A stationary phase method for oscillatory Riemann Hilbert problems

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March 2010

- The Fourier transform:

$$\widehat{f}(\xi) = \int e^{-ix\xi} f(x) dx.$$

- The spatial decomposition $f = f1_{\mathbb{R}_+} + f1_{\mathbb{R}_-}$ leads to the Hardy decomposition:

$$\widehat{f} = H_-(\widehat{f}) + H_+(\widehat{f})$$

- Can invert the Fourier transform via this Hardy decomposition:

$$f(0) = \lim_{z\to\infty} -izH_+(\widehat{f})(z) = \lim_{z\to\infty} izH_-(\widehat{f})(z)$$

if f is sufficiently smooth. To get $f(x)$, we modulate $\widehat{f}$ before the decomposition.

Nonlinear theory

- The one-dimensional scattering transform is a nonlinear Fourier transform. It can be inverted via a *multiplicative* Hardy decomposition (Riemann-Hilbert problem):

$$J(\xi) = M_{-}(\xi)M_{+}(\xi), \ \ \xi \in \mathbb{R}$$

- M_{+}, M_{-} have analytic continuation to opposite half planes and are properly normalized.
- Typically $J = \begin{pmatrix} 1 + p(\xi)q(\xi) & p(\xi) \\ q(\xi) & 1 \end{pmatrix}$ where p,q are nonlinear Fourier data of a function u. Then

$$u(0) = \lim_{z\to\infty} -izM_{+}^{(12)}(z) = \lim_{z\to\infty} izM_{-}^{(12)}(z)$$

To get $u(x)$, modulate the scattering data appropriately before the factorization: $p(\xi) \to e^{ix\xi}p(\xi)$, $q(\xi) \to e^{-ix\xi}q(\xi)$.

The classical stationary phase method

- Using the Fourier transform, solutions of a linear PDE with constant coefficients can be written as oscillatory integrals

$$\int e^{-it\theta(\xi)} g(\xi) d\xi$$

- Asymptotics as $t \to \infty$ of such integrals can be studied by exploiting rapid oscillation of $e^{-it\theta(\xi)}$ away from the stationary points.
- If θ is real valued with stationary points $\lambda_1, \ldots, \lambda_N$ of order $k_1, \ldots, k_N$, then, assuming enough regularity and decay,

$$\int e^{-it\theta(\xi)} g(\xi) d\xi = \sum_{j=1}^{N} \Big[\frac{A_j e^{-it\theta(\lambda_j)}}{t^{\frac{1}{k_j+1}}} + O(t^{-\frac{2}{k_j+1}}) \Big] + O(t^{-k}).$$

Oscillatory Riemann-Hilbert problems

- For many integrable PDEs (NLS, mKdV), the nonlinear Fourier data p, q of the nonlinear solution u and the Fourier transform of the linear solution evolve similarly.

$$J(\xi, 0) \mapsto J(\xi, t) = \begin{pmatrix} 1 + p(\xi)q(\xi) & e^{-it\theta(\xi)}p(\xi) \\ e^{it\theta(\xi)}q(\xi) & 1 \end{pmatrix}$$

- Oscillation of J should lead to cancellation, and thus extraction of asymptotics of u as $t \to \infty$.
- Oscillatory RHPs appear in other nonlinear settings (e.g. random matrix theory).

Methods for oscillatory RHPs

- Deift and Zhou (93): θ analytic (nonlinear steepest descent).
- Varzugin (96): θ non-analytic with stationary points of first order.
- McLaughlin and Miller (06): θ with two Lipschitz derivatives (nonlinear $\overline{\partial}$-steepest descent).
- D. (09): θ with stationary points of arbitrary order: three locally integrable derivatives, Hölder-type continuity condition at stationary points.

Assumptions

In general, $1 + pq > 0$ and p, q have sufficient decay, and

- p, q have two L^2 derivatives;
- Near any stationary point of order ≥ 3, p, q have three L^2 derivatives and $pq < 1$.

These assumptions can be improved if every stationary point has order ≤ 2. For instance,

- For *NLS*: one L^2 derivative for p and q;
- For *mKdV*: one L^2 derivative everywhere and two L^2 derivatives near the stationary points for p and q.

Main Theorem

- θ real valued with stationary points $\lambda_1, \dots, \lambda_N$ of order $k_1, \dots, k_N$. Assume *for simplicity* that $\theta^{(k_j+1)}$ is Lipschitz near λ_j.

Theorem (D.)

For large t our Riemann-Hilbert problem is uniquely solvable; furthermore there are constants $B_j \in \mathbb{R}$, $A_j \in \mathbb{C}$ such that

$$u(t) = \sum_{j=1}^{N} \left[\frac{A_j e^{-it\theta(\lambda_j) + iB_j \ln t}}{t^{\frac{1}{k_j+1}}} + O_\epsilon(t^{-\frac{3}{2(k_j+1)}+\epsilon}) \right]$$

- $B_j = 0$ if k_j is even.
- Interaction of stationary points (hidden in A_j).
- For nice p, q, we can improve $O_\epsilon(t^{-\frac{3}{2(k_j+1)}+\epsilon})$ to $O(t^{-\frac{2}{k_j+1}+\epsilon})$, even $O(t^{-\frac{2}{k_j+1}} \ln t)$.

Ideas of proof

- Follow the spirit of linear theory: Sequence of reductions.

Step 1. Localize $p(\xi), q(\xi)$ to small neighborhood of the stationary points.

Step 2. Near each stationary point, reduce θ to a nice analytic function.

Step 3. The contribution of each stationary point is separated. We then arrive at simpler RHPs, each is localized to one stationary point.

Step 4. The steepest descent argument of Deift and Zhou is applied to each localized analytic RHP, giving a model RHP which can be explicitly studied.

- Step 2 and Step 3 are interchangeable.

Main tool: Beals-Coifman's operator formulation

- For a pair of bounded L^2 weights $w = (w^-, w^+)$, Beals and Coifman introduced the operator

$$C_w f = H_+(fw^-) - H_-(fw^+)$$

  here H_+, H_- are the Hardy projections and act entry-wise on matrices.
- If $J = (I - w^-)^{-1}(I + w^+)$ and if $1 - C_w$ is invertible on L^2 then the respective RHP is uniquely solvable, furthermore the Riemann-Hilbert factors $M_\pm$ can be formulated using C_w.

Perturbation schemes

- To localize p and q, one wants to perturb the weights by oscillatory function $e^{it\Theta}h$ where $\Theta = \pm\theta$. One can assume Θ' keeps the same sign on $supp(h)$.
- We decompose

$$e^{it\Theta}h = H_-(\dots) + H_+(\dots)$$

  and handle the Hardy components separately.
- One Hardy component is *small* on $\mathbb{R}$, the other is *large but very well-structured*: oscillates as $t \to \infty$, and enjoys good analytic continuation.
- Two schemes: direct (for the small component) and indirect (for the large component).

Origin of the interaction of stationary points

- It is essential in our perturbation schemes that:

(i) The oscillating phase in w^- is (locally) decreasing; and

(ii) The oscillating phase in w^+ is (locally) increasing.

- In the canonical factorization

$$J = \begin{pmatrix} 1 & e^{-it\theta} p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{it\theta} q & 1 \end{pmatrix}$$

  the phase-weight relation is incorrect in $\{\theta' < 0\}$.

- Using the solution $\delta_\pm$ of a suitable scalar RHP, we can achieve

$$(w^-, w^+) = \begin{cases} \left(\begin{pmatrix} 0 & e^{-it\theta} P \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ e^{it\theta} Q & 0 \end{pmatrix} \right), & \text{on } \{\theta' > 0\}; \\ \left(\begin{pmatrix} 0 & 0 \\ e^{it\theta} Q & 0 \end{pmatrix}, \begin{pmatrix} 0 & e^{-it\theta} P \\ 0 & 0 \end{pmatrix} \right), & \text{on } \{\theta' < 0\}. \end{cases}$$

- δ_-,δ_+ appear in P, Q and are the source of the $\ln t$ term and the interaction of stationary points.