CLT for first-passage time along thin cylinders

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March 18, 2010
Consider the $d$-dimensional square lattice $\mathbb{Z}^d$ where each edge $e$ has i.i.d. nonnegative passage time $\omega_e$ from a fixed distribution $F$. 
For any path $\mathcal{P}$, define the **passage time** for $\mathcal{P}$ by

$$\omega(\mathcal{P}) := \sum_{e \in \mathcal{P}} \omega_e.$$
For two vertices $x, y \in \mathbb{Z}^d$, the first-passage time $a(x, y)$ is defined as the minimum passage time over all paths from $x$ to $y$. 
Known results: mean behavior

- This model was introduced by Hammersley and Welsh ('65) to model flow of liquid through random media.

- When $\mathbb{E}[^\omega] < \infty$, by subadditivity

$$\nu(x) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[a(0, nx)]$$

exists and is finite for all $x \in \mathbb{Z}^d$ (HW('65)).
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- Kesten ('86) proved that,

\[
\nu(x) > 0 \text{ iff } F(0) = \mathbb{P}(\omega = 0) < p_c(d)
\]

where $p_c(d)$ is the critical probability for bond percolation in $\mathbb{Z}^d$. 
Known results: fluctuation bounds

- Bounds on $\text{Var}(a(0, nx))$ when $F(0) < p_c(d)$:
  - lower bound of $\log n$ for $d = 2$
    due to Pemantle and Peres('94), Newman and Piza('95) and Zhang('08).
  - upper bound of $cn / \log n$ for general $d$
    due to Benjamini, Kalai and Schramm('03).
  - conjectured bound for $d = 2$, $\text{Var}(a(0, nx)) \approx n^{2/3}$.

- It is also conjectured that the distance between the minimizing path and the straight line path joining $0$ to $nx$ is $\approx n^{2/3}$ for $d = 2$. 
Known results: fluctuation bounds

- Bounds on $\text{Var}(a(0, nx))$ when $F(0) < p_c(d)$:
  - **Lower** bound of $c \log n$ for $d = 2$
    due to Pemantle and Peres('94), Newman and Piza('95) and Zhang('08).
  - **Upper** bound of $cn / \log n$ for general $d$
    due to Benjamini, Kalai and Schramm('03).
  - **Conjectured** bound for $d = 2$, $\text{Var}(a(0, nx)) \approx n^{2/3}$.

- It is also conjectured that the **distance** between the minimizing path and the straight line path joining 0 to $nx$ is $\approx n^{2/3}$ for $d = 2$.

- **Nothing** is known about the limiting distribution of $a(0, nx)$ when $F(0) < p_c(d)$. 
Main result: Gaussian Limit

Consider the first-passage time $a_n(h)$ from 0 to $(n, 0, \ldots, 0)$ in the graph $\mathbb{Z} \times [-h, h]^{d-1}$. 
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Theorem (Chatterjee and D.’09)

Suppose $F(0) < p_c(d)$ and $E[\omega^k] < \infty$ for all $k$. Let $\{h_n\}$ be a sequence of integers satisfying

$$h_n \ll n^{\frac{1}{d+1}}.$$

Then

$$\frac{a_n(h_n) - E[a_n(h_n)]}{\sqrt{\text{Var}(a_n(h_n))}} \Rightarrow N(0, 1) \text{ as } n \to \infty.$$
Results: Moment bounds

We have,

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[a_n(h_n)] = \nu(1, 0, \ldots, 0)$$

when $h_n \to \infty$. 
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- For all \( h_n \leq n \),
  \[ \frac{c n}{h_n^{d-1}} \leq \text{Var}(a_n(h_n)) \leq \frac{C n}{1 + \log h_n} \]
  where \( c, C \) depend only on \( F \) and \( d \).
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- For all \( h_n \leq n \),
  \[ \mathbb{E} |a_n(h_n) - \mathbb{E}[a_n(h_n)]|^k \leq c n^{k/2} \]
  where \( c \) depends only on \( F \) and \( d \).
Reason for the CLT: \( d = 2 \) case

- Write \( n = ml \) with \( l \geq h_n \).
- Break \([0, n] \times [-h_n, h_n]\) into \( m \) blocks

\[ B_i = [(i-1)l, il] \times [-h_n, h_n] \text{ for } 1 \leq i \leq m. \]
Reason for the CLT: $d = 2$ case

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- Break $[0, n] \times [-h_n, h_n]$ into $m$ blocks
  
  $$B_i = [(i-1)l, il] \times [-h_n, h_n] \text{ for } 1 \leq i \leq m.$$ 

- Let $X_i$ be the minimum passage time over all paths joining left boundary of $B_i$ to its right boundary inside the block $B_i$.
- $X_i$’s are i.i.d. for $1 \leq i \leq m$. 
We have

\[ a_n(h_n) \geq X_1 + X_2 + \cdots + X_m. \]
Approximation as i.i.d. sum

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We also have

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\[
\mathbb{E}\left[ \frac{a_n(h_n) - \mathbb{E}[a_n(h_n)]}{\sqrt{\text{Var}(a_n(h_n))}} \right] - \sum_{i=1}^{m} \frac{X_i - \mathbb{E}[X_i]}{\sqrt{\text{Var}(a_n(h_n))}} \leq \frac{4\mathbb{E}[Z^2]}{\text{Var}(a_n(h_n))}. 
\]
Thus $a_n(h_n)$ is approximately a sum of i.i.d. random variables when

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We need the condition

$$h_n^3 \ll \frac{n}{m^2} \text{ or } h_n \ll \frac{n^{1/3}}{m^{2/3}}.$$
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Actual proof involves a renormalization argument and the moment bounds.
Open problems

- What is the threshold for CLT? Note that $(d + 1)^{-1} \to 0$ as $d \to \infty$. Is it possible to derive CLT upto $n^\alpha$, where $\alpha$ is uniformly away from zero for all $d$.

- For oriented percolation we have a limiting Tracy Widom distribution. How to explain the transition?

- Results on the order of the variance will give nontrivial bound for the unrestricted case.

- Finally, the structure of the minimizing path is mostly unknown. The path is known to be chaotic.
Thank you!