CLT for first-passage time along thin cylinders

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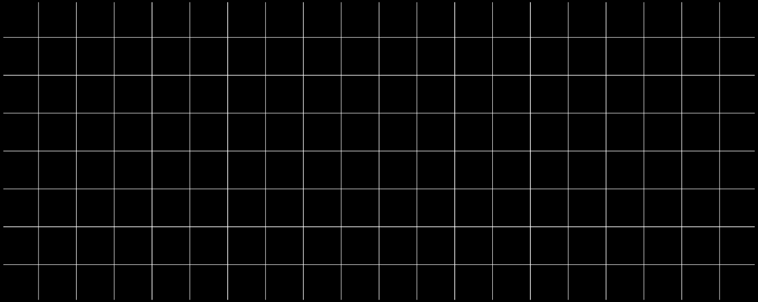
joint work with Sourav Chatterjee, UCB

AZ School of Analysis
March 18, 2010

The model



- Consider the d-dimensional square lattice $\mathbb{Z}^d$
  where each edge e has i.i.d. nonnegative passage time ω_e
  from a fixed distribution F.

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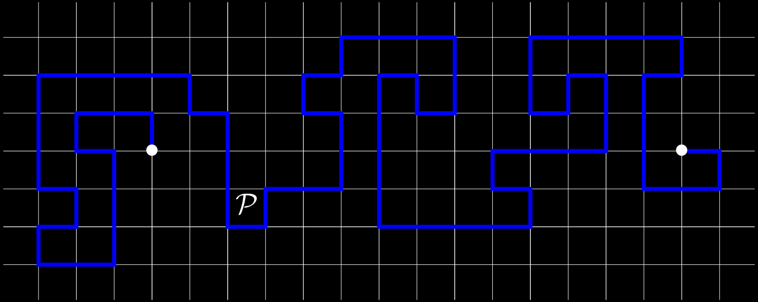

- For any path $\mathcal{P}$, define the passage time for $\mathcal{P}$ by

$$\omega(\mathcal{P}) := \sum_{e \in \mathcal{P}} \omega_e.$$

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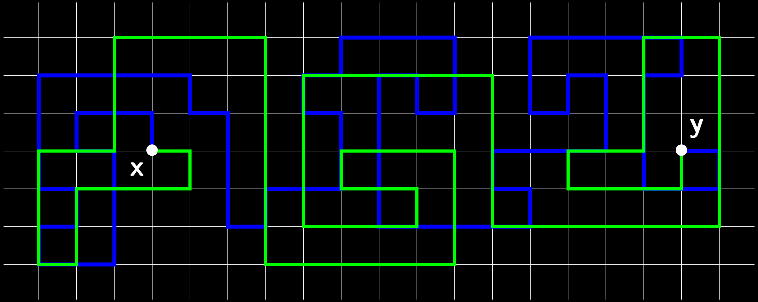

- For two vertices $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, the first-passage time $a(\mathbf{x}, \mathbf{y})$ is defined as the minimum passage time over all paths from $\mathbf{x}$ to $\mathbf{y}$.

Known results: mean behavior

- This model was introduced by Hammersley and Welsh('65) to model flow of liquid through random media.

- When $\mathbb{E}[\omega] < \infty$, by subadditivity

$$\nu(\mathbf{x}) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}[a(\mathbf{0}, n\mathbf{x})]$$

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- Kesten('86) proved that,

$$\nu(\mathbf{x}) > 0 \text{ iff } F(0) = \mathbb{P}(\omega = 0) < p_c(d)$$

where $p_c(d)$ is the critical probability for bond percolation in $\mathbb{Z}^d$.

Known results: fluctuation bounds

- Bounds on $\mathbb{V}\text{ar}(a(\mathbf{0}, n\mathbf{x}))$ when $F(0) < p_c(d)$:
  - lower bound of $c \log n$ for $d = 2$
    due to Pemantle and Peres('94), Newman and Piza('95) and Zhang('08).
  - upper bound of $cn/\log n$ for general d
    due to Benjamini, Kalai and Schramm('03).
  - conjectured bound for $d = 2$, $\mathbb{V}\text{ar}(a(\mathbf{0}, n\mathbf{x})) \approx n^{2/3}$.
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- It is also conjectured that the distance between the minimizing path and the straight line path joining $\mathbf{0}$ to $n\mathbf{x}$ is $\approx n^{2/3}$ for $d = 2$.
- Nothing is known about the limiting distribution of $a(\mathbf{0}, n\mathbf{x})$ when $F(0) < p_c(d)$.

Main result: Gaussian Limit

Consider the first-passage time $a_n(h)$ from $\mathbf{0}$ to $(n, 0, \ldots, 0)$ in the graph $\mathbb{Z} \times [-h, h]^{d-1}$.

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Theorem (Chatterjee and D.'09)

Suppose $F(0) < p_c(d)$ and $\mathbb{E}[\omega^k] < \infty$ for all k. Let $\{h_n\}$ be a sequence of integers satisfying

$$h_n \ll n^{\frac{1}{d+1}}.$$

Then

$$\frac{a_n(h_n) - \mathbb{E}[a_n(h_n)]}{\sqrt{\mathbb{V}\mathrm{ar}(a_n(h_n))}} \Longrightarrow N(0,1) \text{ as } n \to \infty.$$

Results: Moment bounds

- We have,

$$\lim_{n\to\infty} \frac{1}{n}\,\mathbb{E}[a_n(h_n)] = \nu(1,0,\ldots,0)$$

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$$\frac{cn}{h_n^{d-1}} \le \mathbb{V}\mathrm{ar}(a_n(h_n)) \le \frac{Cn}{1+\log h_n}$$

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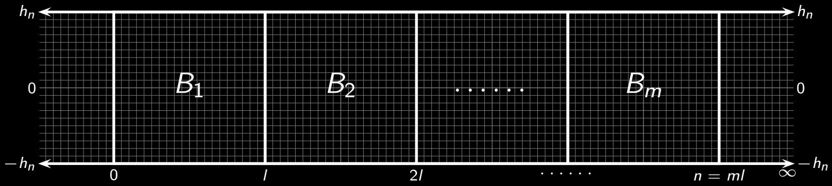
- For all $h_n \le n$,

$$\mathbb{E}\left|a_n(h_n) - \mathbb{E}[a_n(h_n)]\right|^k \le cn^{k/2}$$

  where c depends only on F and d.

Reason for the CLT: $d = 2$ case



- Write $n = ml$ with $l \geq h_n$.
- Break $[0, n] \times [-h_n, h_n]$ into m blocks

$$B_i = [(i-1)l, il] \times [-h_n, h_n] \text{ for } 1 \leq i \leq m.$$

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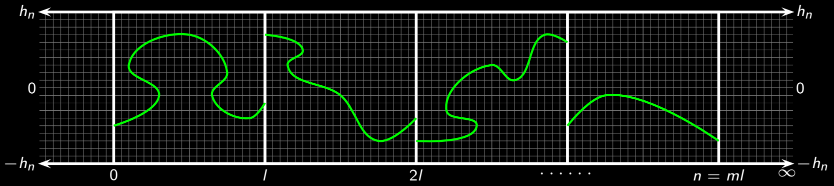

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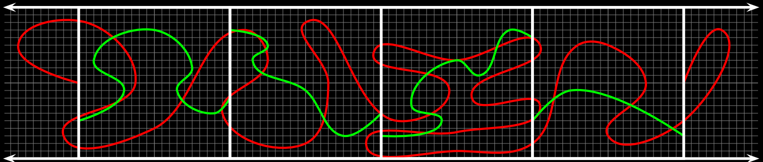
- Let X_i be the minimum passage time over all paths joining left boundary of B_i to its right boundary inside the block B_i.
- X_i's are i.i.d. for $1 \leq i \leq m$.

Approximation as i.i.d. sum



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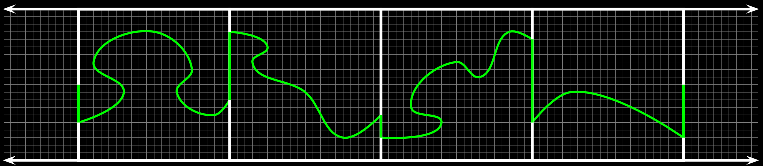

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- We also have
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where Z is sum of all edge-weights in the left/right boundaries of B_i.

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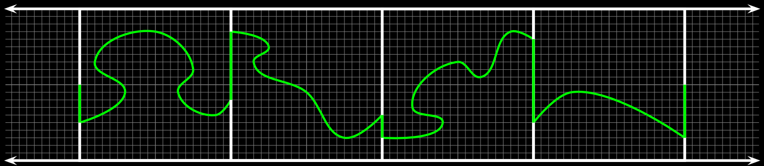

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- $$\mathbb{E}\left|\frac{a_n(h_n) - \mathbb{E}[a_n(h_n)]}{\sqrt{\mathbb{V}\mathrm{ar}(a_n(h_n))}} - \sum_{i=1}^{m} \frac{X_i - \mathbb{E}[X_i]}{\sqrt{\mathbb{V}\mathrm{ar}(a_n(h_n))}}\right|^2 \leq \frac{4\mathbb{E}[Z^2]}{\mathbb{V}\mathrm{ar}(a_n(h_n))}.$$

Approximation as i.i.d. sum (contd.)

- Thus $a_n(h_n)$ is approximately a sum of i.i.d. random variables when

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- Actual proof involves a renormalization argument and the moment bounds.

Open problems

- What is the threshold for CLT? Note that $(d+1)^{-1} \to 0$ as $d \to \infty$. Is it possible to derive CLT upto n^{α}, where α is uniformly away from zero for all d.
- For oriented percolation we have a limiting Tracy Widom distribution. How to explain the transition?
- Results on the order of the variance will give nontrivial bound for the unrestricted case.
- Finally, the structure of the minimizing path is mostly unknown. The path is known to be chaotic.

Thank you!