-ip \cdot x}$$

and it is an isometry

$$\varepsilon \sum_{x \in \Lambda} \overline{\psi(x)} \phi(x) = \sum_{p \in \Lambda^*} \overline{\mathcal{F}_N\psi(p)} \mathcal{F}_N\phi(p).$$

In our case,  $x = k/N$  and define

$$F_W(x) := NW^{-1}f(xN/W).$$

Then, for  $p \in \Lambda^*$ , we have

$$(\mathcal{F}_N F_W)(p) = \sum_{x \in \Lambda} F_W(x) e^{-ip \cdot x} = \sum_{k=1}^N W^{-1}f(k/W) e^{-i(Wp/N) \cdot (k/W)} = \widehat{f}(q) + o(1), \quad q = Wp/N,$$

where the error term vanishes as  $W \rightarrow \infty$  and  $\widehat{f}$  denotes the usual Fourier transform in  $L^1(\mathbb{R})$

$$\widehat{f}(q) = \int f(y) e^{-iqy} dy.$$

With this formula, and with the notation  $\psi_N(j) := \psi(j/N)$  for any  $\psi$  defined on  $\Lambda$ , we have

$$(B\psi_N)(k) = \varepsilon \sum_{\ell=1}^N NW^{-1}f((k-\ell)/W)\psi_N(\ell) = \varepsilon \sum_{y \in \Lambda} F_W\left(\frac{k}{N}-y\right)\psi(y) = \sum_{p \in \Lambda^*} e^{ipk/N} \mathcal{F}_N F_W(p) \mathcal{F}_N\psi(p)$$

and

$$\sum_{p \in \Lambda^*} |\mathcal{F}_N\psi(p)|^2 = \varepsilon \sum_{j=1}^N |\psi_N(j)|^2$$

which is normalized to be 1. Hence  $B$  on the Fourier side acts as a multiplication by the function  $\mathcal{F}_N F_W$ , so

$$\text{Spec} B = \text{Range } \mathcal{F}_N F_W \subset \text{supp } \widehat{f} + o(1).$$

Since  $f$  is nonnegative, symmetric function and  $\int f = 1$ , we have  $\widehat{f}$  is real and

$$\inf \widehat{f} > -1 + \delta$$

for some  $\delta > 0$ , which completes the proof.

## B Large Deviation Estimates

In this Appendix we prove two large deviation results. They are weaker than the corresponding results of Hanson and Wright [59], used in [39, 44], but they require only independent, not necessarily identically distributed random variables with subexponential decay, moreover the proofs are much simpler. Thus the Gaussian decay requirement of Hanson and Wright is relaxed to subexponential, but the tail probability estimate is weaker.

**Lemma B.1** *Let  $a_i$  ( $1 \leq i \leq N$ ) be  $N$  independent complex random variables with mean zero, variance  $\sigma^2$  and uniform subexponential decay, i.e., there exist  $\alpha, \beta > 0$  that for any  $x > 0$*

$$\mathbb{P}(|a_i| \geq x^\alpha) \leq \beta e^{-x}. \quad (\text{B.1})$$

Then for any  $A_i \in \mathbb{C}$  ( $1 \leq i \leq N$ ), we have,

$$\mathbb{P} \left\{ \left| \sum_i a_i A_i \right| \geq (\log N)^{\left(\frac{3}{2} + \alpha\right)} \sigma \left( \sum_i |A_i|^2 \right)^{1/2} \right\} \leq C N^{-\log \log N} \quad (\text{B.2})$$

for some constants  $C$  depending on  $\alpha$  and  $\beta$  in (B.1).

*Proof of Lemma B.1.* Without loss of generality, we may assume that  $\sigma = 1$ . The assumption (B.1) implies that the  $k$ -th moment of  $a_i$  is bounded by:

$$\mathbb{E}|a_i|^k \leq (Ck)^{\alpha k} \quad (\text{B.3})$$

for some  $C > 0$ .

First, for  $p \in \mathbb{N}$ , we estimate

$$\mathbb{E} \left| \sum_{i=1}^N a_i A_i \right|^p. \quad (\text{B.4})$$

With the Marcinkiewicz–Zygmund inequality, for  $p \geq 2$ , we have

$$\mathbb{E} \left| \sum_i a_i A_i \right|^p \leq (Cp)^{p/2} \mathbb{E} \left( \sum_i |a_i A_i|^2 \right)^{p/2} \quad (\text{B.5})$$

(for the estimate of the constant, see e.g. Exercise 2.2.30 of [90]). Inserting (B.3) into (B.5), we have

$$\mathbb{E} \left| \sum_i a_i A_i \right|^p \leq (Cp^{\frac{1}{2} + \alpha})^p \left( \sum_i |A_i|^2 \right)^{p/2}, \quad (\text{B.6})$$

which implies (B.2) by choosing  $p = \log N$  and applying a high moment Markov inequality.  $\square$

**Lemma B.2** Let  $a_i$  ( $1 \leq i \leq N$ ) be  $N$  independent random complex variables with mean zero, variance  $\sigma^2$  and having the uniform subexponential decay (B.1). Let  $B_{ij} \in \mathbb{C}$  ( $1 \leq i, j \leq N$ ). Then we have that

$$\mathbb{P} \left\{ \left| \sum_{i=1}^N \bar{a}_i B_{ii} a_i - \sum_{i=1}^N \sigma^2 B_{ii} \right| \geq (\log N)^{\frac{3}{2}+2\alpha} \sigma^2 \left( \sum_{i=1}^N |B_{ii}|^2 \right)^{1/2} \right\} \leq CN^{-\log \log N} \quad (\text{B.7})$$

and

$$\mathbb{P} \left\{ \left| \sum_{i \neq j} \bar{a}_i B_{ij} a_j \right| \geq (\log N)^{3+2\alpha} \sigma^2 \left( \sum_{i \neq j} |B_{ij}|^2 \right)^{1/2} \right\} \leq CN^{-\log \log N} \quad (\text{B.8})$$

for some constants  $C$  depending on  $\alpha$  and  $\beta$  in (B.1).

*Proof of Lemma B.2.* Without loss of generality, we may assume that  $\sigma = 1$ . First, we prove (B.7). Notice that the random variables  $|a_i|^2 - 1$  ( $1 \leq i \leq N$ ) are independent random variables with mean 0 and variance less than some constants  $C$ . Furthermore, the  $k$ -th moment of  $|a_i|^2 - 1$  is bounded as

$$\mathbb{E}(|a_i|^2 - 1)^k \leq (Ck)^{2\alpha k}. \quad (\text{B.9})$$

Then following the proof of the Lemma B.1 with  $|a_i|^2 - 1$  replacing  $a_i$ , we obtain (B.7).

Next, we prove (B.8). For any  $p \in \mathbb{N}$ ,  $p \geq 2$ , we estimate

$$\mathbb{E} \left| \sum_{i>j} \bar{a}_i \xi_i \right|^p \equiv \mathbb{E} \left| \sum_{i>j} \bar{a}_i B_{ij} a_j \right|^p \quad (\text{B.10})$$

where  $\xi_i := \sum_{j<i} B_{ij} a_j$ . Note that  $a_i$  and  $\xi_i$  are independent for any fixed  $i$ . By the definition,

$$X_n \equiv \sum_{i=1}^n \bar{a}_i \xi_i \quad (\text{B.11})$$

is martingale. Using the Burkholder inequality, we have that

$$\mathbb{E} \left| \sum_i \bar{a}_i \xi_i \right|^p \leq (Cp)^{3p/2} \mathbb{E} \left( \sum_i |\bar{a}_i \xi_i|^2 \right)^{p/2} \quad (\text{B.12})$$

(for the constant, see Section VII.3 of [82]). By the generalized Minkowski inequality, by the independence of  $a_i$  and  $\xi_i$  and using (B.3), we have

$$\left[ \mathbb{E} \left( \sum_i |\bar{a}_i \xi_i|^2 \right)^{p/2} \right]^{2/p} \leq \sum_i \left[ \mathbb{E} |\bar{a}_i \xi_i|^p \right]^{2/p} = \sum_i \left[ \mathbb{E} (|\bar{a}_i|^p) \mathbb{E} (|\xi_i|^p) \right]^{2/p} \leq (Cp)^{2\alpha} \sum_i \left[ \mathbb{E} (|\xi_i|^p) \right]^{2/p}.$$

Using (B.6), we have

$$\mathbb{E}(|\xi_i|^p) \leq (Cp^{\frac{1}{2}+\alpha})^p \left( \sum_j |B_{ij}|^2 \right)^{p/2}.$$

Combining this with (B.12) we obtain

$$\mathbb{E} \left| \sum_i \bar{a}_i \xi_i \right|^p \leq (Cp)^{2p(1+\alpha)} \left( \sum_i \sum_j |B_{ij}|^2 \right)^{p/2}. \quad (\text{B.13})$$

Then choosing  $p = \log N$  and applying Markov inequality, we obtain (B.8) . □

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