A stationary phase method for oscillatory Riemann Hilbert problems

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March 2010
The Fourier transform:

\[ \hat{f}(\xi) = \int e^{-ix\xi} f(x) \, dx. \]

The spatial decomposition \( f = f1_{\mathbb{R}^+} + f1_{\mathbb{R}^-} \) leads to the Hardy decomposition:

\[ \hat{f} = H_- (\hat{f}) + H_+ (\hat{f}) \]

Can invert the Fourier transform via this Hardy decomposition:

\[ f(0) = \lim_{z \to \infty} -izH_+ (\hat{f})(z) = \lim_{z \to \infty} izH_- (\hat{f})(z) \]

if \( f \) is sufficiently smooth. To get \( f(x) \), we modulate \( \hat{f} \) before the decomposition.
The one-dimensional scattering transform is a nonlinear Fourier transform. It can be inverted via a multiplicative Hardy decomposition (Riemann-Hilbert problem):

\[ J(\xi) = M_-(\xi)M_+(\xi), \quad \xi \in \mathbb{R} \]

- \( M_+, M_- \) have analytic continuation to opposite half planes and are properly normalized.
- Typically \( J = \begin{pmatrix} 1 + p(\xi)q(\xi) & p(\xi) \\ q(\xi) & 1 \end{pmatrix} \) where \( p,q \) are nonlinear Fourier data of a function \( u \). Then

\[ u(0) = \lim_{z \to \infty} -izM_+^{(12)}(z) = \lim_{z \to \infty} izM_-^{(12)}(z) \]

To get \( u(x) \), modulate the scattering data appropriately before the factorization: \( p(\xi) \to e^{i\xi}p(\xi), \quad q(\xi) \to e^{-i\xi}q(\xi) \).
The classical stationary phase method

- Using the Fourier transform, solutions of a linear PDE with constant coefficients can be written as oscillatory integrals

\[ \int e^{-it\theta(\xi)} g(\xi) d\xi \]

- Asymptotics as \( t \to \infty \) of such integrals can be studied by exploiting rapid oscillation of \( e^{-it\theta(\xi)} \) away from the stationary points.

- If \( \theta \) is real valued with stationary points \( \lambda_1, \ldots, \lambda_N \) of order \( k_1, \ldots, k_N \), then, assuming enough regularity and decay,

\[ \int e^{-it\theta(\xi)} g(\xi) d\xi = \sum_{j=1}^{N} \left[ A_j e^{-it\theta(\lambda_j)} t^{-k_j} \right] + O(t^{-k}) \]
For many integrable PDEs (NLS, mKdV), the nonlinear Fourier data \( p, q \) of the nonlinear solution \( u \) and the Fourier transform of the linear solution evolve similarly.

\[
J(\xi, 0) \mapsto J(\xi, t) = \begin{pmatrix} 1 + p(\xi)q(\xi) & e^{-it\theta(\xi)}p(\xi) \\ e^{it\theta(\xi)}q(\xi) & 1 \end{pmatrix}
\]

- Oscillation of \( J \) should lead to cancellation, and thus extraction of asymptotics of \( u \) as \( t \to \infty \).
- Oscillatory RHPs appear in other nonlinear settings (e.g. random matrix theory).
Methods for oscillatory RHPs

- Deift and Zhou (93): $\theta$ analytic (nonlinear steepest descent).
- Varzugin (96): $\theta$ non-analytic with stationary points of first order.
- McLaughlin and Miller (06): $\theta$ with two Lipschitz derivatives (nonlinear $\overline{\partial}$-steepest descent).
- D. (09): $\theta$ with stationary points of arbitrary order: three locally integrable derivatives, Hölder-type continuity condition at stationary points.
In general, $1 + pq > 0$ and $p, q$ have sufficient decay, and
- $p, q$ have two $L^2$ derivatives;
- Near any stationary point of order $\geq 3$, $p, q$ have three $L^2$ derivatives and $pq < 1$.

These assumptions can be improved if every stationary point has order $\leq 2$. For instance,
- For NLS: one $L^2$ derivative for $p$ and $q$;
- For mKdV: one $L^2$ derivative everywhere and two $L^2$ derivatives near the stationary points for $p$ and $q$. 
Main Theorem

- \( \theta \) real valued with stationary points \( \lambda_1, \ldots, \lambda_N \) of order \( k_1, \ldots, k_N \). Assume for simplicity that \( \theta^{(k_j+1)} \) is Lipschitz near \( \lambda_j \).

**Theorem (D.)**

*For large \( t \) our Riemann-Hilbert problem is uniquely solvable; furthermore there are constants \( B_j \in \mathbb{R}, A_j \in \mathbb{C} \) such that*

\[
    u(t) = \sum_{j=1}^{N} \left[ \frac{A_j e^{-it \theta(\lambda_j) + iB_j \ln t}}{t^{k_j + 1}} + O_\epsilon(t^{-\frac{3}{2(k_j + 1)} + \epsilon}) \right]
\]

- \( B_j = 0 \) if \( k_j \) is even.
- Interaction of stationary points (hidden in \( A_j \)).
- For nice \( p, q \), we can improve \( O_\epsilon(t^{-\frac{3}{2(k_j + 1)} + \epsilon}) \) to \( O(t^{-\frac{2}{k_j + 1} + \epsilon}) \), even \( O(t^{-k_j + 1} \ln t) \).
Ideas of proof

- Follow the spirit of linear theory: Sequence of reductions.

**Step 1.** Localize \( p(\xi), q(\xi) \) to small neighborhood of the stationary points.

**Step 2.** Near each stationary point, reduce \( \theta \) to a nice analytic function.

**Step 3.** The contribution of each stationary point is separated. We then arrive at simpler RHPs, each is localized to one stationary point.

**Step 4.** The steepest descent argument of Deift and Zhou is applied to each localized analytic RHP, giving a model RHP which can be explicitly studied.

- Step 2 and Step 3 are interchangeable.
For a pair of bounded $L^2$ weights $w = (w^-, w^+)$, Beals and Coifman introduced the operator

$$C_w f = H_+(fw^-) - H_-(fw^+)$$

here $H_+, H_-$ are the Hardy projections and act entry-wise on matrices.

If $J = (I - w^-)^{-1}(I + w^+)$ and if $1 - C_w$ is invertible on $L^2$ then the respective RHP is uniquely solvable, furthermore the Riemann-Hilbert factors $M_{\pm}$ can be formulated using $C_w$. 

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A stationary phase method for oscillatory RHPs
To localize $p$ and $q$, one wants to perturb the weights by oscillatory function $e^{it\Theta}h$ where $\Theta = \pm \theta$. One can assume $\Theta'$ keeps the same sign on $\text{supp}(h)$.

We decompose

$$e^{it\Theta}h = H_-(...) + H_+(...)$$

and handle the Hardy components separately.

One Hardy component is small on $\mathbb{R}$, the other is large but very well-structured: oscillates as $t \to \infty$, and enjoys good analytic continuation.

Two schemes: direct (for the small component) and indirect (for the large component).
Origin of the interaction of stationary points

- It is essential in our perturbation schemes that:
  (i) The oscillating phase in \(w^-\) is (locally) decreasing; and
  (ii) The oscillating phase in \(w^+\) is (locally) increasing.

- In the canonical factorization
  \[
  J = \begin{pmatrix} 1 & e^{-it\theta} p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{it\theta} q & 1 \end{pmatrix}
  \]

  the phase-weight relation is incorrect in \(\{\theta' < 0\}\).

- Using the solution \(\delta_\pm\) of a suitable scalar RHP, we can achieve
  \[
  (w^-, w^+) = \begin{cases} 
  \begin{pmatrix} 0 & e^{-it\theta} P \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ e^{it\theta} Q & 0 \end{pmatrix} \end{cases}, & \text{on } \{\theta' > 0\}; \\
  \begin{pmatrix} 0 & 0 \\ e^{it\theta} Q & 0 \end{pmatrix}, \begin{pmatrix} 0 & e^{-it\theta} P \\ 0 & 0 \end{pmatrix} \end{cases}, & \text{on } \{\theta' < 0\}.
  \]

- \(\delta_-, \delta_+\) appear in \(P, Q\) and are the source of the \(\ln t\) term and the interaction of stationary points.